

Unusual Implications of Analytic and Structural Relationships between Hypergeometric Differential Equations and Hypergeometric Functions in the Complex Plane

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Abstract: - This academic research first provides a foundational overview of hypergeometric differential equations and hypergeometric functions in specific regions of the complex plane. Building on this foundation, it develops a comprehensive set of results that reveal various relationships between the relevant classes of equations and functions, supported by carefully formulated propositions, deductions, and illustrative examples. In particular, this study emphasizes the logical consistency among the analysis, theoretical results, and applications, demonstrating how these relationships extend existing perspectives in hypergeometric theory. Moreover, the study concludes with key deductions and recommendations based on the findings, suggesting potential directions for future research. In light of the references cited, the contributions of this work aim to promote a broader understanding of hypergeometric structures and their applications in complex analysis and related fields.

Key-Words: - Real-Analytic functions, Special functions, Hypergeometric equations and functions, Real-Complex series expansions, Homogeneous and nonhomogeneous differential equations, Equations and inequalities in certain domains of the complex plane.

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1 Introduction, Motivation, and Preliminaries

The well-established concepts of mathematical equations, inequalities, and functions are fundamental tools in both theoretical and applied research across nearly all scientific disciplines. Even when only two of these tools are considered, the significance of their applications and implications in relevant studies becomes clearly evident. Consequently, all three concepts play

essential roles throughout the course of this investigation.

Another important tool employed in this special study is the class of special functions. These are specific mathematical functions that have well-established names and notations due to their significance in mathematical analysis, functional analysis, physics, and various other fields. Although these functions are widely recognized and used by consensus, there is no

universally accepted formal definition of a *special* function.

Nevertheless, the mathematical literature generally includes certain functions that are commonly regarded as Special Functions. Many of these functions frequently arise as solutions to particular types of differential equations or as integrals of elementary functions. Owing to their broad range of applications, especially in symbolic computation, most symbolic computation engines are designed to recognize and manipulate the majority of these functions. For further information and various types of examples, the privately conducted studies presented in [1], [2], [3], [4], [5], [6], [7], [8], [9] may be consulted.

In particular, nearly all well-known special functions are also considered as functions of complex variables. As such, these complex functions may be analytic in certain regions of the (complex) z –plane, their singularities and branch cuts can be described, and their differential and integral representations are typically known.

Additionally, they can often be expanded into Taylor-Maclaurin series, and in some cases, into asymptotic series or similar expansions. Furthermore, there are often many relationships between these functions — or between subsets of them — and other special functions. In many cases, a more complex special function can be expressed in terms of simpler ones. These relationships provide various representations that are useful for evaluating certain expressions or solving differential equations. In particular, one of the most straightforward ways to evaluate a special function is through a suitable series expansion. This is critically important for our investigation. Although the special functions hold the theoretical study of special functions hold significant intellectual appeal, there are also practical motivations worth noting. For decades, special functions have played an essential role in applied mathematics. Their real-complex type implications and applications in various sciences and engineering have also been extensively recognized in the literature [10], [11], [12], [13], [14], [15], [16].

In this context, under certain additional conditions involving specific inequalities, which are expected to be significant for potential scientific outcomes, we emphasize that particular types of special equations and functions will play a key role in our investigation. Specifically, the first group consists of certain special differential equations, while the second comprises specific

special functions that serve as solutions to these equations.

Specially, in applied mathematical analysis, well-known differential equations and special functions are frequently encountered. As is widely recognized, differential equations involve one or more functions along with their derivatives. In particular, their special solutions are both prominent and highly suitable for mathematical investigation. Numerous examples can be cited in this context, including Struve's differential equations, hypergeometric differential equations, and Bessel differential equations, all of which are commonly studied in both theoretical and applied settings.

At the same time, several well-known functions — such as the Struve, hypergeometric, and Bessel functions — can also be cited as examples. Various different type functions are, respectively, particular solutions of the corresponding differential equations mentioned earlier pointed out the earlier studies referenced in [17], [18], [19], [20], [21], [22], [23], [24], [25] may be considered for more detailed information on various type functions and some of their applications.

As previously mentioned, by considering hypergeometric complex differential equations and complex hyper geometric functions — both of which are special complex functions representing particular solutions to these differential equations — various comprehensive results will be established. These results will be based on specific relationships involving the hypergeometric complex differential equations, the hypergeometric functions themselves, and their mathematical implications.

Accordingly, it is necessary to provide more detailed information related to our fundamental results and their (possible) implications. We will now introduce the essential notations and concepts concerning the aforementioned complex differential equations, complex functions, and relevant background.

Here and throughout this document, it is necessary to recall the following familiar notations:

$$\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C} \text{ and } \mathbb{U}$$

denote, respectively, the set of natural numbers, the set of integers, the set of real numbers, the set of complex numbers, and the open unit disc in the complex plane, namely the open unit disk given by

$$\mathbb{U} = \{ z : |z| < 1 \ (z \in \mathbb{C}) \}.$$

Also, let

$$\mathbb{Z}_0 := \mathbb{Z} \cup \{0\}, \quad \mathbb{Z}_0^+ := \mathbb{N} \cup \{0\}$$

and

$$\mathbb{Z}_0^- := \{z \in \mathbb{Z} : z \leq 0\}.$$

The first important topic concerns complex functions defined by power series, to which we now turn our attention. In this context, the following functional series in the complex variable z , commonly known as the (Gauss, Gaussian, or ordinary) hypergeometric function (of the second kind), is denoted by one of the following notations:

$$F(\alpha, \delta; \gamma; z) \quad \text{and} \quad {}_2F_1(\alpha, \delta; \gamma; z),$$

and also defined by

$$F(\alpha, \delta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\delta)_n}{(\gamma)_n} \frac{z^n}{n!}, \quad (1)$$

where $\alpha \in \mathbb{C} - \mathbb{Z}_0^-$, $\delta \in \mathbb{C} - \mathbb{Z}_0^-$, $\gamma \in \mathbb{C} - \mathbb{Z}_0^-$ and $z \in \mathbb{U}$, and, in terms of familiar Gamma function $\Gamma(\cdot)$, the (rising) Pochhammer symbol, namely, the symbol $(v)_n$ is also defined by the form:

$$(v)_n = \frac{\Gamma(v+n)}{\Gamma(v)} \\ = \begin{cases} 1 & \text{when } n=0 \\ v(v+1) \cdots (v+n-1) & \text{when } n \in \mathbb{N} \end{cases}, \quad (2)$$

where $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $v \in \mathbb{C} - \mathbb{Z}_0^-$. Concisely; $(v)_1 = v$, $(v)_2 = v(v+1)$, ... Cf., e.g., [26].

