On the Designing of Fractional Order FIR Differentiator Using Radial Basis Function and Window

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Abstract: The previous work in [1], Tseng et al. have designed a fractional order differentiator using radial basis function by directly truncating the coefficients to approximate the fractional order derivative D^{α} of the given digital signal. This paper presents the designing of fractional order differentiator using radial basis function and window. Three design examples are given to illustrate that the use of window along with radial basis function method, improve the frequency response characteristics and minimize the integral root square error than the existing radial basis function method.

Key–Words: Fractional order derivative, Digital differentiator, Grunwald-Letnikov derivative, Radial basis function (RBF).

1 Introduction

In recent years, fractional order differentiators and integrators have been an important topic of research in fractional calculus. A digital fractional order differentiator can estimate the fractional order derivative of a digital signal [2]. From the last few decades, the concept of fractional derivative has received great attention in many applications of engineering, science and technology including image processing [3], fractional order PID controller, automatic control [4], fluid dynamics [4], electromagnetic theory [5], phase lock loop [6], electrical networks and probability [7] etc.

In the area of fractional calculus, the integer order derivative of $D^n f(x) = \frac{d^n f(x)}{dx^n}$ (nth order derivative of the function $f(x)$) is generalized to fractional order derivative $D^{\alpha} f(x) = \frac{d^{\alpha} f(x)}{dx^{\alpha}}$, where n is an integer and α is a real number [2][4]. The fractional operator D^{α} is implemented both in continuous and digital domain. An excellent survey of this implementation has been presented in [8]. Some methods have been developed to design digital FIR and IIR filters such as Taylor series method [9], Newton series [10], fractional fourier transform [11], impulse invariant method [12], least squares method [13], fractional sample delay [14], and Savitzky-Golay method [15][16].

In [1], Tseng et al. have designed a fractional order differentiator using radial basis function by directly truncating the coefficients to approximate the fractional order derivative D^{α} of the given digital signal. We are applied window for truncating the filter coefficients.

The rest of the paper is organized as follows: In Section 2, the definition of Grnwald-Letnikov derivative and the design of fractional order FIR differentiator using window for truncation are explained. In Section 3, three numerical examples demonstrate that the fractional derivative of given signal can estimated accurately. Finally, conclusions are made.

2 Design of Fractional Order FIR Differentiator

The details of RBF interpolation method can be found in [1]. For the completeness of the paper, we are briefly explaining the RBF design method, which is taken from [1]. The ideal frequency response of fractional order differentiator is given by $(jw)^{\alpha}$, where α is a fractional number in the range $(0, 1)$. We are designing a fractional order digital differentiator that approximates the given frequency response

$$H_d(e^{j\omega}) = (j\omega)^{\alpha} e^{-j\omega I} \tag{1}$$

where I is a prescribed delay value. There are three most frequently used definitions for the general fractional differintegral are: the Riemann-Liouville (R-L),

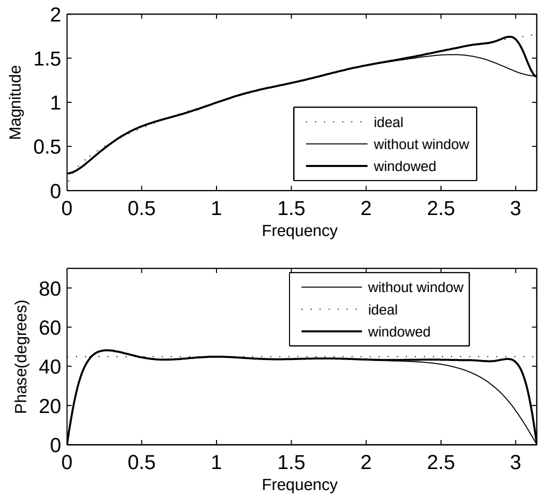

Figure 1: Magnitude and phase response of fractional order differentiator using direct truncation and windowed gaussian RBF with $\sigma = 2.3$ and $\alpha = 0.5$.

