# On Weakly Prime $k$-Ideals and Weakly Primary $k$-Ideals over Ternary Semirings

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*Abstract:* In this paper, we introduce and investigate the notions of weakly prime k-ideals and weakly primary k-ideals in commutative ternary semirings, extending several fundamental results from the theory of semirings to ternary algebraic structures. Characterizations of weakly prime k-ideals are provided via ideal quotients, and the behavior of these ideals under intersection is studied. In particular, we prove that non-prime weakly prime or non-primary weakly primary k-ideals will satisfy $A^3 = \{0\}$ and $\text{rad}(A) = \text{rad}(\{0\})$. Furthermore, we show that the ternary product involving two non-prime weakly prime k-ideals and any k-ideal contained within the larger one results in zero. We also demonstrate that the property of being weakly primary is preserved under the intersection of a family of non-primary weakly primary k-ideals. The investigation is complemented by auxiliary lemmas concerning ideal quotients and subtractive properties.

*Key-Words:* - weakly primary, weakly prime, ternary semirings, k-ideals, radical, ideal quotients.

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## 1 Introduction

The study of algebraic structures with several operations has led to many interesting generalizations. Ternary algebraic structures naturally extend binary ones, and a ternary semiring is simply a semiring where the multiplication is a ternary operation instead of a binary one. This paper was motivated by the introduction of weakly prime ideals, [1], and the subsequent development of the theory of weakly primary ideals in commutative rings, [2]. These developments motivate the study of analogous notions in ternary semirings. Many related notions have been studied in different algebraic settings. A systematic ideal theory for semirings was developed, [3], while regular and bi-ideal structures in ternary semirings were studied, [4], [5]. More recently, $k$-ideals (subtractive ideals) have become important in the study of semirings and their ternary versions because they preserve useful properties, such as the subtractive law. At the same time, fuzzy algebraic structures and graded ring theory have seen significant progress. Fuzzy proper exact sequences and fuzzy projective semimodules over semirings were studied, [6], providing new tools for homological algebra in semiring categories. Furthermore, graded rings in which every graded semi-primary ideal is graded 1-absorbing primary were characterized, [7], a result that encourages the analysis of primary-like ideals in ternary structures. These recent developments led us to extend weakly prime and weakly primary concepts to the ternary setting, adding the $k$-ideal condition.

In this paper, we introduce and study the notions of weakly prime $k$-ideals and weakly primary $k$-ideals in commutative ternary semirings. We prove several fundamental results that generalize what is known in the binary case, for instance characterizations via ideal quotients, behavior under intersection, and the

fact that a non-primary weakly primary $k$-ideal $A$ must satisfy $A^3 = \{0\}$ and $\mathrm{rad}(A) = \mathrm{rad}(\{0\})$. We also show that the product of non-prime weakly prime $k$-ideals is zero. These results extend the classical theory of weakly prime and weakly primary ideals in rings, [1], [2], and provide a starting point for future work on ternary semirings, including possible extensions to fuzzy and graded contexts, [6], [7].

Throughout this paper, we adopt the following definition: A non-empty set $R$ together with a binary operation, called addition (denoted by $+$), and a ternary operation, called ternary multiplication (denoted by juxtaposition), is called a *ternary semiring* if it satisfies the following conditions:

(1) $(R, +)$ is a commutative semigroup with an identity element $0 \in R$ such that $r + 0 = r$ for all $r \in R$.

(2) The ternary multiplication is associative, i.e., $(abc)de = a(bcd)e = ab(cde)$ for all $a, b, c, d, e \in R$.

(3) The ternary multiplication is distributive with respect to addition, i.e., $(a + b)cd = acd + bcd$, $a(b+c)d = abd+acd$, and $ab(c+d) = abc+abd$ for all $a, b, c, d \in R$.

(4) $0ab = a0b = ab0 = 0$ for all $a, b \in R$.

