On the Linear Relation of Signals

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Abstract: - In this paper two signals over finite and closed interval of time are considered. A linear relation of the two with shifting is defined. In general, when a signal is expressed linearly of another with shifting, this gives an error. The absolute error should be minimized to have the best approximation. For the case of discrete signals without shifting, minimizing this error is just the well-known least square method. Hence, the proposed relation of signals is a generalization of the least square method. Criterion of approximating a signal linearly of the other is based on the relative error. The relative error is defined by comparing the norm of absolute error with the norm of the normalized approximated signal. Geometric interpretation and the applications in finance are also discussed. Especially in finance, the relation of signals can be applied to predict the dynamics of stocks, exchange rates and commodity prices. Predicting such dynamics on itself is very difficult, especially for high frequency data.

Key-Words: - Relation of signals, Norm, Relative error, Dynamics of stocks, exchange rates, Commodity prices

1 Introduction

The dynamics of stocks, exchange rates and commodity prices have been the interest of researchers in various fields including economists, mathematicians, even physicists. However, predicting such dynamics on itself is difficult or probably impossible, especially for high frequency data.

Economists often study the dynamics and relate it to the underlying fundamental economy or perception. Among others, Evans and Lyons [9] studied the effect of macro news on the exchange rates. Their findings are that macro news flow increased order flow volatility and the induced order flow has signed (first moment) effects on the exchange rate. In the other paper [10], they studied the linkages between transaction flows, exchange rates, and future fundamentals. Pao *et al*. [17] studied the effect of bond premium and capital regulatory on loan and deposit rates, as an option-pricing model.

The dynamical relationship among Italian, Spanish, and United Kingdom prices over the period of 1874–1998 was studied by Gadea *et al*. in [11]. It was found that the price of differential dynamics in the period of 1940–1998 was captured by deterministic trends. The cross-dynamics of volatility term structure slopes implied by foreign exchange options was examined by Krylova *et al*. [14]. Their empirical findings demonstrated that a few principal components could explain a vast proportion of the variation in volatility term structure slopes across the major exchange rates. Muchnik *et al*. [16] also studied volatility, but focused on the return of an investment on shares and exchange rates.

Finding a trend of the dynamics of stocks, exchange rates and commodity prices is often used to understand the behavior of the dynamics. Béreau *et al*. [3] studied the nonlinear dynamics of the real exchange rate towards its behavioral equilibrium value. Their finding was that the real exchange rate convergence process in the long-run was characterized by nonlinearities for emerging economies, but industrialized countries exhibited a linear pattern. Cahyono *et al*. [7] applied higher order polynomial to model the dynamics of Jakarta Composite Index. On the other hand, Adam *et al*. [1] considered candle representation data to obtain the so called temporal probability density function (t-pdf) of the dynamics. The behavior of such dynamics is often applied in modeling of stock market. Take an example modeling of Cyprus stock market by Bougioukou [4].

Theoretical scientists, such as mathematicians and physicists, are mostly interested in creating linkages of the dynamics of stocks, exchange rates and commodity prices with the existing theories or to develop a new theory. Often, theorists apply or develop existing mathematical models for the dynamics. For example, in the case of Italian stock market, D'Amigo and Petriani [2] uses semi-Markov environment to model stock market, and shows that it performed better than simple Markov model.

Physicists often model such dynamics by applying well-known physics related phenomena. Some applied continuous time random walk (CTRW), which is interpreted with a model of anomalous diffusion, where a cloud of particles spread at a rate different from the classical Brownian motion. A review of the CTRW for asset dynamics can be found in [12]. Budinski-Petkovića *et al*. [5] presented analysis of the USA stock market using fractal function. Their finding was a good agreement with the S&P 500 data when a complete financial growth is considered. However, moving the final time of the fitting interval towards earlier dates caused growing discrepancy between two curves. Xia *et al*. [20] applied the multi-scale entropy and multi-scale time irreversibility to analyze the financial time series. Both methods had nearly the same classification results, which mean that they were capable of distinguishing different series in a reliable manner. They observed that effects of noise on Americas markets and Europe markets were relatively more significant than on Asia markets.