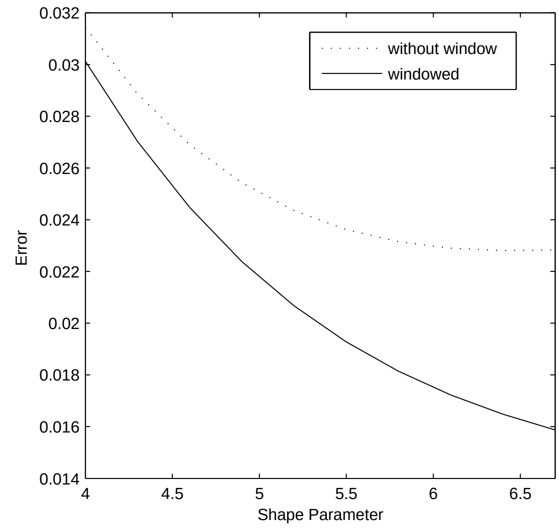

Figure 2: Integral root-squared error E of the fractional order differentiator using gaussian RBF with different shape parameter σ

the Grnwald- Letnikov (G-L) and the Caputo definitions [2][4][17][18]. The Grnwald- Letnikov derivative is given by

$$D^{\alpha}f(t) = \lim_{\Delta\to 0}\sum_{k=0}^{\infty}\frac{(-1)^k C_k^{\alpha}}{\Delta^{\alpha}}f(t-k\Delta) \quad (2)$$

where c_k^{α} is the binomial coefficient. For calculating the value of c_k^{α}, we can use the relation between Eulers Gamma function and factorial, defined as

$$c_k^{\alpha} = \left(\begin{array}{c}\alpha\\ k\end{array}\right) = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} \quad (3)$$

$$c_k^{\alpha} = \begin{cases} 1 & k=1 \\ \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{1.2.3.\cdots.k} & k \geq 1 \end{cases} \quad (4)$$

The above notation $\Gamma(.)$ is the gamma function. First, let us define coefficients $a(k)$ below

$$a(k) = (-1)^k C_k^{\alpha} \quad (5)$$

then the fractional derivative in eq.(2) can be rewritten as

$$D^{\alpha}f(t) = \lim_{\Delta\to 0}\sum_{k=0}^{\infty}\frac{a(k)}{\Delta^{\alpha}}f(t-k\Delta) \quad (6)$$

The right hand side of this equation is of infinite length. In [1], Tseng et al. have taken first L terms only and ignore the remaining terms. This is equivalent to multiplying the right hand side of eq. (6) by rectangular window of length L. We can apply tapered windows to obtain finite number of terms. Consider a Hamming window of length M defined as

$$w(n) = \left[0.54 - 0.46\cos\left(\frac{2\pi n}{M-1}\right)\right]$$
$$0 \leq n \leq M-1 \quad (7)$$

Thus, by using Hamming window for truncating the filter coefficients, the fractional order derivative $D^{\alpha}f(t)$ in eq. (6) can be approximated as

$$D^{\alpha}f(t) \approx \lim_{\Delta\to 0}\sum_{k=0}^{M}\frac{a(k)w(k)}{\Delta^{\alpha}}f(t-k\Delta) \quad (8)$$

Moreover, by removing limit, the fractional order derivative $D^{\alpha}f(t)$ can be further approximated as

$$D^{\alpha}f(t) \approx \sum_{k=0}^{M}\frac{a(k)w(k)}{\Delta^{\alpha}}f(t-k\Delta) \quad (9)$$

A smaller value of Δ needs to be chosen for reducing the approximation error of eq. (9). By taking $t = n - I$, the discrete-time derivative signal $D^{\alpha}f(n-I)$ can be obtained as

$$D^{\alpha}f(n-I) \approx \sum_{k=0}^{M}\frac{a(k)w(k)}{\Delta^{\alpha}}f(n-I-k\Delta) \quad (10)$$

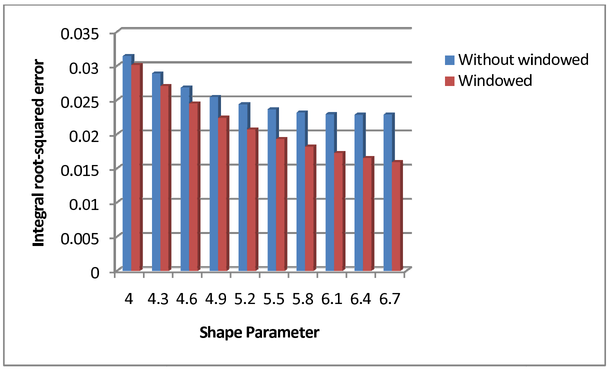

Figure 3: Integral root-squared error E of the fractional order differentiator using gaussian RBF with different shape parameter σ