A ternary semiring $R$ is called *commutative* if its ternary multiplication is commutative, i.e., $abc = acb = bac = bca = cab = cba$ for all $a, b, c \in R$. Next, we recall some basic definitions in ternary semirings, [8]. A *left ideal* [*right ideal*, *middle ideal*, respectively] of a ternary semiring $R$ is a non-empty subset $I$ of $R$ such that if $a, b \in I$ and $r, s \in R$, then $a + b \in I$ and $rsa \in I$ [$ars \in I, ras \in I$, respectively]. A subset $I$ of $R$ is an *ideal* of $R$ if $I$ is a left ideal, right ideal and middle ideal of $R$. A $k$-*ideal* (or a *subtractive ideal*) is an ideal $K$ such that if $x, x + y \in K$ then $y \in K$. A *prime ideal* $P$ of $R$ is a proper ideal for which $abc \in P$ implies $a \in P$ or $b \in P$ or $c \in P$. A *primary ideal* $P$ of $R$ is a proper ideal of $R$ such that if $abc \in P$, then $a \in \mathrm{rad}(P)$ or $b \in \mathrm{rad}(P)$ or $c \in \mathrm{rad}(P)$. The *radical* of an ideal $I$, denoted by $\mathrm{rad}(I)$, is the set of all $x \in R$ such that $x^n \in I$ for some odd positive integer $n$. Here, $x^n$ denotes the $n$-th power of $x$, defined inductively by $x^1 = x$, $x^3 = xxx$, and $x^n = x^{n-2}xx$ for $n \geq 5$. On the other hand, additive repetition $na = a+a+\cdots+a$ ($n$ times) is well-defined for any positive integer $n$, because addition is binary and associative. We have that $\mathrm{rad}(I)$ is an ideal of $R$ containing $I$.

For ideals $I, J, K$ of a ternary semiring $R$, we define their ternary product as

$$IJK = \Big\{ \sum_{i=1}^{n} a_i b_i c_i \mid n \in \mathbb{N}, \; a_i \in I,\ b_i \in J,\ c_i \in K \text{ for all } i \Big\}. \tag{1}$$

In particular, from (1), we write $I^3 = III$. For an element $x \in R$, the ideal generated by $x$ is denoted by $\langle x \rangle = \{abx + nx \mid a, b \in R, n \in \mathbb{N}\}$.

The purpose of this paper is to extend the results obtained in the previous study, [9], to the context of ternary semirings. In Section 2, we prove the main theorems, establishing properties of weakly primary $k$-ideals and weakly prime $k$-ideals that generalize the binary case.

## 2 Main Results

We define a proper ideal $A$ of $R$ to be *weakly primary* (resp. *weakly prime*) if $0 \neq abc \in A$ implies $a \in \mathrm{rad}(A)$ or $b \in \mathrm{rad}(A)$ or $c \in \mathrm{rad}(A)$ (resp. $a \in A$ or $b \in A$ or $c \in A$). The zero ideal $\{0\}$ is always weakly primary and weakly prime by definitions. Every primary (resp. prime) ideal is clearly weakly primary (resp. weakly prime), but the converse need not be true.

**Example 1.** Consider the commutative ternary semiring $R = \mathbb{Z}_8 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}, \overline{7}\}$, the integers modulo 8, with ternary multiplication defined by the usual product of three elements modulo 8. Let $A = \{\overline{0}, \overline{4}\}$. Then $A$ is obviously a $k$-ideal of $R$. We have that $A$ is neither weakly prime nor prime, because $\overline{2} \cdot \overline{2} \cdot \overline{1} = \overline{4} \in A$ but $\overline{1}, \overline{2} \notin A$. It is easy to show that $\mathrm{rad}(A) = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}\}$. Now suppose $\overline{0} \neq \overline{x} \cdot \overline{y} \cdot \overline{z} \in A$. Then $\overline{x} \cdot \overline{y} \cdot \overline{z} = \overline{4}$. Thus at least one of $x, y, z$ must be even. Every even element of $\mathbb{Z}_8$ belongs to $\mathrm{rad}(A) = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}\}$. Therefore, $A$ is weakly primary.

Let $A$ be a $k$-ideal of a commutative ternary semiring $R$, and let $I, J$ be ideals of $R$. Then the *ideal quotient*, [10], of $A$ over $I$ and $J$ is defined by

$$(A : I, J) = \{r \in R \mid rIJ \subseteq A\}. \tag{2}$$

For $a, b \in R$, the *ideal quotient* of $A$ over $a$ and $b$ is defined by

$$(A : a, b) = \{r \in R \mid rab \in A\}. \tag{3}$$

The following lemma shows some properties of the ideal quotients defined in (2) and (3).