In special, in the expression given by (1), the complex series becomes finite if either α or β is a non-positive integer, in which case the function reduces to a polynomial. In particular, when $-\ell = : \alpha$ ($\ell \in \mathbb{N}$), it is of the binomial-type form

$$F(-\ell, \delta; \gamma; z) \\ = \sum_{n=0}^{\infty} (-1)^n \binom{\ell}{n} \frac{(\delta)_n}{(\gamma)_n} \frac{z^n}{n!}. \quad (3)$$

In this case, up to constant factors and scaling, these functions correspond to the Jacobi polynomials. Moreover, several other classes of orthogonal polynomials are special cases of the Jacobi polynomials (again, up to normalization constants), and thus can also be expressed in terms of $F(-, -; -; z)$. Especially, this includes the Legendre and Chebyshev polynomials, and some of related implications.

It is also worth noting that, for the complex variable z , the above hypergeometric-type function admits analytic continuation along any

path in the complex plane that avoids the branch points at 1 and ∞ . In special, in practical applications, most computational implementations adopt a branch cut along the real axis for z with $z \geq 1$.

As some of fundamental properties of the hypergeometric functions, under admissible conditions of the relevant parameters, from (1), it follows that

$$F(\alpha, \delta; \gamma; z) = F(\delta, \alpha; \gamma; z), \quad (4)$$

and, by the help of the private identity given by (2) and, of course, for admissible values of the mentioned parameters α , δ and γ , it is easily obtained that

$$\frac{d}{dz} (F(\alpha, \delta; \gamma; z)) = \frac{\alpha\delta}{\gamma} F(\alpha, \delta; \gamma; z), \quad (5)$$

and, more generally,

$$\frac{d^n}{dz^n} (F(\alpha, \delta; \gamma; z)) \\ = \frac{(\alpha)_n (\delta)_n}{(\gamma)_n} \\ \times F(n + \alpha, n + \delta; n + \gamma; z) \quad (6)$$

for all $n \in \mathbb{Z}_0^+$.

In particular, various properties (or relationships) in relations with different type derivatives, such as (5) and (6), will be useful for simplifying complex expressions involving equations or inequalities that contain derivatives. For detailed information on hypergeometric functions and related special functions with complex (or real) parameters, see the studies, [17], [18], [19], [20].

The second important topic concerns a second-order complex differential equation with variable coefficients. Accordingly, it is necessary to provide some preliminary information. An analytic function defined by the series expansion in the form of (1) serves as a solution to the following homogeneous differential equation with variable coefficients:

$$(1-z) \frac{d^2 w}{dz^2} + [\gamma - (\alpha + \delta + 1)z] \\ \times \frac{dw}{dz} - \alpha\delta w = 0, \quad (7)$$

where $w := w(z)$, $\alpha \in \mathbb{C} - \mathbb{Z}_0^-$, $\delta \in \mathbb{C} - \mathbb{Z}_0^-$, $\gamma \in \mathbb{C} - \mathbb{Z}_0^-$ and $z \in \mathbb{C} - [1, \infty)$.

In addition, we specially note that the mentioned complex differential equation given in

(6) is also known in the mathematical literature as the (Euler) hypergeometric differential equation. A simple examination reveals that the differential equation in (7) has three singular points: 0, 1, and ∞ .

Indeed, for the comprehensive relationships between (1) and (7), first of all, we see useful to point some extra information about them out. For analytic property of the complex series given in (1) along with (2), it is absolutely and uniformly convergent in the unit open disc $\mathbb{U}$.

Of course, the convergence also extends over the unit circle when $\Re(\alpha + \delta - \gamma) < 0$, it converges at all the points of the unit circle except the point $z = 1$ when $0 \leq \Re(\alpha + \delta - \gamma) < 1$ of the unit set with the slit $(1, \infty)$. The function defined by the complex series given by (1), i.e., $F(\alpha, \delta; \gamma; z)$ is an univalent-analytic function in the complex plane with the slit $(1, \infty)$. When α or δ are zero or negative integers, the mentioned series (1) terminates after a finite number terms and the hypergeometric function is a polynomial of the parameter z . Further, when $n \in \mathbb{Z}_0^-$, the complex function given by (1) is not defined but the limit can be considered there.

Since there are a large number of relations between the special function given by (1) and the special equation given by (7), naturally, there may be a wide range of possible special equations and/or functions determined by the function given (1).

So, they will also include many significant relationships with several special functions and equations, and also their implications.

It is particularly noteworthy that the essential works [1] and [2] constitute only two of the key sources for the foundational material used in this investigation and are closely related to the present study.

We can now continue to determine those (relations or properties) in the next section.

For convenience, we also need to recall only one of general properties of the differential equations with complex variable coefficient. For it, we take into account the *second-order* differential equation given by the form:

$$\lambda(z) \frac{d^2 w}{dz^2} + \beta(z) \frac{dw}{dz} + \mu(z)w + \sigma(z) = 0, \quad (8)$$

where $w := w(z)$ and also each one of the complex expressions $A(z)$, $\beta(z)$, $C(z)$ and $D(z)$ is an analytic function in any suitable domain of the complex set $\mathbb{C}$.

In special, when an analytic function like the form $w := w(z)$ is any (particular) solution for the *second-order* differential equation being of the form given in (8), the complex function w also is a (particular) solution for the *third-order* differential equation given by

$$\frac{d}{dz} \left(\lambda(z) \frac{d^2 w}{dz^2} + \beta(z) \frac{dw}{dz} + \mu(z)w + \sigma(z) \right) = 0, \quad (9)$$

or, equivalently,

$$e(z) \frac{d^3 w}{dz^3} + g(z) \frac{d^2 w}{dz^2} + k(z) \frac{dw}{dz} + h(z)w + t(z) = 0, \quad (10)$$

where each one of the mathematical expressions $e(z)$, $g(z)$, $k(z)$, $h(z)$ and $t(z)$ is analytic in appropriate complex regions of the familiar plane.