Table 1: Integral root-squared error E with Gaussion RBF in Example 1

Shape parameter σ	Error using Gaussian RBF	Error using window Gaussian RBF
4.0	0.03143	0.03012
4.3	0.02887	0.02703
4.6	0.02680	0.02447
4.9	0.02542	0.2237
5.2	0.02436	0.02066
5.5	0.02362	0.01927
5.8	0.02315	0.01815
6.1	0.02290	0.01723
6.4	0.02281	0.01648
6.7	0.02283	0.01587

Because $f(n-I-k\Delta)$ are non-integer delay samples of signal $f(n)$, the $f(n-I-kh)$ needs to be estimated by using non-integer delay sample formula in [1], defined as

$$f(n-I-k\Delta) = \sum_{m=0}^{N} g(m, I+k\Delta) f(n-m) \tag{11}$$

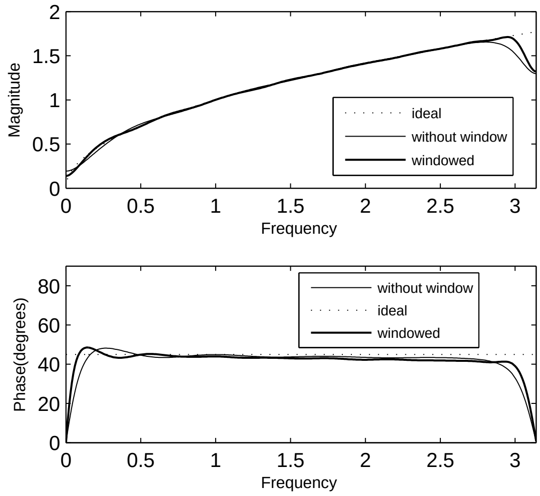

Figure 4: Magnitude and phase response of fractional order differentiator using direct truncation and windowed inverse multi-quadric RBF with $\sigma = 6.4$ and $\alpha = 0.5$.

Substituting eq. (11) into eq. (10), we get

$$D^{\alpha} f(n-I) \approx \sum_{k=0}^{M} \frac{a(k)w(k)}{\Delta^{\alpha}} \sum_{m=0}^{N} g(m, I+k\Delta) f(n-m) \tag{12}$$

$$\approx \sum_{m=0}^{N} \left[\frac{1}{\Delta^{\alpha}} \sum_{k=0}^{M} a(k)w(k)g(m, I+k\Delta) \right] f(n-m) \tag{13}$$

Defining the filter coefficients as

$$h(m) = \frac{1}{\Delta^{\alpha}} \sum_{k=0}^{M} a(k)w(k)g(m, I+k\Delta) \tag{14}$$

then eq. (13) can be rewritten in the form of convolution as

$$D^{\alpha} f(n-I) \approx \sum_{m=0}^{N} h(m) f(n-m) \tag{15}$$

$$D^{\alpha} f(n-I) = h(n) * f(n) \tag{16}$$

where $*$ denotes the convolution sum operator. The z-transform of eq. (15) yields

$$Y(z) = \left[\sum_{m=0}^{N} h(m) z^{-m} \right] F(z) = H(z)F(z) \tag{17}$$

where $Y(z)$ is the z-transform of $D^{\alpha} f(n-I)$, $F(z)$ is the z-transform of $f(n)$ and $H(z)$ is the transfer function of designed fractional order differentiator using window, whose frequency response will approximate the ideal response of fractional order differentiator.