**Lemma 2.** *Let $A$ be a $k$-ideal of a commutative ternary semiring $R$, and let $I, J$ be ideals of $R$. Then $(A : I, J)$ is a $k$-ideal of $R$. In particular, for any $a, b \in R$, $(A : a, b)$ is a $k$-ideal of $R$.*

*Proof.* By Proposition 11, [10], we know that $(A : I, J)$ is a $k$-ideal of $R$. The special case $(A : a, b)$ follows by taking $I = \langle a \rangle$ and $J = \langle b \rangle$.

**Lemma 3.** *Let $R$ be a ternary semiring. If a $k$-ideal $H$ of $R$ is the union of two $k$-ideals $K$ and $L$, then $H = K$ or $H = L$.*

*Proof.* Suppose, to the contrary, that $H = K \cup L$ but $H \neq K$ and $H \neq L$. Then there exist elements $u \in H \setminus K$ and $v \in H \setminus L$. Since $H = K \cup L$, the element $u$ (not being in $K$) must belong to $L$, so $u \in L$. Similarly, $v \notin L$ implies $v \in K$.

Now consider the sum $u + v$. Since $H$ is an ideal (in particular, closed under addition), we have $u+v \in H = K \cup L$. Thus, $u + v \in K$ or $u + v \in L$. We examine both possibilities.

**Case 1:** $u + v \in K$. We know that $v \in K$ and $u + v \in K$. Since $K$ is a $k$-ideal, the subtractive property implies that $u \in K$. This contradicts the choice $u \in H \setminus K$.

**Case 2:** $u+v \in L$. Similarly, since $u \in L$, $u+v \in L$ and $L$ is a $k$-ideal, we obtain $v \in L$, contradicting $v \in H \setminus L$.

Both cases lead to a contradiction. Hence, $H = K$ or $H = L$.

**Example 4.** (1) Consider the commutative ternary semiring $R = \mathbb{Z}_8$ described in Example 1. Let $A = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}\}$. Then $A$ is a weakly prime $k$-ideal of $R$. We have that $R \setminus A = \{\overline{1}, \overline{3}, \overline{5}, \overline{7}\}$. For any $a, b \in R \setminus A$, $(A : a, b) = A$ and $(\{\overline{0}\} : a, b) = \{\overline{0}\}$.

(2) Let $R = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}\} \subseteq \mathbb{Z}_8$ be a commutative ternary semiring induced by the ternary multiplication on $\mathbb{Z}_8$, and let $A = \{\overline{0}, \overline{4}\}$. In this case, $A$ is a weakly prime $k$-ideal of $R$, in contrast to Example 1. Moreover, $R \setminus A = \{\overline{2}, \overline{6}\}$ and for any $a, b \in R \setminus A$, $(A : a, b) = (\{\overline{0}\} : a, b) = R$.

For a weakly prime $k$-ideal $A$ of $R$, the following proposition establishes the relationships among $A, (A : a, b)$ and $(\{0\} : a, b)$ for $a, b \in R \setminus A$.

**Proposition 5.** *Let $A$ be a proper $k$-ideal of a commutative ternary semiring $R$. The following statements are equivalent.*

*(1) $A$ is weakly prime.*

*(2) For any $a, b \in R \setminus A$,*

$$(A : a, b) = A \cup (\{0\} : a, b). \tag{4}$$

*(3) For any $a, b \in R \setminus A$,*

$$(A : a, b) = A \quad or \quad (A : a, b) = (\{0\} : a, b). \tag{5}$$

*Proof.* We prove the equivalence by showing that $(1) \Rightarrow (2)$, $(2) \Rightarrow (3)$, and $(3) \Rightarrow (1)$.

**Proof of $(1) \Rightarrow (2)$:** Assume that $A$ is a weakly prime $k$-ideal of $R$. Let $a, b \in R \setminus A$. We need to establish the equality in Equation (4).

First, let $r \in A$. Since $A$ is an ideal, for any elements $a, b \in R$, the ternary product $rab$ involves the element $r$ from $A$, and therefore $rab \in A$. This means $r \in (A : a, b)$. Hence, $A \subseteq (A : a, b)$.

Next, let $r \in (\{0\} : a, b)$. By definition, $rab = 0$ and hence $rab \in A$ since every ideal contains 0. Thus $r \in (A : a, b)$. Hence, $(\{0\} : a, b) \subseteq (A : a, b)$. Therefore, $A \cup (\{0\} : a, b) \subseteq (A : a, b)$.