Other theorists deal with high frequency data. Todorova and Vogt [19] analyzed high frequency financial data from XETRA and the NYSE using maximum likelihood estimation and the Kolmogorov–Smirnov statistic to test whether the power law hypothesis. They found that the universality and scale invariance properties of the power law were violated. Ponta *et al*. [18] conducted a simulation of high-frequency market data which was performed with the Genoa Artificial Stock Market. Their simulation results showed that this mechanism could reproduce fat-tailed distributions of returns without ad-hoc behavioral assumptions on agents. Çağlar *et al*. [6] applied a statistical methodology to estimate the model parameters with an application on high-frequency price data, and to validate the model by simulations with the estimated parameters. They found that the statistical properties of agent level behavior were reflected on the stock price, and could affect the entire process.

The main interest of the investors, however, is the optimal return of an investment portfolio. Chang *et al.* in [8] reported a study of the effect of liquidity on the stock returns, where they found that the liquidity effect was a significant investment style in stock market. On the other hand, Li *et al.* in [15] studied an optimal investment problem which considered taxes, dividends transaction cost. According to their study, taxes and dividends had positive effects on the optimal investment strategies, whereas transaction costs exerted a negative effect on the optimal strategies. Optimizing return of an investment portfolio needs the ability to predict the dynamics of stocks, exchange rates and commodity prices considered in the portfolio. The prediction may be based on the dynamics on itself, which is very difficult, or based on its relation of the dynamics of the other(s).

2 Linear Relation of Signals

In this section a linear relation of two signals with shifting will be discussed.

Definition 1 *Let a,b,T,τ be real numbers, and $T>0$. Continuous real functions f and g are defined in a closed interval $[0,T]$. Function g is defined to be in a linear form of f shifted by τ in the interval $[T_1,T_2]$ if*

$$g(t)=a\cdot f(t-\tau)+b, \tag{1}$$

for any $t\in[T_1,T_2]$ and $T_1,T_2,T_1-\tau,T_2-\tau$ are in $[T_1,T_2]$. The parameters a, b and τ are called the slope, the intercept and the time lag of the linear relation (1).

Plotting the function in Cartesian coordinate *t*-*y*, *a* represents dilatation parameter with respect to horizontal axis *t*. The parameter τ represents horizontal translation, to the right for $\tau>0$ and to the left for $\tau<0$. The parameter *b* represents vertical translation, upward for $b > 0$, and downward for $b < 0$. Figure 1 shows an illustrative plot of functions f and g, where

$$g(t)=3\cdot f(t-20)-275. \tag{2}$$

Moreover, in this figure g is the dynamics of Japanese yen (JPY) relative to Indonesian rupiah (IDR) in the year 2012.

Eqn (1) can be applied in finance as follows. Suppose *f* and *g* be the dynamics of stock A and stock B, respectively. If $\tau > 0$ as illustrated in Fig.1, function *g* is shifted to the right from function *f*. The value of *g* can be predicted based on the value of *f* at the previous time, $g(t)$ depends on $f(t-\tau)$. Hence, the dynamics of stock B can be predicted by applying the dynamics of stock A. Predicting the dynamics of stock A on itself is very difficult, especially for high frequency data, see [13]. It can also be interpreted that the dynamics of stock A has influence to the dynamics of stock B with the time lag τ.

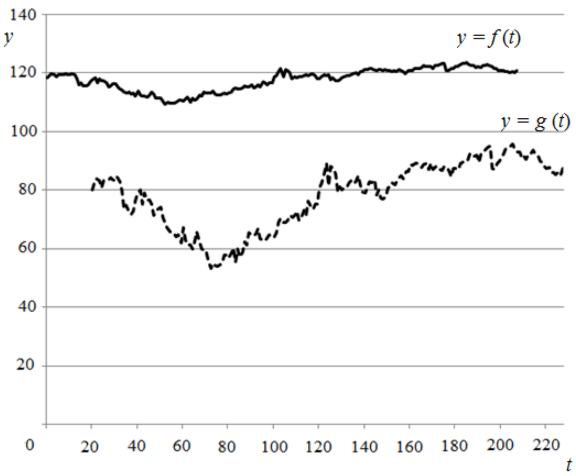

Fig. 1 Linear relation of two signals with shifting.