To obtain various types of novel results, we apply reasoning similar to that used between (8) and (10) to the homogeneous second-order hypergeometric differential equation (with complex variable coefficients). This also yields additional material supporting our main results and some of their possible implications, which will be presented in the next section. Of course, this approach further leads to the homogeneous third-order hypergeometric differential equation (with complex variable coefficients) in the following differential form:

$$(1-z)z^2 \frac{d^3 w}{dz^3} + z[1+\gamma - (3+\alpha+\delta)z] \frac{d^2 w}{dz^2} - [\gamma - (1+\alpha+\delta+\alpha\delta)]z \frac{dw}{dz} = 0 \quad (11)$$

can be easily determined, where $w := w(z)$, $\alpha \in \mathbb{C} - \mathbb{Z}_0^-$, $\delta \in \mathbb{C} - \mathbb{Z}_0^-$, $\gamma \in \mathbb{C} - \mathbb{Z}_0^-$ and $z \in \mathbb{C} - [1, \infty)$.

For further (specialized) information on the aforementioned differential equations (with variable coefficients) and the related special functions, the reader is referred to earlier results reported in the references cited in this paper, [27], [28], [29], [30], [31].

2 Essential Results and Some of Their Related Implications along with Applications

As is well known, the methods used in the proofs of mathematical theories are also of great importance for theoretical studies in mathematics. First of all, since our essential results are directly related to the theory of complex functions, the fundamental references given in the references, [32], [33], [34], [35], [36], [37], [38] will play various roles in the study of special complex-type functions and/or certain classes of complex functions analytic in certain domains of the mentioned plane $\mathbb{C}$.

In particular, for the *Carathéodory* class (or Carathéodory functions), we denote by the notation $\mathcal{N}$ the class of all analytic functions in the open unit disc $\mathbb{U}$ that are normalized by the condition $v(0) = 1$.

Furthermore, an analytic function $v = v(z)$ belonging to the class $\mathcal{N}$ is generally called a *carathéodory* function if it satisfies the condition given by

$$\Re\{v(z)\} > 0 \quad (z \in \mathbb{U}).$$

In addition, we note that this important class is often referred to as the *Carathéodory* class, or the class of functions with positive real part, as introduced by C. Carathéodory in the mathematical literature as indicated in the essential papers, [37], [38]. This provides the fundamental theoretical background for our investigations aimed at establishing various results concerning the relationships between the hypergeometric function and the hypergeometric equation.

At the same time, the auxiliary theorem presented below will play an important role in proving our main results. For further details and some of related implications (or applications), see [39], [40], [41], [42].

Lemma 1 [43]. Let $v := v(z)$ be a non-constant function belonging to the family $\mathcal{N}$, consisting of analytic functions represented by the series

$$v(z) = 1 + t_j z^j + t_{j+1} z^{j+1} + \dots + \dots + t_{j+n} z^{j+n} + \dots, \quad (12)$$

where $j \in \mathbb{N}$, $n \in \mathbb{N}$, $t_j \in \mathbb{C}$, and $z \in \mathbb{U}$. Suppose also that v is not a *Carathéodory* function in the domain $\mathbb{U}$. Then there is a point $z_0 \in \mathbb{U}$ such that

$$v(z_0) = iL \quad (i^2 = -1), \quad (13)$$

and

$$z_0 v'(z_0) = M, \quad (14)$$

where

$$L = \mathbb{R} - \{0\} \quad \text{and} \quad M \leq -\frac{1+L^2}{2}j. \quad (15)$$

Within the scope of this academic investigation, by means of the relevant lemma and by taking into account the (complex-type) differential equations given in (7) and (11), we want to begin by presenting a number of comprehensive results that also encompass various analytic and geometric properties, and then proceed to prove them.

The first essential result is given in the following form.

Theorem 1. Let $w := w(z) := F(\alpha, \delta; \gamma; z)$, and also let $\kappa(z)$ be a complex-valued function that is analytic in the domain $\mathbb{U}$, and satisfies the specific form of the following nonhomogeneous differential equation:

$$z(1-z) \frac{d^2 w}{dz^2} + \gamma \frac{dw}{dz} - \alpha \delta w + \kappa(z) = 0 \quad (z \in \mathbb{U}). \quad (16)$$

For admissible values of the parameters α , β and γ , and for any $z \in \mathbb{U}$, suppose that one of the following conditions holds:

$$|\kappa(z)| < \frac{|1+\vartheta|}{2}, \quad (17)$$

$$-1 + 2 \Re\{\kappa(z)\}$$

$$\begin{cases} < \Re(\vartheta) & \text{when } \Re(\vartheta) > -1 \\ \neq -1 & \text{when } \Re(\vartheta) = -1 \\ > \Im(\vartheta) & \text{when } \Re(\vartheta) < -1 \end{cases}, \quad (18)$$

and

$$-1 + 2 \Im\{\kappa(z)\}$$

$$\begin{cases} < \Im(\vartheta) & \text{when } \Im(\vartheta) > 0 \\ \neq -1 & \text{when } \Im(\vartheta) = 0 \\ > \Im(\vartheta) & \text{when } \Im(\vartheta) < 0 \end{cases}. \quad (19)$$

Then

$$\Re\{v(z)\} > 0, \quad (20)$$

where $\vartheta := \alpha + \delta$ and $z \in \mathbb{U}$.

Proof. Both the related lemma and the relationship highlighted between the complex-type differential equations given in (8) and (10) will be instrumental in the key steps of the forthcoming proof.

In light of the special information in (1), (2), (5) and (6), we want to proceed with the construction of these steps. Since the hypergeometric function $F(\alpha, \delta; \gamma; z)$, defined in (1), is a solution of the complex differential equation given in (7), it admits the following series representation:

$$\begin{aligned} F(\alpha, \delta; \gamma; z) &= 1 + \frac{\alpha\delta}{2}z \\ &+ \frac{\alpha(\alpha+1)\delta(\delta+1)}{\gamma(\gamma+1)} \frac{z^2}{2!} \\ &+ \frac{\alpha(\alpha+1)(\alpha+2)\delta(\delta+1)(\delta+2)}{\gamma(\gamma+1)(\gamma+2)} \frac{z^3}{3!} \\ &+ \dots \quad (z \in \mathbb{U}). \end{aligned}$$

For the proof of Theorem 1, we take the complex function given by

$$w := w(z) := F(\alpha, \delta; \gamma; z) \quad (z \in \mathbb{U}).$$

Then, clearly, the function w satisfies the hypotheses of Lemma 1; namely, $w(0) = 1$ and w is analytic in $\mathbb{U}$.