Now let $r \in (A : a, b)$. Then, by definition, $rab \in A$. We distinguish two cases based on the value of $rab$.

*Subcase 1:* $rab \neq 0$. Since $0 \neq rab \in A$ and $A$ is weakly prime, we obtain that $r \in A$ or $a \in A$ or $b \in A$. By hypothesis, $a \notin A$ and $b \notin A$. Therefore, the only possibility is $r \in A$.

*Subcase 2:* $rab = 0$. Then $r \in (\{0\} : a, b)$.

Hence, $r \in A \cup (\{0\} : a, b)$, proving $(A : a, b) \subseteq A \cup (\{0\} : a, b)$.

Combining the two inclusions, we obtain Equation (4), which is statement (2).

**Proof of $(2) \Rightarrow (3)$:** Assume that statement (2) holds. Let $a, b \in R \setminus A$. Then we have Equation (4). By Lemma 2, both $(A : a, b)$ and $(\{0\} : a, b)$ are $k$-ideals of $R$. Thus, the $k$-ideal $(A : a, b)$ is expressed as the union of two $k$-ideals $A$ and $(\{0\} : a, b)$. Applying Lemma 3, we conclude that:

$$(A : a, b) = A \quad \text{or} \quad (A : a, b) = (\{0\} : a, b). \tag{6}$$

Equation (6) is precisely condition (5).

**Proof of $(3) \Rightarrow (1)$:** Assume that statement (3) holds. To prove that $A$ is weakly prime, let $x, y, z \in R$ be arbitrary elements such that $0 \neq xyz \in A$. We will show that $x \in A$ or $y \in A$ or $z \in A$.

Suppose, for the sake of contradiction, that none of $x, y, z$ belongs to $A$. That is, $x \notin A, \quad y \notin A, \quad z \notin A$. Since $y, z \notin A$, statement (3) applied to $a = y$ and $b = z$ gives condition (5).

*Case (i):* $(A : y, z) = A$. From the given condition $xyz \in A$, we have $x \in (A : y, z) = A$, contradicting $x \notin A$.
*Case (ii):* $(A : y, z) = (\{0\} : y, z)$. Again, from $xyz \in A$, we have $x \in (A : y, z)$. Hence, $x \in (\{0\} : y, z)$. It follows that $xyz = 0$, contradicting $xyz \neq 0$.

Both cases lead to a contradiction. Hence, at least one of $x, y, z$ must belong to $A$, and so $A$ is weakly prime.

The chain of implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$ establishes the equivalence of the three statements, completing the proof. □

**Example 6.** Let $R = \mathbb{Z}_8$ and $A = \{\overline{0}, \overline{4}\}$ as in Example 1. Recall that $A$ is a weakly primary $k$-ideal of $R$ and $\operatorname{rad}(A) = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}\}$. Hence, $R \setminus \operatorname{rad}(A) = \{\overline{1}, \overline{3}, \overline{5}, \overline{7}\}$. We obtain the following:

(1) $(A : \overline{1}, \overline{1}) = A$ and $(A : \overline{1}, \overline{1}) \neq \operatorname{rad}(A)$.

(2) For every $a, b \in R \setminus \operatorname{rad}(A)$, $(A : a, b) = A$ and $(\{\overline{0}\} : a, b) = \{\overline{0}\}$.

The following proposition shows the relationships between a weakly primary $k$-ideal $A$ of $R$, and the conditions involving $A$, $(A : a, b)$ and $(\{0\} : a, b)$ for $a, b \in R \setminus \operatorname{rad}(A)$.

**Proposition 7.** *Let $A$ be a proper $k$-ideal of a commutative ternary semiring $R$.*

*(i) If $A$ is weakly primary, then for every $a, b \in R \setminus \operatorname{rad}(A)$,*

$$(A : a, b) \subseteq \operatorname{rad}(A) \cup (\{0\} : a, b). \tag{7}$$

*(ii) If for every $a, b \in R \setminus \operatorname{rad}(A)$ we have*

$$\begin{aligned} (A : a, b) = A \quad \text{or} \\ (A : a, b) = (\{0\} : a, b), \end{aligned} \tag{8}$$

*then $A$ is weakly primary.*

*Proof.* **(i)** Assume $A$ is weakly primary and take $a, b \notin \operatorname{rad}(A)$. For any $r \in (A : a, b)$, we have $rab \in A$. If $rab = 0$, then $r \in (\{0\} : a, b)$. If $rab \neq 0$, weak primariness forces $r \in \operatorname{rad}(A)$ (since $a, b \notin \operatorname{rad}(A)$). Thus $r \in \operatorname{rad}(A) \cup (\{0\} : a, b)$. Hence, the inclusion (7) holds.