If f can be expressed in a linear form of g, then the vice versa also applies. In this case, (1) can be written in the form of

$$f(t') = \frac{1}{a} \cdot g(t' + \tau) - \frac{b}{a}, \tag{2}$$

where $t' \in [T_1 - \tau, T_2 - \tau] \subseteq [0, T]$. Hence, if g is obtained by translating f to the right direction as far as $|\tau|$, then f can also be obtained by translating g to the left direction as far as $|\tau|$ (with multiplication and addition related to coefficient a and b).

3 Linear Approximation of a Signal

Previous section discusses two signals that can be expressed linearly to the other. In general, however, one signal is not exactly in the linear form of the other. In this case, a signal, say g, will be approximated linearly by the other signal f. The definition is presented in the following.

Definition 2 *Let f and g be continuous functions define in* $[0,T]$. *Approximating g linearly in f with shifting τ is writing g in the form of*

$$g(t) = a\ f(t - \tau) + b + \varepsilon(t) \tag{3}$$

for real numbers a, b and τ, $T_1 \leq t \leq T_2$ and T_1, T_2, $T_1 - \tau$, $T_2 - \tau$ in $[0,T]$. *The term $\varepsilon(t)$ is the error of the approximation.*

To have best approximation, one needs to minimize the error

$$\varepsilon(t) = g(t) - \left(a\ f(t - \tau) + b\right). \tag{4}$$

To do so, a norm of $\varepsilon(t)$ should be defined.

3.1 Problem Formulation

Minimizing the error (4), should be done by minimizing its norm. In this paper, the norm of the error $\varepsilon(t)$ is defined in the form

$$\|\varepsilon(t)\| = \int_{T_1}^{T_2} \left(g(t) - \left(a\ f(t - \tau) + b\right)\right)^2 dt\,. \tag{5}$$

To find the best approximation of g linearly in f with shifting is summarized in the following problem.

Problem (*) *Let f and g be continuous functions defined in* $[0,T]$. *For T_1, T_2 in [0,T] and $T_1 \leq t \leq T_2$, f and g satisfy (3). Find a, b and τ for $T_1 - \tau$, $T_2 - \tau$ in [0,T], such that (5) minimum.*

3.2 Existence and uniqueness of a, b and τ

One may ask the existence and uniqueness of a, b and τ in the problem above. For existence, this summarized in the following theorem.

Theorem 1 *There exist a, b and τ for the problem (*).*

Proof : Since $f(t-\tau)$ and $g(t)$ are continuous functions in $[T_1,T_2]$, this implies that $\left(g(t) - \left(a\ f(t-\tau) + b\right)\right)^2$ is also continuous function in $[T_1,T_2]$. Hence, the integral (5) exists for any real numbers a, b and τ. Moreover, the integral (5) is non negative. This guarantees the existence of a, b and τ such that (5) is minimum.

□

For uniqueness, it will be restricted for the case of $\tau = 0$.