Accordingly, for the desired proof, we define the function $v(z)$, as in Lemma 1, by

$$v(z) := w(z) := F(\alpha, \delta; \gamma; z) \quad (z \in \mathbb{U}). \quad (21)$$

Our aim is to show that the relevant function $v := v(z)$, defined in (1), has a positive real part. Furthermore, from definition (11), we obtain

$$\begin{aligned} v'(z) &\equiv w'(z) \\ &:= F'(\alpha, \delta; \gamma; z) \\ &\equiv \frac{d}{dz}(F(\alpha, \delta; \gamma; z)) \quad (z \in \mathbb{U}). \end{aligned} \quad (22)$$

By means of the homogeneous hypergeometric differential equation given in (7), we also define an analytic function $\kappa(z)$ as follows:

$$\begin{aligned} z(1-z) \frac{d^2v}{dz^2} + \gamma \frac{dv}{dz} - \alpha\delta v \\ &= (1 + \alpha + \delta)z \frac{dv}{dz} \\ &=: -\kappa(z), \end{aligned} \quad (23)$$

where $w := v(z)$, $\alpha \in \mathbb{C} - \mathbb{Z}_0^-$, $\delta \in \mathbb{C} - \mathbb{Z}_0^-$, $\gamma \in \mathbb{C} - \mathbb{Z}_0^-$ and $z \in \mathbb{U}$.

Now suppose that the function $v := v(z)$ does not have positive real part in the open complex set $\mathbb{U}$. Then, under condition (15) with $j := 1$, it follows from (13) and (14) of the lemma that there exists a point z_0 in $\mathbb{U}$ such that

$$v(z_0) = iL \quad \text{and} \quad z_0 v'(z_0) = M, \quad (24)$$

where $i^2 = -1$, and the mentioned parameters j, L and M satisfy the mentioned conditions given in (15).

Furthermore, for admissible values of the relevant parameters α , δ and γ , and by applying Lemma 1 (see the information in (14) and (15)), the combination of the expressions given in (22) and (23) yields the following inequalities:

$$\begin{aligned} |-\kappa(z_0)| &= |\kappa(z_0)| \\ &= |1 + \vartheta| |z_0 v'(z_0)| \\ &= |M| |1 + \vartheta| \\ &\geq |1 + \vartheta| (1 + L^2) \frac{j}{2} \\ &\geq \frac{|1 + \vartheta|}{2}, \end{aligned} \quad (25)$$

$$\begin{aligned} -\Re\{\kappa(z_0)\} &= \Re\{(1 + \vartheta)z_0 v'(z_0)\} \\ &= M(1 + \Re\{\vartheta\}) \\ &\begin{cases} \leq -\frac{1}{2}(1 + L^2)\Re(1 + \vartheta) & \text{if } \Re(\vartheta) > -1 \\ = 0 & \text{if } \Re(\vartheta) = -1 \\ \geq -\frac{j}{2}(1 + L^2)\Im m(\vartheta) & \text{if } \Re(\vartheta) < -1 \end{cases} \\ &\begin{cases} \leq -\frac{1+\Re(\vartheta)}{2} & \text{if } \Re(\vartheta) > -1 \\ = 0 & \text{if } \Re(\vartheta) = -1 \\ \leq -\frac{1+\Re(\vartheta)}{2} & \text{if } \Re(\vartheta) < -1 \end{cases}, \end{aligned} \quad (26)$$

and

$$\begin{aligned} -\Im\{\kappa(z_0)\} &= \Im\{(1 + \vartheta)z_0 v'(z_0)\} \\ &= M(1 + \Im\{\vartheta\}) \\ &\begin{cases} \leq -\frac{1}{2}(1 + L^2)\Im(1 + \vartheta) & \text{if } \Im(\vartheta) > 0 \\ = 0 & \text{if } \Im(\vartheta) = 0 \\ \geq -\frac{j}{2}(1 + L^2)\Im m(\vartheta) & \text{if } \Im(\vartheta) < 0 \end{cases} \\ &\begin{cases} \leq -\frac{1+\Im(\vartheta)}{2} & \text{if } \Im(\vartheta) > 0 \\ = 0 & \text{if } \Im(\vartheta) = 0 \\ \leq -\frac{1+\Im(\vartheta)}{2} & \text{if } \Im(\vartheta) < 0 \end{cases} \end{aligned} \quad (27)$$

can be easily obtained, where $\vartheta := \alpha + \delta$ and $z_0 \in \mathbb{U}$. However, the related inequalities (or the

corresponding cases) obtained in (25) – (27) contradict the hypotheses of the related theorem, which are given in (17) – (19), respectively.

Therefore, there exists no point $z_0 \in \mathbb{U}$ such that the function v , belonging to the class N , satisfies any of the conditions given in (13) and (14). This implies that v satisfies the condition given in (12). Consequently, the function v , defined by (21), immediately yields that the inequality:

$$\Re\{v(z)\} = \Re\{F(\alpha, \delta; \gamma; z)\} > 0,$$

which establishes the conclusion of the theorem as stated in (20). Hence, the proof of Theorem 1 is complete.

The second essential result is presented in Theorem 2 below. For admissible values of the parameters given in (1) – (7), the result can be established by an argument similar to that used in the proof of Theorem 1. To this end, it suffices to consider the following equivalent form of the homogeneous hypergeometric differential equation given in (7):

$$(1-z)z^2 \frac{d^2 w}{dz^2} - (1+\alpha+\delta)z^2 \frac{dw}{dz} - \alpha\delta zw = -\gamma z \frac{dw}{dz}, \quad (28)$$

and then introduce the analytic function as the form:

$$v(z) := w(z) := F(\alpha, \delta; \gamma; z) \quad (z \in \mathbb{U}),$$

which is defined as in (21) and is a solution of the second-order complex-type differential equation given in (7). Since the remaining steps follow closely those in the proof of Theorem 1 and lead directly to the desired conclusion, the details of the proof of Theorem 2 are omitted.