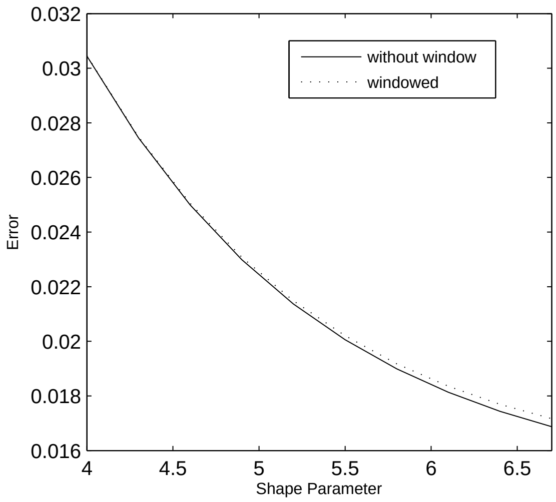

Figure 5: Integral root-squared error E of the fractional order differentiator using inverse multi-quadric RBF with different shape parameter σ.

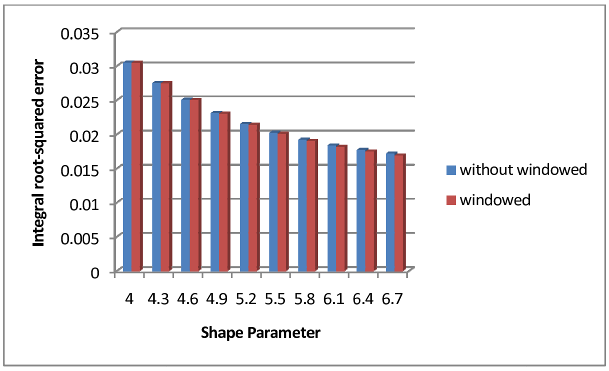

Figure 6: Integral root-squared error E of the fractional order differentiator using inverse multi-quadric RBF with different shape parameter σ.

3 Design Examples

To demonstrate the effectiveness of the tapered windowed method, several examples are presented in this section. To evaluate the performance, the integral root-squared error of frequency response is defined by

Table 2: Integral root-squared error E with inverse multi-quadratic RBF in Example 2

Shape parameter σ	Error using inverse multi-quadratic RBF	Error using window inverse multi-quadratic RBF
4.0	0.03046	0.03044
4.3	0.02749	0.02745
4.6	0.02505	0.02499
4.9	0.02308	0.02299
5.2	0.02149	0.02137
5.5	0.02020	0.02005
5.8	0.01918	0.01899
6.1	0.01835	0.01813
6.4	0.01769	0.01744
6.7	0.01716	0.01687

$$E = \sqrt{\int_0^{\lambda\pi} \left|H(e^{j\omega}) - H_d(e^{j\omega})\right|^2 d\omega} \tag{18}$$

The error is computed in the range $[0, \pi]$.

Example 1: The design of Example 1 as given in [1] is repeated where $N = 60$, $I = 30$, $\Delta = 0.05$, $\sigma = 2.3$, $\alpha = 0.5$ and $\lambda = 0.9$ using Hamming window of length $M = 620$, equal to the length L of the rectangular window used in [1]. Here, we have applied Gaussian RBF and Hamming window. Fig. 1 (a) and (b) shows the magnitude response and phase response of fractional order differentiator, respectively. Fig. 2 and 3 shows the integral root-squared error. Table I lists the integral root-squared error E of frequency response for different values of shape parameter $\sigma \in [4, 6.7]$. It is observed that the performance of the Hamming windowed method is better than that of [1].

Example 2: This example deals with the inverse multi-quadric RBF design of fractional order differentiators along with Hamming window. The design of Example 2 as given in [1] is repeated where $N = 60$, $I = 30$, $\Delta = 0.05$, $\sigma = 6.4$, $\alpha = 0.5$ and $\lambda = 0.9$ using Hamming window of length $M = 620$. The magnitude response and phase response of fractional order differentiator are shown in Fig. 4(a) and (b), respectively. Fig. 5 and 6 shows the integral root-squared error. Table II lists the integral root-squared error E of frequency response for different values of shape parameter $\sigma \in [4, 6.7]$. The method in [1], yields integral root-squared error $E = 0.01716$, which shows that the Hamming windowed method is better for the design.