Theorem 1 *If $\tau = 0$, there exist a unique value of a and b for the problem (*).*

Proof : It will be easier to analyzed the uniqueness of *a* and *b*, by writing (5) in the form

$$\|\varepsilon(t)\| = a^2 \int_{T_1}^{T_2} \big(f(t-\tau)\big)^2 dt + b^2 - 2a \int_{T_1}^{T_2} f(t-\tau) g(t) dt + 2ab \int_{T_1}^{T_2} f(t-\tau) dt + \int_{T_1}^{T_2} \big(g(t)\big)^2 dt. \quad (6)$$

For the case $\tau = 0$, (6) becomes

$$\|\varepsilon(t)\| = a^2 \int_{T_1}^{T_2} \big(f(t)\big)^2 dt + b^2 - 2a \int_{T_1}^{T_2} f(t) g(t) dt + 2ab \int_{T_1}^{T_2} f(t) dt + \int_{T_1}^{T_2} \big(g(t)\big)^2 dt. \quad (7)$$

Observe that (7) is a quadratic function of two variables *a* and *b*. The coefficient of terms a^2 and b^2 are positive. Hence, there is only one global minimum point.

□

In general Problem (*) does not yield a unique value of a, b and τ. Take a simple example for $f(t) = \cos t$ and $g(t) = \sin t$. In this case, $a = 1$, $b = 0$, but τ may be equal to several values, namely $\tau = \frac{1}{2}\pi, 2\frac{1}{2}\pi, 4\frac{1}{2}\pi, \cdots$.

3.3 Case of Discrete Signal

For digital signal, *f* and *g* are given in time series, namely at discrete time t_i. Suppose $\Delta t = t_{i+1} - t_i$, $t_1 = T_1 + \Delta t$, $\tau = n_\tau \cdot \Delta t$ and $T_2 = T_1 \cdot n \cdot \Delta t = t_n$. The error $\|\varepsilon(t)\|$ is computed by using its Riemann sum

$$\|\varepsilon(t)\| = \sum_{i=1}^{n} \big(g(t_i) - (a\ f(t_i - \tau) + b)\big)^2 \Delta t\,. \quad (8)$$

For the case of $\tau = 0$, (8) becomes

$$\|\varepsilon(t)\| = \sum_{i=1}^{n} \big(g(t_i) - (a\ f(t_i) + b)\big)^2 \Delta t\,. \quad (9)$$

Minimizing $\|\varepsilon(t)\|$ in (9) means seeking *a*, *b* such that

$$\|\varepsilon(t)\| = \sum_{i=1}^{n} \big(g(t_i) - (a\ f(t_i) + b)\big)^2 \quad (10)$$

minimum. This is nothing else, but the least square method. Therefore, Problem (*) is a generalization of the well-known least square method.

4 Relative Error of Approximation

Although the best approximation has already obtained, it might not be a good approximation. It is because, the given signal cannot be approximated linearly by the other. An illustration for the case of least square method is given in Fig. 2. The solid line is the best linear approximation of the given points, compared to the dotted line. The solid line, however, does not very well represent the trend of the points. It is because the points do not yield a trend, at least not a linear trend.

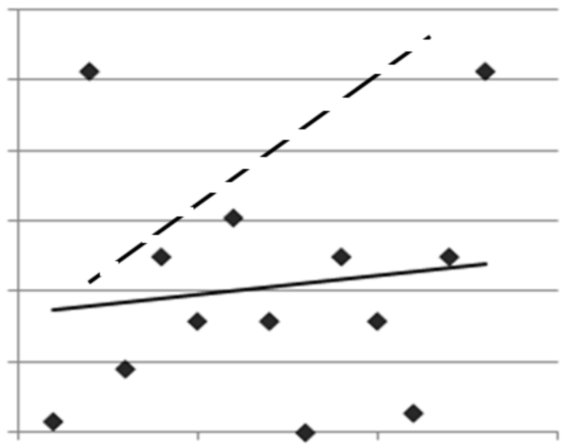

Fig. 2 Illustration of the best linear approximation by using least square (solid line), and other approximation line (dotted line).

Relative error is applied for a criterion whether a signal is a good linear approximation of the other. The relative error is defined by comparing the absolute error to the norm of the normalized approximated signal. The normalized approximated signal *g* is defined by

$$g_{normal}(t) = g(t) - \frac{\int_{T_1}^{T_2} g(t) dt}{T_2 - T_1}. \quad (11)$$

Illustrative plot of the normalized g is presented in Fig. 3. The curves of (11) pass through the horizontal axis, creating area at above of and below horizontal axis. The area above the horizontal axis is equal to the area below the horizontal axis.