Theorem 2. Let $w := w(z) := F(\alpha, \delta; \gamma; z)$ and $z \in \mathbb{U}$, and also let $\rho(z)$ be analytic in the domain $\mathbb{U}$, and take the form specified by the following complex nonhomogenous differential equation given by

$$(1-z)z^2 \frac{d^2 w}{dz^2} - (1+\alpha+\delta)z^2 \frac{dw}{dz} - \alpha\delta zw + \rho(z) = 0. \quad (29)$$

Then, for admissible values of the complex parameters α , δ , and γ , and for any $z \in \mathbb{U}$, if

any one of the inequalities (or the related cases) given by

$$|\rho(z)| < \frac{|\gamma|}{2}, \quad (30)$$

$$\Re\{\rho(z)\} \begin{cases} > -\frac{\Re(\gamma)}{2} & \text{if } \Re(\gamma) > 0 \\ \neq 0 & \text{if } \Re(\gamma) = 0 \\ < -\frac{\Re(\gamma)}{2} & \text{if } \Re(\gamma) < 0 \end{cases}, \quad (31)$$

and

$$\Im\{\rho(z)\} \begin{cases} > -\frac{\Im(\gamma)}{2} & \text{if } \Im(\gamma) > 0 \\ \neq 0 & \text{if } \Im(\gamma) = 0 \\ < -\frac{\Im(\gamma)}{2} & \text{if } \Im(\gamma) < 0 \end{cases}, \quad (32)$$

is ensured, then

$$\Re\{v(z)\} > 0$$

is also ensured, where $z \in \mathbb{U}$.

The third essential result is presented in Theorem 3 below. It directly concerns a type of homogeneous third-order complex differential equation with variable coefficients, which has the form given by

$$(1-z)z^2 \frac{d^3 w}{dz^3} + z[1+\gamma - (3+\alpha+\delta)z] \frac{d^2 w}{dz^2} = (1+\alpha+\delta)z \frac{dw}{dz} \quad (z \in \mathbb{U}). \quad (33)$$

It is evident that the differential equation given in (33) is of a type similar to that given in (10), as in (8) – (10). This complex-type differential equation is incorporated into the formulation of the theorem, which is expressed in the following form:

Theorem 3. Let $w := w(z) := F(\alpha, \delta; \gamma; z)$, and also let $\tau(z)$ be analytic in the disc $\mathbb{U}$, and consider the form specified by the complex-type nonhomogeneous differential equation given by

$$(1-z)z^2 \frac{d^3 w}{dz^3} + z[1+\gamma - (3+\alpha+\delta)z] \frac{d^2 w}{dz^2} \times \frac{dw}{dz} + \tau(z) = 0 \quad (z \in \mathbb{U}). \quad (34)$$

Then, for admissible values of the complex parameters α , δ , and γ , and for any $z \in \mathbb{U}$, if

any one of the inequalities (or the related cases) given by

$$|\tau(z)| < \frac{|1+\eta|}{2}, \quad (35)$$

$$\Re\{\tau(z)\} \begin{cases} < \frac{1+\Re(\eta)}{2} & \text{if } \Re(\eta) > -1 \\ \neq 0 & \text{if } \Re(\eta) = -1 \\ > \frac{1+\Re(\eta)}{2} & \text{if } \Re(\eta) < -1 \end{cases}, \quad (36)$$

and

$$\Im\{\tau(z)\} \begin{cases} < \frac{\Im(\eta)}{2} & \text{if } \Re(\eta) > 0 \\ \neq 0 & \text{if } \Re(\eta) = 0 \\ > \frac{\Im(\eta)}{2} & \text{if } \Re(\eta) < 0 \end{cases}, \quad (37)$$

is provided, then

$$\Re\{w(z)\} > 0$$

is also provided, where $\eta := \alpha + \delta + \alpha\delta$, and for $z \in \mathbb{U}$.

Proof. The proof of this theorem proceeds by reasoning similar to that employed in Theorems 1 and 2. As in (8) – (10), if an analytic function is a solution of the complex-type differential equation given in (7), then it also satisfies the equation given in (11).

Since the analytic function $w := w(z)$, as stated in Theorem 3, coincides with the hypergeometric-type function $F(\alpha, \delta; \gamma; z)$ — a solution of the hypergeometric differential equation given in (7) — it may be identified with the function $v := v(z)$ introduced in Lemma 1. Referring to Lemma 1 and to (33) (or its equivalent form given in (34)), we consider v as defined in (11) and utilize its derivative as given in (22). By following the similar steps as considered in the proof of Theorem 1 and applying analogous reasoning, contradictions to the hypotheses in (35) – (37) can be derived. Therefore, with only minor modifications, the proof is completed. The detailed steps are omitted for brevity.

As a result of this extensive study, the first part provides general information on various essential mathematical tools.

In the second chapter, three comprehensive theorems are presented, together with second- and third-order complex differential equations and additional conditions for their possible solutions.

Focusing on the results established in Chapter 2, it is evident that many special cases can be derived from these main results.

Consequently, by considering the key results in Section 2 and the related mathematical concepts, a large number of special results can be obtained or reformulated in different forms, including numerous novel relationships between mathematical theories and their potential applications.

To assist researchers, we now organize this material to highlight some of these special results.

In particular, by choosing appropriate values for the relevant parameters in the hypergeometric-type function given in (1), and consequently in the hypergeometric-type differential equation given in (7), a wide variety of special complex-type functions can be generated (or reformulated).

In light of the fundamental information provided in (1) – (7), for some of these cases, the resulting complex-type special functions and the associated differential-type complex equations derived from (7) can also be presented as representative examples (or implications). We now want to proceed to outline some of these cases in detail below.

i) Some Hypergeometric Forms Associated with Elementary Functions (with Respect to the Complex Variable z): For all valid values of the natural numbers m and for some complex number z ;

$$\frac{1}{(1 \pm z)^m} = F(m, 1; 1; \mp z) \quad (38)$$

$$= \sum_{k=0}^{\infty} \frac{(\mp m)_k (1)_k (\mp z)^k}{(1)_k k!},$$

$$\exp(\pm z) = \lim_{\delta \rightarrow \infty} F\left(1, \delta; 1; \frac{\mp z}{\delta}\right) \quad (39)$$

$$= \lim_{\delta \rightarrow \infty} \sum_{k=0}^{\infty} \frac{(1)_k (\delta)_k (\mp z/\delta)^k}{(1)_k k!},$$

$$\ln(1 \pm z) = z F(1, 1; 2; \mp z) \quad (40)$$

$$= \sum_{k=0}^{\infty} \frac{(\mp m)_k (1)_k (\mp z)^k}{(1)_k k!},$$

$$\ln\left(\frac{1+z}{1-z}\right) = 2z F\left(\frac{1}{2}, 1; \frac{3}{2}; z^2\right) \quad (41)$$

$$= \sum_{k=0}^{\infty} \frac{(1/2)_k (1)_k z^{2k}}{(3/2)_k k!},$$

$$\cos(nz) = F\left(\frac{n}{2}, -\frac{n}{2}; \frac{1}{2}; \sin^2 z\right) \quad (42)$$

$$= \sum_{k=0}^{\infty} \frac{(n/2)_k (-n/2)_k}{(1/2)_k} \frac{\sin^{2k}(z)}{k!},$$

$$\arcsin(z) = z F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2\right) \quad (43)$$

$$= \sum_{k=0}^{\infty} \frac{(1/2)_k (1/2)_k}{(3/2)_k} \frac{z^{2k}}{k!}$$

and

$$\arctan(z) = z F\left(\frac{1}{2}, 1; \frac{3}{2}; -z^2\right) \quad (44)$$

$$= \sum_{k=0}^{\infty} \frac{(1/2)_k (1)_k}{(3/2)_k} \frac{(-z)^{2k}}{k!},$$

where $m \in \mathbb{N}$ and $z \in \mathbb{C}$.