**(ii)** Suppose the condition (8) holds. Let $x, y, z \in R$ satisfy $0 \neq xyz \in A$. Then $z \in (A : x, y)$ because $zxy = xyz \in A$. Assume that $x \notin \operatorname{rad}(A)$ and $y \notin \operatorname{rad}(A)$. If $(A : x, y) = (\{0\} : x, y)$, then $xyz = 0$, a contradiction. Thus $(A : x, y) = A$, so $z \in A \subseteq \operatorname{rad}(A)$. Consequently, $0 \neq xyz \in A$ implies $x \in \operatorname{rad}(A)$ or $y \in \operatorname{rad}(A)$ or $z \in \operatorname{rad}(A)$. Therefore, $A$ is weakly primary.

**Lemma 8.** *Let $R$ be a commutative ternary semiring. Assume that $A$ is a weakly primary $k$-ideal of $R$. If $a \in A$ and $a + b \in \operatorname{rad}(A)$, then $b \in \operatorname{rad}(A)$.*

*Proof.* Since $a + b \in \operatorname{rad}(A)$, there exists an odd positive integer $m$ such that $(a + b)^m \in A$. Expanding $(a+b)^m$ using the distributive laws of the ternary semiring, we can write

$$(a + b)^m = S + b^m, \tag{9}$$

where $b^m$ is the $m$-fold ternary product of $b$ with itself, and $S$ in Equation (9) is the sum of all terms that contain at least one factor equal to $a$. Because $a \in A$ and $A$ is an ideal, each term in $S$ lies in $A$, hence $S \in A$. Now $(a + b)^m \in A$ and $S \in A$, and $A$ is a $k$-ideal, so the subtractive property yields $b^m \in A$. Since $m$ is odd, this means $b \in \operatorname{rad}(A)$. □

**Example 9.** As in Example 4 (2), consider $R = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}\} \subseteq \mathbb{Z}_8$ and $A = \{\overline{0}, \overline{4}\}$. Although $A$ is a weakly prime $k$-ideal of $R$, it is not prime, since $\overline{2} \cdot \overline{2} \cdot \overline{2} = \overline{0} \in A$ while $\overline{2} \notin A$. We note that $A^3 = \{\overline{0}\}$, which motivates the next theorem.

**Theorem 10.** *Let $R$ be a commutative ternary semiring.*

*(i) Let $A$ be a weakly primary $k$-ideal of $R$. If $A$ is not primary, then $A^3 = \{0\}$.*

*(ii) Let $A$ be a weakly prime $k$-ideal of $R$. If $A$ is not prime, then $A^3 = \{0\}$.*

*Proof.* Assume that $A$ is a weakly primary $k$-ideal of $R$ and suppose that $A^3 \neq \{0\}$. We will show that $A$ must be primary.

Let $x, y, z \in R$ satisfy $xyz \in A$. We prove that $x \in \operatorname{rad}(A)$ or $y \in \operatorname{rad}(A)$ or $z \in \operatorname{rad}(A)$.

**Case 1:** $xyz \neq 0$. Since $A$ is weakly primary, the result follows.

**Case 2:** $xyz = 0$. We consider several subcases.

*Subcase 2.1:* There exists $a \in A$ such that $xay \neq 0$. Then

$$\begin{aligned} xy(z + a) &= xyz + xya \\ &= 0 + xya = xya \neq 0. \end{aligned} \tag{10}$$

Since $xya \in A$ and is nonzero, weak primariness applied to $xy(z + a)$ yields $x \in \operatorname{rad}(A)$ or $y \in \operatorname{rad}(A)$ or $z + a \in \operatorname{rad}(A)$. If $z + a \in \operatorname{rad}(A)$, then because $a \in A$, by Lemma 8, we obtain $z \in \operatorname{rad}(A)$. Thus the conclusion holds.