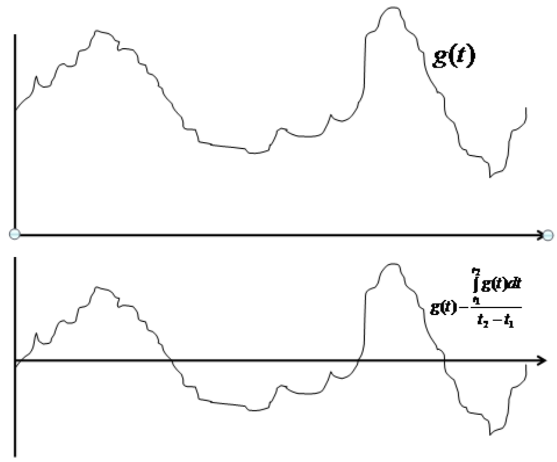

Fig. 3 Illustrative plot of the normalized signal.

Hence mathematically, the relative error is in the form

$$E_{rel} = \frac{\|\varepsilon(t)\|}{\|g_{normal}(t)\|} = \frac{\int_{T_1}^{T_2} \left(g(t) - \left(a\ f(t-\tau) + b\right)\right)^2 dt}{\int_{T_1}^{T_2} \left(g(t) - \frac{\int_{T_1}^{T_2} g(t)dt}{T_2 - T_1} \right)^2 dt}. \qquad (12)$$

The relative error may exceed 100%, if the absolute error is larger than the norm of the normalized approximated signal. There is no standard for the goodness of relative error, the less, the better.

For the application in derivative trading, the interpretation of the norm of normalized signal (12) is the following. The norm (12) related to the area of the normalized signal with the horizontal axis. It is related to the total of difference of the adjacent local minimum and local maximum of the curve. These differences may yield the total profit. Hence, the relative error is related to the percentage error of the total profit.

5 Conclusion

A linear relation of two signals with shifting is introduced. The absolute value of the shift is often called as the time lag. In this relation, a signal can be predicted by applying the other. Therefore, it will be very useful for the application for predicting the dynamics of stocks, exchange rates and commodity prices. Predicting such dynamics is very difficult on itself, especially for high frequency data.

In general, a signal is not in a linear expression of the other. In this case, a signal is approximated in a linear expression of the other which yields an (absolute) error. To have the best approximation, the error should be minimized to obtain the intercept, the slope and the time lag. The existence of the parameters is guaranteed. For the case of zero time lag, there are unique values of slope and intercept. For discrete signals, minimizing the error is the well-known least square method. Therefore, approximating a signal linearly by the other signal with shifting is a generalization of the least square method.

The relative error is defined by comparing the absolute error with the norm of normalized approximated signal. This error is applied to determine whether a signal can be approximated linearly by the other signal with shifting. This criterion still needs to be investigated further in the future research.

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References:

[1] P. Adam, La Gubu and E. Cahyono, Statistical Characteristics of Jakarta Composite Index (JCI) Dynamics based on Short Term Data Represented in Candles, *International Journal of Economics, Finance and Management Sciences*. Vol. 2, No. 2, 2014, pp. 138-142. doi: 10.11648/j.ijefm.20140202.14

[2] G. D'Amigo and F. Petroni, A semi-Markov model for price returns, *Physica A Statistical Mechanics and its Applications*, Vol. 391, No. 20, 2012, pp.4867–4876.

[3] S. Béreau, A.L. Villavicencio and V. Mignon, Nonlinear adjustment of the real exchange rate towards its equilibrium value: A panel smooth transition error correction modeling, *Economic Modelling*, Vol. 27, No. 1, 2010, pp. 404-416.

[4] A. Bougioukou, Modeling Cyprus stock market, *WSEAS Transactions on Mathematics*, Vol. 11, Issue 12, 2012, pp. 1076-1084.

