A new method judging the interpolation direction

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Abstract: - The discrete Fourier transform (DFT) algorithms are commonly used in frequency estimation. The interpolation direction of the interpolated algorithms based on DFT is always defined by the information of spectral amplitude, such as the Rife algorithm. In the case of low signal to noise ratio (SNR), false decision of the interpolation direction occurs sometimes. However, decreasing false rate of the interpolation direction decision can improve the frequency estimation accuracy directly. In this paper, a new method to judge the interpolation direction based on spectral phase information instead of amplitude is presented, which is helpful to improve the original Rife algorithm or other interpolated algorithms based on DFT. The simulation results demonstrate that the false rate of interpolation direction judged by the proposed method is obviously lower than that by the original Rife algorithm in the case of low SNR, and also show that its achievement of a low variance close to the Cramer-Rao lower bound.

Key-Words: - Frequency estimation; Rife algorithm; interpolation direction

1 Introduction

Estimating the frequency of signals with low signal to noise ratio (SNR) is an important issue in many applications such as communications, radar and sonar signal processing, and deep space exploration. Searching the spectral peak of a sinusoidal signal is the simplest and most efficient technique of frequency estimation [1-3]. This method is called discrete Fourier transform algorithm (DFT). The data processing includes signal truncating, sampling, Fourier transform and spectral peak search in frequency domain. In most cases, the spectral peak searched is not the exact frequency of the input signal because of “picket-fence effect” and noise. Namely the frequency estimate resulting from the DFT are normally different from the real one, as mentioned in [4].

Some interpolated algorithms based on DFT have been studied to get higher frequency estimate accuracy [1-2, 5-9]. The Rife algorithm is a classic one of interpolated algorithms. The interpolation value is found between the two highest spectral lines to get signal frequency estimation based on periodogram and discrete Fourier coefficient. The Rife algorithm includes two different search stages: coarse search and fine search. The coarse search finds the general location of frequency by seeking the spectral peak. The fine search uses the secant method to compute the precise location of the frequency estimated. Some modified Rife algorithms (M-Rife algorithms) have also been studied to improve the performance in the case of low SNR [10-12]. These M-Rife algorithms fall into two categories. One keeps the interpolation value between 0.4 and 0.5 by shifting the total frequency spectra of the input signal [10-11]; the other makes the two highest spectral lines located in the main lobe by using window function [12].

For the Rife algorithm, the determination of interpolation direction plays a key role. In the case of low SNR, the Rife algorithm sometimes gets false interpolation direction. And the interpolation in the false direction may lead to greater estimation errors.

Some window functions have been discussed to reduce the sidelobe [4, 13-14]. In these window functions, the Kaiser-Bessel or the Blackman-Harris window was declared the top performer [14]. However, the original formula calculating the interpolation value is no longer applicable due to adding the window function. So a scaling constant is applied to the formula to deal with the effects of adding different window functions [15]. Moreover, much more computation has been cost to find the modified scaling constant [16].

However few have considered how to improve interpolation direction decision in frequency estimation algorithms based on DFT in the case of low SNR. In this paper, we propose a new method that the interpolation direction is judged by phase information instead of amplitude in frequency spectra. The false rate of interpolation direction decision is effectively reduced and the frequency estimation accuracy is then improved in the case of low SNR.

The next content is structured as follows. In section 2, the method that judges the interpolation direction in Rife algorithm is described. Then, a new method to judge the interpolation direction is proposed in section 3. Next, the new interpolation direction judging method is applied to the original Rife algorithm. And the comparison of the improved Rife algorithm with the original Rife algorithm is studied by in Section 4. Finally some conclusion remarks are given in Section 5.

2 The interpolation of Rife algorithm

Assuming there is a discrete signal $y(n)$ with length N and sampling rate f_s. Rife algorithm can be simply described as follows:

(1) Calculate its frequency spectra $Y(k) = FFT(y(n))$, here $n = 0,1,2,\cdots,N-1$, $k = 0,1,2,\cdots,N-1$, and find the spectral peak position $k_{\max}$ in N points with $\max(|Y(k)|)$;

(2) Compare $|Y(k_{\max}-1)|$ with $|Y(k_{\max}+1)|$, if $|Y(k_{\max}-1)| > |Y(k_{\max}+1)|$, then $r = -1$; Otherwise, $r = 1$;