ii) Some Hypergeometric Forms Associated with the Elliptic functions (with Respect to the Complex Variable z) :

$$K(z) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; z^2\right) \quad (45)$$

$$= \sum_{k=0}^{\infty} \frac{(1/2)_k (1/2)_k}{(1)_k} \frac{z^{2k}}{k!}$$

and

$$E(z) = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; z^2\right) \quad (46)$$

$$= \sum_{k=0}^{\infty} \frac{(-1/2)_k (1/2)_k}{(1)_k} \frac{z^{2k}}{k!}.$$

iii) Some Hypergeometric Forms Associated with the (Adjoint) Legendre-type functions (with Respect to the Complex Variable z) : For all $n, m \in \mathbb{N}$,

$$\begin{aligned} P_m^n(z) &= \frac{(z+1)^{n/2}}{(z-1)^{n/2}} \frac{1}{\Gamma(1-n)} \\ &\quad \times F\left(-m, m+1; 1-m; \frac{1-z}{2}\right) \\ &= \frac{(z+1)^{n/2}}{(z-1)^{n/2}} \frac{1}{\Gamma(1-n)} \\ &\quad \times \sum_{k=0}^{\infty} \frac{(-n)_k (n)_k}{(1/2)_k} \frac{[(1-z)/2]^k}{k!}. \end{aligned} \quad (47)$$

iv) Hypergeometric Forms Associated with the Chebyshev Polynomials-type functions (with Respect to the Complex Variable z) : For all $n \in \mathbb{N}$,

$$T_n(z) = F\left(-n, n; \frac{1}{2}; \frac{1-z}{2}\right)$$

$$= \sum_{k=0}^{\infty} \frac{(-n)_k (n)_k}{(1/2)_k} \frac{[(1-z)/2]^k}{k!}. \quad (48)$$

v) Hypergeometric Forms Associated with the Legendre Polynomials-type functions (with Respect to the Complex Variable z) : For all $n \in \mathbb{N}$,

$$\begin{aligned} P_n(z) &= F\left(-n, n+1; 1; \frac{1-z}{2}\right) \\ &= \sum_{k=0}^{\infty} \frac{(-n)_k (n+1)_k}{(1)_k} \frac{[(1-z)/2]^k}{k!}. \end{aligned} \quad (49)$$

vi) Some Hypergeometric Forms Associated with the Gegenbauer (Ultraspherical) Polynomials type functions (with Respect to the Complex Variable z) : For all $n \in \mathbb{N}$ and for some $\alpha \in \mathbb{C} - \mathbb{Z}_0^-$,

$$\begin{aligned} C_n^\alpha(z) &= \frac{(1+\alpha)_n}{\Gamma(n+1)} \\ &\quad \times F\left(-n, n+\alpha; \alpha+\frac{1}{2}; \frac{1-z}{2}\right) \\ &= \frac{(1+\alpha)_n}{n!} \\ &\quad \times \sum_{k=0}^{\infty} \frac{(-n)_k (n+\alpha)_k}{(\alpha+1/2)_k} \frac{[(1-z)/2]^k}{k!}. \end{aligned} \quad (50)$$

vii) Hypergeometric Forms Associated with Jacobi Polynomial-type functions (with Respect to the Complex Variable z) : For all $n \in \mathbb{N}$ and for some $\alpha, \beta \in \mathbb{C} - \mathbb{Z}_0^-$,

$$\begin{aligned} P_n^{\alpha, \beta}(z) &= \frac{(1+\alpha)_n}{\Gamma(n+1)} \\ &\quad \times F\left(-n, n+1+\alpha+\beta; \alpha+1; \frac{1-z}{2}\right) \\ &= \frac{(1+\alpha)_n}{n!} \\ &\quad \times \sum_{k=0}^{\infty} \left(\frac{(-n)_k (n+1+\alpha+\beta)_k}{(\alpha+1/2)_k} \times \frac{[(1-z)/2]^k}{k!} \right). \end{aligned} \quad (51)$$

viii) Hypergeometric Forms Associated with the Confluent Hypergeometric functions (with Respect to the Complex Variable z) :

$$\begin{aligned} &\lim_{\delta \rightarrow \infty} F\left(\alpha, \gamma; \delta; \frac{z}{\gamma}\right) \\ &= \lim_{\delta \rightarrow \infty} \sum_{k=0}^{\infty} \frac{(\alpha)_k (\gamma)_k}{(\delta)_k} \frac{z^k}{k!} \end{aligned}$$

$$\equiv {}_1F_1(\alpha, \delta; z), \quad (52)$$

where the notation ${}_1F_1(-, -; z)$ is called the confluent hypergeometric function or Kummer's function in the mathematical literature. In general, it is denoted by the symbol ${}_1F_1(-, -; z)$ and defined by

$$\begin{aligned} {}_1F_1(\alpha, \delta; z) &= \sum_{k=0}^{\infty} \frac{(\alpha)_k z^k}{(\delta)_k k!} \\ &= 1 + \frac{\alpha}{\delta} \frac{z}{1!} + \frac{\alpha(\alpha+1)}{\delta(\delta+1)} \frac{z^2}{2!} + \dots \end{aligned} \quad (53)$$

for the admissible values of the parameters α ($\alpha \in \mathbb{C} - \mathbb{Z}_0^-$), δ ($\delta \in \mathbb{C} - \mathbb{Z}_0^-$), and z ($z \in \mathbb{U}$).