*Subcase 2.2:* $xay = 0$ for all $a \in A$.

*Subcase 2.2(a):* Suppose there exists $a \in A$ such that $xaz \neq 0$. Then

$$\begin{aligned} x(y+a)z &= xyz + xaz \\ &= 0 + xaz = xaz \neq 0. \end{aligned} \tag{11}$$

Weak primariness gives $x \in \mathrm{rad}(A)$ or $y + a \in \mathrm{rad}(A)$ or $z \in \mathrm{rad}(A)$. If $y + a \in \mathrm{rad}(A)$, then by Lemma 8, we obtain $y \in \mathrm{rad}(A)$. Hence, we are done.

*Subcase 2.2(b):* Symmetrically, if there exists $a \in A$ with $yaz \neq 0$, we use $(x + a)yz$ and the same reasoning to get the result.

Thus we may now assume $xay = xaz = yaz = 0$ for all $a \in A$.

*Subcase 2.2(c):* There exist $b, c \in A$ such that $xbc \neq 0$. Then

$$\begin{aligned} x(y+b)(z+c) &= xyz + xyc + xbz + xbc \\ &= 0 + 0 + 0 + xbc \\ &= xbc \neq 0. \end{aligned} \tag{12}$$

By weak primariness and Lemma 8, we obtain $x \in \mathrm{rad}(A)$ or $y \in \mathrm{rad}(A)$ or $z \in \mathrm{rad}(A)$. Thus the conclusion follows.

*Subcase 2.2(d):* There exist $a, c \in A$ such that $ayc \neq 0$. Using $(x + a)y(z + c)$:

$$\begin{aligned} (x+a)y(z+c) &= xyz + xyc + ayz + ayc \\ &= 0 + 0 + 0 + ayc \\ &= ayc \neq 0. \end{aligned} \tag{13}$$

Similarly, weak primariness, together with Lemma 8, yields $x \in \mathrm{rad}(A)$ or $y \in \mathrm{rad}(A)$ or $z \in \mathrm{rad}(A)$.

*Subcase 2.2(e):* There exist $a, b \in A$ such that $abz \neq 0$. The symmetric computation with $(x + a)(y + b)z$ gives the same result.

*Subcase 2.2(f):* If none of the above cases occurs, then we have $xAy = xAz = yAz = \{0\}$ and $xAA = yAA = zAA = \{0\}$.

Since $A^3 \neq \{0\}$, there exist $a, b, c \in A$ with $abc \neq 0$. Consider

$$p := (x + a)(y + b)(z + c). \tag{14}$$

Expanding using distributivity, we obtain

$$\begin{aligned} p = xyz + xyc + xbz + xbc \\ + ayz + ayc + abz + abc. \end{aligned} \tag{15}$$

By the vanishing conditions, all terms except $abc$ are zero. Thus $p = abc \neq 0$. Since $p \in A$, we obtain that $x + a \in \mathrm{rad}(A)$ or $y + b \in \mathrm{rad}(A)$ or $z + c \in \mathrm{rad}(A)$. It follows that $x \in \mathrm{rad}(A)$ or $y \in \mathrm{rad}(A)$ or $z \in \mathrm{rad}(A)$ as $a, b, c \in A$. The computations in Equations (10), (11), (12), (13), (14) and (15) cover all subcases.

In every subcase we have shown that $xyz \in A$ implies one of $x, y, z$ lies in $\mathrm{rad}(A)$. Hence, $A$ is primary. This completes the proof.

For part (ii), the same argument applies with $A$ in place of $\mathrm{rad}(A)$, using the subtractive property of the $k$-ideal $A$. The same case analysis shows that if $A$ is weakly prime and not prime, then $A^3 = \{0\}$. □

**Proposition 11.** *Let $A$ be a weakly primary $k$-ideal of a commutative ternary semiring $R$. If $A$ is not primary, then every element of $A$ is nilpotent, that is, $a^3 = 0$ for all $a \in A$.*

*Proof.* By Theorem 10(i), $A^3 = \{0\}$. Hence, for any $a \in A$, we have $a^3 = aaa \in A^3 = \{0\}$. Then $a^3 = 0$, and hence $a$ is nilpotent. □