(3) Find the interpolation value

$$\delta = \frac{|Y(k_{\max}+r)|}{|Y(k_{\max}+r)| + |Y(k_{\max})|} \tag{1}$$

(4) Estimate the frequency finally

$$\hat{f} = \frac{k_{\max} + r\cdot\delta}{N} f_s \tag{2}$$

The interpolation direction in Rife algorithm is judged by checking the two highest spectral lines in the mainlobe of amplitude spectra. In the case of high SNR, for example 0dB or above, the determination of interpolation direction in this case is usually accurate. However, when the real frequency is quite close to the spectral peak line, the left spectral line and the right one have about the same amplitude. So it is possible to get a false interpolation direction for noise interference in the case of low SNR.

Similar to the above Rife algorithm, other interpolation methods mostly make use of the amplitude spectral information of DFT, and also taking into no consideration of phase information. Next we will present a new method which judges the interpolation direction of Rife algorithm based on the phase information of the spectral lines.

3 A new method judging the interpolation direction

In fact, the interpolation direction can also be judged by the phase information of the spectral lines in main lobe. In the previous study, we find that if the real frequency of the signal lies between two spectral lines, the product of their imaginary parts is negative. In other words, the imaginary part signs of the two spectral lines are different. So are their real parts.

Assuming there is a discrete signal $x(n)$ with length N,

$$x(n) = Ae^{j(2\pi f_0 n/f_s + \phi)}, n = 0,1,2,\cdots,N-1 \tag{3}$$

Where A is the amplitude, φ is the initial phase, f_s is the sampling rate, and f_0 is the frequency to be estimated.

Let its Fourier Transform denote as $X(k)$, namely

$$X(k) = \sum_{n=0}^{N-1} x(n)\exp(-j2\pi nk/N) \tag{4}$$

And it is assumed that its $k_{\max}$th spectral line is the highest one of $X(k)$. According to the geometric series summation formula, the Fourier Transform in (4) can be rewritten as

$$\begin{aligned} X(k_{\max}) &= A\exp(j\phi)\frac{\exp(j2\pi N\Delta)-1}{\exp(j2\pi\Delta)-1} \\ &= A\frac{\cos(2\pi N\Delta+\phi)-\cos(\phi)}{\cos(2\pi\Delta)-1+j\sin(2\pi\Delta)} \\ &\quad + jA\frac{\sin(2\pi N\Delta+\phi)-\sin(\phi)}{\cos(2\pi\Delta)-1+j\sin(2\pi\Delta)} \end{aligned} \tag{5}$$

where $\Delta = \frac{f_0}{f_s} - \frac{k_{\max}}{N}$.

Ignoring the amplitude, the imaginary and real parts of (5) can be expressed as

$$\begin{aligned} \mathrm{Im}(X(k_{\max})) &= \frac{\sin(2(N-1)\pi\Delta+\phi)}{2-2\cos(2\pi\Delta)} \\ &+ \frac{\sin(\phi)-\sin(2N\pi\Delta+\phi)-\sin(\phi-2\pi\Delta)}{2-2\cos(2\pi\Delta)} \end{aligned} \tag{6}$$

$$\begin{aligned} \mathrm{Re}(X(k_{\max})) &= \frac{\cos(2(N-1)\pi\Delta+\phi)}{2-\cos(2\pi\Delta)} \\ &+ \frac{\cos(\phi)-\cos(2N\pi\Delta+\phi)-\cos(\phi-2\pi\Delta)}{2-2\cos(2\pi\Delta)} \end{aligned} \tag{7}$$

The numerators of imaginary and the real parts are similar. Firstly let's consider the imaginary part. The numerator of the imaginary part in (6) can be rewritten as below

$$\begin{aligned}
&\sin(2(N-1)\pi\Delta+\phi)+\sin(\phi)-\sin(2N\pi\Delta+\phi)\\
&-\sin(\phi-2\pi\Delta)\\
&=\sin((N-1)\pi\Delta+\phi+(N-1)\pi\Delta)\\
&+\sin((N-1)\pi\Delta+\phi-(N-1)\pi\Delta)\\
&-\sin\left(\left(N-1\right)\pi\Delta+\phi+\left(N+1\right)\pi\Delta\right)\\
&-\sin((N-1)\pi\Delta+\phi-(N+1)\pi\Delta)\\
&=2\sin((N-1)\pi\Delta+\phi)\cos((N-1)\pi\Delta)\\
&-2\sin((N-1)\pi\Delta+\phi)\cos((N+1)\pi\Delta)\\
&=2\sin((N-1)\pi\Delta+\phi)(\cos((N-1)\pi\Delta)-\cos((N+1)\pi\Delta))
\end{aligned} \tag{8}$$