In special, we note that each one of the familiar notations $(\alpha)_k$ and $(\delta)_k$ denotes the same symbol in connection with the Pochhammer symbol, as introduced in (2). Naturally, there is a fundamental relationship between the definitions given in (2) and (51), namely

$$F(-, \alpha; \delta; z) \equiv {}_1F_1(\alpha, \delta; z).$$

In the literature, the confluent hypergeometric function

$$F(-; -; z) (\equiv {}_1F_1(-, -; z))$$

is also known as degenerate form of the hypergeometric function

$$F(-, -; -; z) (\equiv {}_2F_1(-, -; -; z))$$

as introduced in (1). Furthermore, in view of the special case in (51), this relationship defines an analytic function for all finite values of z , which is closely related to the hypergeometric function in (1). It can also be regarded as the limiting form:

$$\lim_{\beta \rightarrow \infty} F(-, \alpha; \delta; z/\alpha)$$

when $\alpha \rightarrow \infty$.

ix) Hypergeometric Forms of the Main Error Functions (with Respect to the Complex Variable z):

$$\begin{aligned} \operatorname{erf}(z) &= \frac{2z}{\sqrt{\pi}} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -z^2\right) \\ &= \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(1/2)_k (-1)^k}{(3/2)_k k!} z^{2k+1} \end{aligned} \quad (54)$$

and

$$\operatorname{erfc}(z) = 1 - \operatorname{erf}(z)$$

$$\begin{aligned} &= 1 - \frac{2z}{\sqrt{\pi}} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -z^2\right) \\ &= 1 - \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(1/2)_k (-1)^k}{(3/2)_k k!} z^{2k+1}, \end{aligned} \quad (55)$$

where $z \in \mathbb{C}$. Here we point out that the special functions $\operatorname{erf}(z)$ and $\operatorname{erfc}(z)$ are known as the (complex) error function and the (complex) complementary error function, respectively for all values of the variable (parameter) z .

To further clarify the relations between the error functions and the hypergeometric functions discussed above, and naturally to facilitate the identification of connections with other types of hypergeometric functions and related equations, we want to present several specific explanations.

For those, as is pointed out the reference [9], various types of special functions are also defined in terms of these fundamental error functions. Each of them can also be represented in different forms and plays an important role, particularly in mathematics and many engineering. Firstly, we briefly recall some relevant information about these special functions.

As more special information reintroduced in the recent papers, [2], [9], the (main) error function $\operatorname{erf}(z)$ is defined by the help of the following integral:

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-q^2} ds, \quad (56)$$

which yields also to define many other special functions. Specially, only two of them are the complementary error function $\operatorname{erfc}(z)$ and the imaginary error function $\operatorname{erfi}(z)$, which are also defined by

$$1 - \operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-q^2} dq \quad (57)$$

and

$$\begin{aligned} \operatorname{erfi}(z) &= -i \operatorname{erf}(iz) \\ &= \frac{2}{\sqrt{\pi}} \int_0^z e^{q^2} dq, \end{aligned} \quad (58)$$

where $z \in \mathbb{C}$.

Moreover, by considering the special information presented by (2) and making use of the identity

$$\begin{aligned} \frac{(1/2)_k}{(3/2)_k} &= \frac{1}{2} \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+\frac{3}{2})} \\ &= \frac{1}{2} \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+\frac{1}{2}) \Gamma(k+\frac{3}{2})} \\ &= \frac{1}{2k+1}, \end{aligned}$$

the following equivalent representations can be then obtained:

$$\begin{aligned}\operatorname{erf}(z) &= \frac{2z}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{1}{2k+1} \frac{(-1)^k}{k!} z^{2k} \\ &= \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)k!} z^{2k+1},\end{aligned}\quad (59)$$

$$\operatorname{erfc}(z) = 1 - \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)k!} z^{2k+1}, \quad (60)$$

and

$$\operatorname{erfi}(z) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)k!} z^{2k+1}, \quad (61)$$

where $z \in \mathbb{C}$.

As previously indicated in Chapter 1, a necessary and sufficient condition for the existence of any complex function series in the range of equations (33) – (61) is that the series be uniformly convergent. Therefore, to evaluate such functions within the framework of the given theorems, the region of absolute convergence must be a suitable set $D \subseteq \mathbb{U} \subset \mathbb{C}$.

Furthermore, by applying the information in (2) to each relevant function series, every series between (38) and (61), including their special cases, can be simplified by simple operations. For instance, starting from the expression in (38), we can easily obtain the following equivalent assertions:

$$\begin{aligned}\omega &:= \omega(z) := F(1,1;1;z) \\ &= \sum_{k=0}^{\infty} \frac{(1)_k (1)_k}{(1)_k} \frac{z^k}{k!} \\ &= \sum_{k=0}^{\infty} (1)_k \frac{z^k}{k!} \\ &= \sum_{k=0}^{\infty} k! \frac{z^k}{k!} \\ &= \sum_{k=0}^{\infty} z^k,\end{aligned}\quad (62)$$

which also is the solution (of the hypergeometric) equation given by

$$z(1-z)\omega'' + (1-3z)\omega' - \omega = 0, \quad (63)$$

where $\omega := \omega(z)$ and $z \in \mathbb{U}$.

By taking into account the specific information provided above, it is clear that there is a meaningful relationship between the complex-type differential equation in (63) and the complex-valued function that solves the equation in (62).

Naturally, each newly derived function will also satisfy a suitable hypergeometric type differential equation. This observation gives rise to various special relationships between the

corresponding complex functions and their associated complex differential equations.

Now, as an extensive example, let us take into account the homogeneous second-order ordinary (complex) differential equation presented by

$$z\omega'' + (\alpha - z)\omega' - \delta\omega = 0, \quad (64)$$

where $\omega := \omega(z)$, $\alpha \in \mathbb{C} - \mathbb{Z}_0^-$, $\delta \in \mathbb{C} - \mathbb{Z}_0^-$ and $z \in \mathbb{C}$.

In addition, we note that this type of complex differential equation is well known in the literature as the confluent hypergeometric differential equation, and one of its (special) solutions is the confluent hypergeometric function, as in (53).

In particular, for detailed information regarding some of the aforementioned complex functions, their associated differential equations, and related topics, the additional studies from [44], [45], [46], [47], [48], [49], [50], [51], [52], [53], [54], [55], [56], [57], [58], [59], [60] can also be consulted. Considering this comprehensive information, infinitely many special results or relationships related to our main findings can be established. Of course, it is not possible to list all of them.