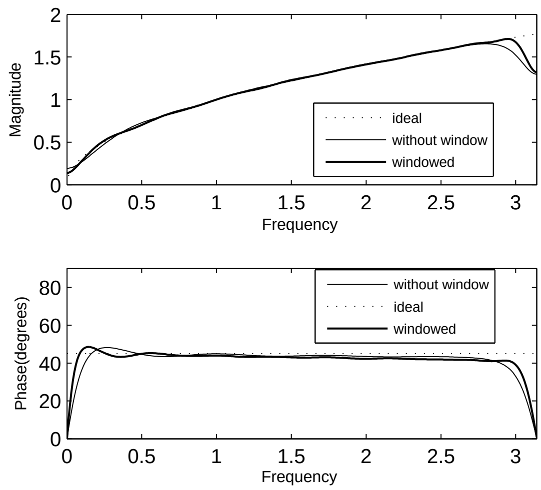

Figure 7: Magnitude and phase response of fractional order differentiator using direct truncation and windowed inverse quadratic RBF with $\sigma = 6.7$ and $\alpha = 0.5$.

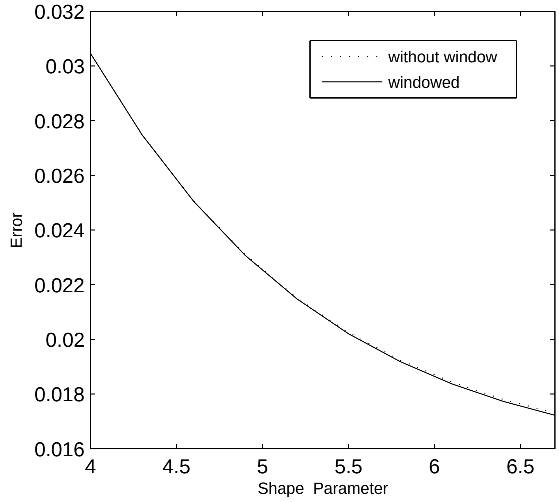

Figure 8: Integral root-squared error E of the fractional order differentiator using inverse quadratic RBF with different shape parameter σ

Example 3: In this example, the performance of fractional order differentiator using inverse multiquadratic RBF and Hamming window is studied for the given design parameters as $N = 60$, $M = 620$, $I = 30$, $\Delta = 0.05$, $\sigma = 6.7$, $\alpha = 0.5$ and $\lambda = 0.9$. The magnitude response and phase response of fractional order differentiator are shown in Fig. 7 (a) and (b), respectively. Fig. 8 and 9 shows the integral root-squared error. Table III lists the integral root-squared error E of frequency response for different values of shape parameter σ. The method in [1], yield integral root-squared error $E = 0.01731$, which shows that the Hamming windowed method is better for the design.

4 Conclusion

In [1], Tseng et al. have designed a fractional order differentiator using radial basis function by directly truncating the filter coefficients. In this paper, we have shown that the use of window along with the RBF achieve higher design accuracy with significant reduction in integral root squared error. We can use other windows also to improve the frequency response.

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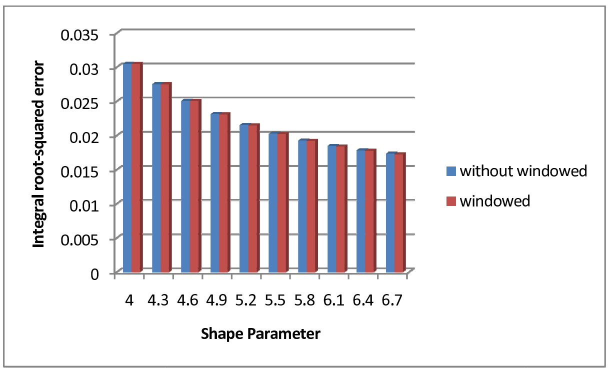

Figure 9: Integral root-squared error E of the fractional order differentiator using inverse quadratic RBF with different shape parameter σ

Table 3: Integral root-squared error E with inverse quadratic RBF in Example 3

Shape parameter σ	Error using inverse quadratic RBF	Error using window inverse quadratic RBF
4.0	0.03046	0.03045
4.3	0.02749	0.02748
4.6	0.02506	0.02504
4.9	0.02310	0.02307
5.2	0.02152	0.02148
5.5	0.02025	0.02020
5.8	0.01924	0.01918
6.1	0.01844	0.01837
6.4	0.01781	0.01773
6.7	0.01731	0.01722

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