If $\Delta > 0$, the real frequency is situated at the right side of the spectral peak line. It means that the interpolation direction is at right. The imaginary part of the $(k_{\max}+1)$th spectral line is:

$$\begin{aligned}
&\mathrm{Im}(X(k_{\max}+1))\\
&=\frac{\sin(2(N-1)\pi(\Delta-\frac{1}{N})+\phi)-\sin(\phi-2\pi(\Delta-\frac{1}{N}))}{2-2\cos(2\pi(\Delta-\frac{1}{N}))}\\
&+\frac{\sin(\phi)-\sin(2N\pi(\Delta-\frac{1}{N})+\phi)}{2-2\cos(2\pi(\Delta-\frac{1}{N}))}
\end{aligned} \tag{9}$$

And its numerator can be expressed as

$$\begin{aligned}
&\sin(2(N-1)\pi(\Delta-\frac{1}{N})+\phi)-\sin(\phi-2\pi(\Delta-\frac{1}{N}))\\
&+\sin(\phi)-\sin(2N\pi(\Delta-\frac{1}{N})+\phi)\\
&=\sin(2(N-1)\pi\Delta+\phi+\frac{2\pi}{N})-\sin(\phi-2\pi\Delta+\frac{2\pi}{N})\\
&\quad+\sin(\phi)-\sin(2N\pi\Delta+\phi)\\
&=2\sin((N-1)\pi\Delta+\phi+\frac{\pi}{N})\cos((N-1)\pi\Delta+\frac{\pi}{N})\\
&\quad-2\sin((N-1)\pi\Delta+\phi+\frac{\pi}{N})\cos((N+1)\pi\Delta-\frac{\pi}{N})\\
&=2\sin((N-1)\pi\Delta+\phi+\frac{\pi}{N})\times\\
&\quad(\cos((N-1)\pi\Delta+\frac{\pi}{N})-\cos((N+1)\pi\Delta-\frac{\pi}{N}))
\end{aligned} \tag{10}$$

Let's consider the signs of the imaginary parts in (6) and (9). When $0<\Delta<\frac{1}{2N}$,

$$0<(N-1)\pi\Delta<(N+1)\pi\Delta<\frac{(N+1)\pi}{2N}<\pi \tag{11a}$$

$$\begin{aligned}
&(N+1)\pi\Delta-\frac{\pi}{N}<N\pi\Delta-\frac{\pi}{2N}\\
&<N\pi\Delta+\frac{\pi}{2N}<(N-1)\pi\Delta+\frac{\pi}{N}<\pi
\end{aligned} \tag{11b}$$

When the length of signal is large enough, we have such an inequality:

$$0<(N+1)\pi\Delta-\frac{\pi}{N}<(N-1)\pi\Delta+\frac{\pi}{N}<\pi \tag{12}$$

Then,

$$\cos((N-1)\pi\Delta)-\cos((N+1)\pi\Delta)>0 \tag{13}$$

$$\cos((N-1)\pi\Delta+\frac{\pi}{N})-\cos((N+1)\pi\Delta-\frac{\pi}{N})<0 \tag{14}$$

$$\sin((N-1)\pi\Delta+\phi)\approx\sin((N-1)\pi\Delta+\phi+\frac{\pi}{N}) \tag{15}$$

The signs of $\sin((N-1)\pi\Delta+\phi)$ and $\sin((N-1)\pi\Delta+\phi+\pi/N)$ are same, while $\cos((N-1)\pi\Delta)-\cos((N+1)\pi\Delta)$ and $\cos((N-1)\pi\Delta+\pi/N)-\cos((N+1)\pi\Delta-\pi/N)$ have different signs. So, when $\Delta>0$, $\mathrm{Im}(X(k_{\max}))\cdot\mathrm{Im}(X(k_{\max}+1))<0$.

We can also get the other result: when $\Delta<0$, $\mathrm{Im}(X(k_{\max}))\cdot\mathrm{Im}(X(k_{\max}+1))>0$.