Nevertheless, we aim to highlight only three of these results, together with one of the related inequalities from the hypotheses of Theorems 1–3, as direct implications.

We also intend to present the others, accompanied by suitable examples, to the attention of researchers working in their related fields.

The first is Implication 1, presented below, which directly follows from one of the special implications of Theorem 1. It can be expressed as follows:

Implication 1. *Let the values of the related parameters α , δ and γ , which were used in Theorem 1, choose as the forms given by*

$$\alpha := 1, \quad \delta := -\frac{5}{2} \quad \text{and} \quad \gamma := -\frac{5}{2}. \quad (65)$$

Then, the second order complex-type differential equation with (complex) variable coefficient given by

$$2z(1-z)\omega'' - 5\omega' + 5\omega + z\omega' = 0, \quad (66)$$

and the following hypergeometric (complex) function:

$$\begin{aligned}F\left(1, -\frac{5}{2}; -\frac{5}{2}; z\right) &\equiv \frac{1}{1-z} \\ &(\equiv: \omega := \omega(z)),\end{aligned}\quad (67)$$

respectively, are easily obtained, Then, according to Theorem 1, the (related) complex function $\omega = : \omega(z)$ is a solution of the (complex-type) differential equation given by (66) and the relevant function $\kappa(z)$, as considered in the proof of Theorem 1, arrives at the form given by

$$\kappa(z) := \frac{z}{2(1-z)^2} \quad (z \in \mathbb{U}). \quad (68)$$

Then, the following statement:

$$\Re\left(\frac{z}{2(1-z)^2}\right) < -\frac{1}{2} \Rightarrow \Re\left(\frac{1}{1-z}\right) > 0 \quad (69)$$

holds, where $z \in \mathbb{U}$.

The second is Implication 2, presented below, which directly follows from one of the special implications of Theorem 2. It can also be expressed as follows:

Implication 2. Let the values of the related parameters α , δ and γ , which were used in Theorem 1, choose as the forms given by

$$\alpha := 1, \delta := 2 \text{ and } \gamma := 1. \quad (70)$$

Then, the second order complex-type differential equation with (complex) variable coefficient given by

$$(1-z)z^2\omega'' + (1-4z)z\omega' - 2z\omega = 0, \quad (71)$$

and the following hypergeometric (complex) function:

$$F(1,2;1;z) \equiv \frac{1}{1-z} \quad (72)$$

$$(\equiv: \omega =: \omega(z)),$$

respectively, are easily obtained, where $z \in \mathbb{U}$. Then, according to Theorem 2, the (related) complex function $\omega =: \omega(z)$ is a solution of the (complex-type) differential equation given by (71) and the relevant function $\rho(z)$, as considered in the proof of Theorem 2, immediately follows that

$$\kappa(z) := -\frac{z}{2(1-z)^2} \quad (z \in \mathbb{U}). \quad (73)$$

Then, the following statement:

$$\Re\left(\frac{2z}{(1-z)^3}\right) < \frac{1}{2} \Rightarrow \Re\left(\frac{1}{(1-z)^2}\right) > 0 \quad (74)$$

is also ensured, where $z \in \mathbb{U}$.

The last is Implication 3, presented below, which directly follows from one of the special

implications of Theorem 3. It can also be expressed as follows:

Implication 3. Let the values of the related parameters α , δ and γ , which were considered in Theorem 3, be in the forms given by

$$\alpha := 1, \delta := 1 \text{ and } \gamma := 1. \quad (75)$$

Then, the second order complex-type differential equation with (complex) variable coefficient given by

$$(1-z)z^2\omega''' + (2-5z)z\omega'' - 3z\omega' = 0, \quad (76)$$

and the following hypergeometric (complex) function:

$$F(1,1;1;z) \equiv \frac{1}{1-z} \quad (77)$$

$$(\equiv: \omega =: \omega(z)),$$

respectively, are easily determined, where $z \in \mathbb{U}$. In the present case, according to Theorem 3, the (related) complex function $\omega =: \omega(z)$ is a solution of the (complex-type) differential equation given by (76) and the mentioned function $\tau(z)$ indicated as in Theorem 3 also own the form :

$$\tau(z) := \frac{3z}{(1-z)^3} \quad (z \in \mathbb{U}). \quad (78)$$

Then, the following statement:

$$\Re\left(\frac{z}{(1-z)^2}\right) < \frac{2}{3} \Rightarrow \Re\left(\frac{1}{1-z}\right) > 0 \quad (79)$$

is also provided, where $z \in \mathbb{U}$.

3 Results and Observations

To evaluate the results of this research from various perspectives, the first part of this study, which is closely related to a broad range of applications in complex function theory and a particular class of complex-type differential equations, provides a fundamental overview of hypergeometric-type differential equations with complex variables and hypergeometric-type series with complex coefficients. The relevant material is primarily presented in (1) – (11), along with comprehensive references.

In the second part, in light of the basic information presented in the first section, the essential results derived from the fundamental analyses summarized in (16) and (64) are presented, together with detailed investigations of their proofs.

In particular, several fundamental theorems establishing special relationships between certain classes of hypergeometric-type functions and their associated differential equations are introduced, with emphasis on rigorous proofs.

Furthermore, a variety of related results stemming from these comprehensive findings are presented, along with numerous implications, applications, and potential consequences, as illustrated in (65) – (79). To this end, particular attention is given to a broad class of hypergeometric-type functions and their associated equations.

Finally, this research is supported by a wide range of references that may be useful for researchers interested in these topics and their applications. These include, for some examples, fundamental hypergeometric-type series, Appell series, two-sided hypergeometric-type series, elliptic hypergeometric-type series, generalized hypergeometric-type functions, generalized hypergeometric-type series, hypergeometric-type distributions, Lauricella hypergeometric-type series, modular hypergeometric-type series, and, as previously mentioned, the Riemann differential-type equation.

In special, the essential purpose is to show how each of these can be related to different types of those differential equations and their solutions in light of the fundamental theorems naturally obtained and their various possible implications. As a notable example, the article in [61] provides detailed information from various perspectives that may be valuable to researchers working in related fields.

Declaration of Generative AI and AI-assisted Technologies in the Writing Process

During the preparation of this work, the author(s) utilized artificial intelligence (AI) technologies solely for the purpose of linguistic refinement and grammatical correction. Following the application of these tools, the author(s) comprehensively reviewed, verified, and edited the resulting text. The author(s) maintain full accountability and responsibility for the accuracy, integrity, and final content of the publication.

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