[5] Lj. Budinski-Petkovića, I. Lončarevića, Z.M. Jakšićb and S.B. Vrhovac, Fractal properties of

financial markets, *Physica A: Statistical Mechanics and its Applications*, Vol. 409, 2014, pp. 43–53.

[6] M. Çağlar, N. Bahtiyar and I. Altıntaş, Parameter estimation of an agent-based stock price, *Applied Stochastic Model in Business and Industry*, 2013, doi: 10.1002/asmb.1968.

[7] E. Cahyono, B. Sarita, P. Adam and F.P. Arisanty, The trend and dynamics distribution of the Jakarta stock exchange (JSX) composite, *International Journal of Economics and Finance Studies*, Vol. 4, No. 2, 2012, pp. 77-85.

[8] K. W. Chang, G. Y. Wang and C. Lu, The Effect of Liquidity on Stock Returns: A Style Portfolio Approach, *WSEAS Transactions on Mathematics*, Vol. 12, Issue 2, 2013, pp. 170-179.

[9] M.D.D. Evans and R.K. Lyons, How is macro news transmitted to exchange rates?, *Journal of Financial Economics*, Vol. 88, No. 1, 2008, pp. 26-50.

[10] M.D.D. Evans and R.K. Lyons, Exchange Rate Fundamentals and Order Flow, *The Quarterly Journal of Finance*, Vol. 2, No. 4, 2012.

[11] M.D. Gadea, M.B. Kaabia and M. Sabaté, Exchange rate regimes and prices: The cases of Italy, Spain and the United Kingdom (187-1998), *Journal of International Financial Markets, Institutions and Money*, Vol. 19, No. 3, 2009, pp. 477-489.

[12] M.M. Meerschaert and E. Scalas, Couple continuous random walks in finance, *Physica A: Statistical Mechanics and its Applications*, Vol. 370, 2006, pp. 114-118.

[13] T. Ito and K. Sato, Exchange rate change and inflation in post-crisis Asian economies: Vector autoregression analysis of exchange rate pass-through. *Journal of Money, Credit and Banking*, Vol. 40, No. 7, 2008, pp. 1407-1437.

[14] E. Krylova, J. Nikkinen and S. Vähämaä, Cross-dynamics of votality term structures implied by foreign exchange options, *Journal of Economics and Business*, Vol. 61, No. 5, pp. 355-375.

[15] D. Li, X. Rong and H. Zao, Optimal investment problem with taxes, dividends and transaction costs under the constant elasticity of variance (CEV) model, *WSEAS Transactions on Mathematics*, Vol. 12, Issue 3, 2013, pp. 243-245.

[16] L. Muchnik, A. Bunde and S. Havlin, Long term memory in extreme returns of financial time series, *Physica A: Statistical Mechanics and its Applications*, Vol. 388, No. 19, 2009, pp. 4145-4150.

[17] S. H. Pao, J. H. Lin and S. H. Chang, The Effects of the Change of Bond Insurance Premium and Capital Regulatory Ratio on Loan and Deposit Rates: An Option-Pricing Model, *WSEAS Transactions on Mathematics*, Vol. 10, Issue 7, 2011, pp. 219-228.

[18] L. Ponta, E. Scalas, M. Raberto and S Cincotti, Statistical analysis and agent-based microstructure modeling of high-frequency financial trading, *IEEE Journal of Selected Topics in Signal Processing*, Vol. 6. No. 4, 2012, pp. 381-387.

[19] L. Todorova and B. Vogt, Power law distribution in high frequency financial data? An econometric analysis, *Physica A: Statistical Mechanics and its Applications*, Vol. 389, Issue 21, 2010, pp. 4433–4444

[20] J. Xia, P. Shang, J. Wang and W. Shi, Classifying of financial time series based on multiscale entropy and multiscale time irreversibility, *Physica A: Statistical Mechanics and its Applications*, Vol. 391, Issue 1-2, 2012, pp. 151–158.