Simultaneously, when $\Delta>0$, $\mathrm{Re}(X(k_{\max}))\cdot\mathrm{Re}(X(k_{\max}+1))<0$; and when $\Delta<0$, $\mathrm{Re}(X(k_{\max}))\cdot\mathrm{Re}(X(k_{\max}-1))<0$.

So far, the assumption in the beginning of Section 3 has been proved. The phase differences between the peak spectral line and its adjacent lines can be applied to the interpolation direction decision. For example, if the phase difference between the peak line and its right line is bigger than 90°, then the interpolation is in the right direction, otherwise it is in the left direction.

4 Simulation

The phase information of spectral lines can be used to judge the interpolation direction when the length of the data is large enough. The phase information has one advantage over the amplitude of frequency spectra. When the estimated frequency is close to the real frequency, the amplitudes of two spectral lines adjacent to the peak one are about identical. However, their imaginary/real parts have different signs. So the phase information is more reliable to judge the interpolation direction than amplitude in theory.

Simulation of judging the interpolation direction was done to show the difference between two methods of spectral amplitude and phase. The signal was added with -30dB Gaussian white noise, and its initial frequency f_0 was randomly set between 50kHz and 60kHz at each time. The simulation results of 10000 runs have been listed in Table 1. The number of false interpolation direction times was 1627 via the amplitude method; however the other number was only 353 via the phase method discussed in Section 3. It is demonstrated that the method via phase is statistically more reliable to judge the interpolation direction than the one via amplitude in the case of SNR=-30dB.

Table 1. The interpolation direction decision false times of two methods.

Decision method	False times
Via amplitude	1627
Via phase	353

In the next simulation, the false decision rate of interpolation direction by spectral phase information is compared with that by spectral amplitude in a wide range of SNR. At first, the SNR was set as -20dB. Similar to the simulation terms above, 5000 runs were done, while the initial frequency f_0 was randomly set between 50kHz and 60kHz with random initial phase. And the sampling rate was $200kSps$. Then, the SNR was set in succession as -19dB, -18dB, and so on, till 20dB. The number of false interpolation direction decision times was recorded with SNR changing. The results in term of false rate have been shown in Figure 1.

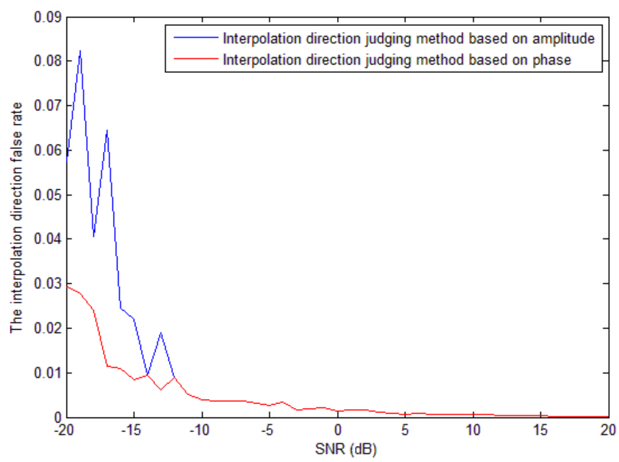

Figure 1.The false rates of two interpolation direction judging methods.

It is obvious that the false rate of interpolation direction decision based on spectral phase information is always lower than that based on spectral amplitude. The more accurate interpolation direction decision brings higher estimation accuracy, especially in the case of low SNR. The simulation of frequency estimate was done by applying the proposed method of judging interpolation direction to the original Rife algorithm.

The signal was generated similarly to that described above at -20dB. The initial frequency was set as 54768Hz. The interpolation value is still gotten by Equation (1). However the interpolation direction is judged by the phase information of the spectral peak line and its two adjacent lines. Run 50 times and got the average estimate bias. Then the initial frequency changed from 54768Hz to 54769Hz with step of 0.001Hz. The average frequency estimate bias of 50 simulation times was recorded with the initial frequency changing.

The frequency estimate bias of the improved Rife algorithm was compared with that by the original Rife algorithm, as shown in Figure 2. No matter what the initial frequency is, the frequency estimate bias of the improved Rife algorithm is obviously lower than that of the original Rife algorithm when the initial frequency is close to a spectral line. However the estimate bias difference is not obvious when the initial frequency is far away from spectral lines.

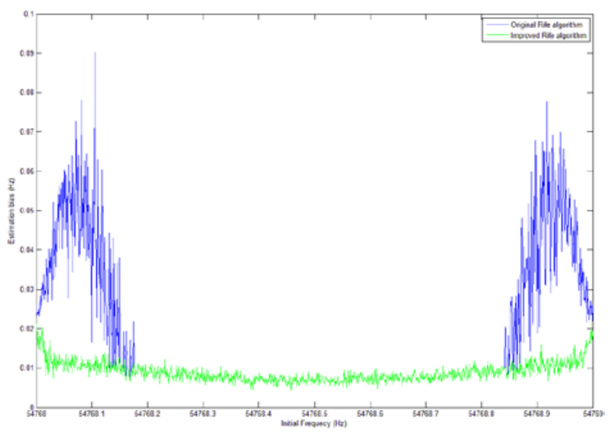

Figure 2.The frequency estimate bias of two algorithms (SNR = -20dB).

The interpolation false rates in above simulation were also recorded, and shown in Figure 3. Obviously, the interpolation false rate accords with the estimate bias in Figure 2. It is also demonstrated that the interpolation false rate plays an important role in frequency estimation.

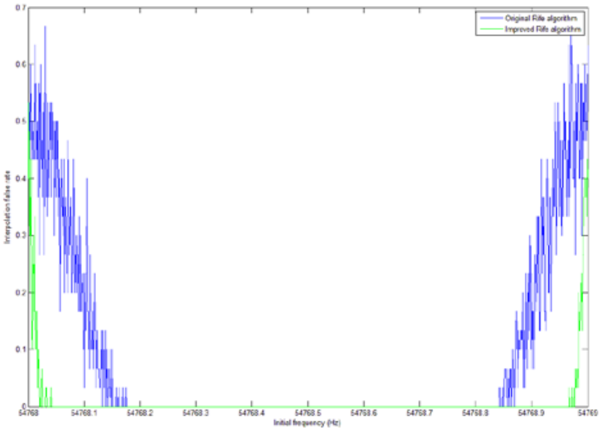

Figure 3.The interpolation false rates of two algorithms with different initial frequencies (SNR = -20dB)

The frequency estimate results of two methods for different SNRs were gotten, as shown in Figure 4. The large estimate biases scatter around spectral lines. It is reasonable for the interpolation direction is hard to judge when the real frequency is quite close to a spectral line. From the results, the frequency estimate bias decrease with the SNR increasing. However, the frequency estimate bias of the improved Rife algorithm is always lower than that of the original algorithm. The RMSE of the frequency estimate bias in Figure 4 is given in Figure 5 correspondingly.

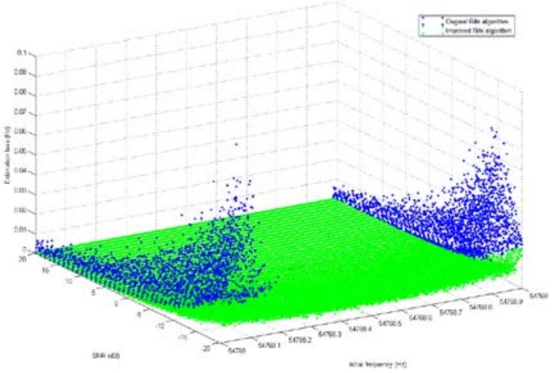

Figure 4. The estimate bias of the original Rife and improved Rife algorithms with different SNRs

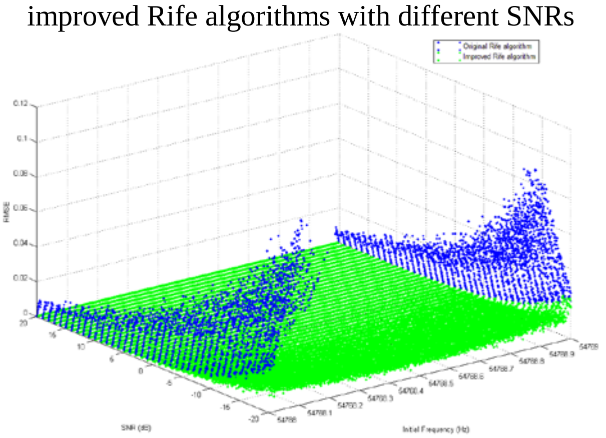

Figure 5. The frequency estimate RMSE of two algorithms for different SNRs.

In order to demonstrate in details, four different initial frequencies were chosen, as shown in Figure 5. The estimate RMSE of each frequency for the improved and original Rife algorithms was illustrated in Figure 6.

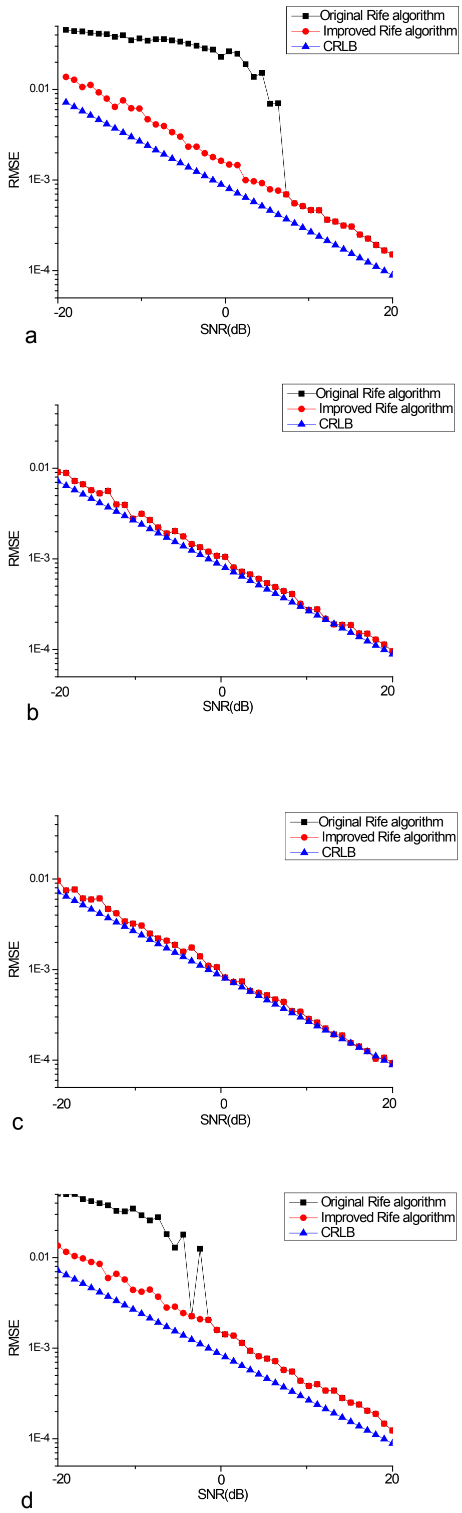

Figure 6. The RMSE of initial frequencies for the improved and original Rife algorithms:
(a) f_0=54768.035Hz, (b) f_0=54768.335Hz,
(c) f_0=54768.635Hz, (d) f_0=54768.935Hz.

For the frequency f_0=54768.035Hz, it is very close to the spectral peak line from the right. The original Rife and improved Rife algorithms have the same amount of RMSE with SNR around 4dB and above. In the case of SNR less than 4dB, however, the RMSE of the improved Rife algorithm was quite closer to the Cramer-Rao lower bound (CRLB) than that of the original Rife algorithm.

For the cases of frequency f_0=54768.335Hz and f_0=54768.635Hz, which are situated in the middle of two spectral lines, the RMSE for two algorithms are same.

For the frequency f_0=54768.935Hz, it is also very close to the spectral peak line but from the left. Two algorithms have the same amount of RMSE with the SNR above 0dB. However the RMSE of the improved Rife algorithm is closer to CRLB than that of the original one when the SNR is lower than -2dB.

In summary, from Figure 6 (a) and (d), our improved Rife algorithm has some advancement in the case of low SNR when the real frequency is close to the spectral peak line of a sinusoidal signal.

4 Conclusion

For the interpolated algorithms based on DFT, the interpolation direction affects the frequency estimate accuracy directly. An interpolation direction judging method from spectral phase information is proposed to improve DFT algorithms. Its false rate was compared with that by spectral amplitude in the paper. It is less sensible to noise interference than that from spectral amplitude in simulation. The algorithm improved by our interpolation direction judging method is more accurate and stable than the original algorithm in the case of low SNR. Therefore, the interpolation direction judging method can be applied to frequency estimation based on the interpolated DFT algorithms, especially in the case of low SNR.

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