**Proposition 12.** *Let $P$ and $Q$ be two weakly prime $k$-ideals of a commutative ternary semiring $R$. If both $P$ and $Q$ are not prime, and if $P \subseteq Q$, then for every $k$-ideal $I$ of $R$ such that $I \subseteq Q$, we have*

$$PQI = \{0\}. \tag{16}$$

*Proof.* By Theorem 10(ii), since $Q$ is weakly prime but not prime, we have $Q^3 = \{0\}$. Because $P \subseteq Q$ and $I \subseteq Q$, for any elements $p \in P$, $q \in Q$, and $i \in I$, we have $p, q, i \in Q$. Thus the ternary product $pqi$ belongs to $Q^3 = \{0\}$, so $pqi = 0$. Therefore, every generator of the ideal $PQI$ is zero, and consequently, Equation (16) holds. □

**Example 13.** With $R = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}\} \subseteq \mathbb{Z}_8$ and $A = \{\overline{0}, \overline{4}\}$ as in Example 9, $A$ is a weakly prime $k$-ideal of $R$ but not prime. Moreover, $\mathrm{rad}(A) = R = \mathrm{rad}(\{0\})$. Indeed, $\overline{2}^3 = \overline{8} = \overline{0}$, $\overline{4}^3 = \overline{64} = \overline{0}$, and $\overline{6}^3 = \overline{216} = \overline{0}$. Thus every element of $R$ is nilpotent, and hence both radicals equal $R$.

**Theorem 14.** *Let $R$ be a commutative ternary semiring. Then:*

*(i) If $A$ is a weakly primary $k$-ideal that is not primary, then $\mathrm{rad}(A) = \mathrm{rad}(\{0\})$.*

*(ii) If $A$ is a weakly prime $k$-ideal that is not prime, then $\mathrm{rad}(A) = \mathrm{rad}(\{0\})$.*

*Proof.* **(i)** The inclusion $\mathrm{rad}(\{0\}) \subseteq \mathrm{rad}(A)$ holds trivially, because $0 \in A$.

For the reverse inclusion, let $x \in \mathrm{rad}(A)$. Then there exists an odd positive integer $n$ such that $x^n \in A$. Since $A$ is not primary, Theorem 10(i) yields $A^3 = \{0\}$. Hence, $x^{3n} = (x^n)^3 = 0$, which implies $x \in \mathrm{rad}(\{0\})$. Therefore, $\mathrm{rad}(A) \subseteq \mathrm{rad}(\{0\})$, and equality follows.
**(ii)** Let $A$ be a weakly prime $k$-ideal that is not prime. By Theorem 10(ii), $A^3 = \{0\}$. The remaining steps are analogous to those in part (i). □

**Theorem 15.** *Let $\{A_i\}_{i\in\mathcal{I}}$ be a family of weakly primary $k$-ideals of a commutative ternary semiring $R$ that are not primary. Then $A = \bigcap_{i\in\mathcal{I}} A_i$ is a weakly primary $k$-ideal of $R$.*

*Proof.* Each $A_i$ is a $k$-ideal, so $A$ is a $k$-ideal. Because each $A_i$ is not primary, Theorem 14(i) gives $\mathrm{rad}(A_i) = \mathrm{rad}(\{0\})$ for every $i$. Hence,

$$\mathrm{rad}(A) = \bigcap_{i\in\mathcal{I}} \mathrm{rad}(A_i) = \mathrm{rad}(\{0\}). \qquad (17)$$

Since each $A_i$ is proper (it is a weakly primary ideal), their intersection $A$ is also proper.

Now suppose $0 \neq abc \in A$ with $a, b, c \in R$. We will show $a \in \mathrm{rad}(A)$ or $b \in \mathrm{rad}(A)$ or $c \in \mathrm{rad}(A)$.

Assume, to the contrary, that $a, b, c \notin \mathrm{rad}(A) = \mathrm{rad}(\{0\})$. Because $\mathrm{rad}(A) = \bigcap_{i\in\mathcal{I}} \mathrm{rad}(A_i)$ as in Equation (17), there exists an index $j$ such that $a \notin \mathrm{rad}(A_j)$. Since $A_j$ is weakly primary, $abc \in A_j$ and $a \notin \mathrm{rad}(A_j)$ force $b \in \mathrm{rad}(A_j)$ or $c \in \mathrm{rad}(A_j)$. But $\mathrm{rad}(A_j) = \mathrm{rad}(\{0\})$, so $b \in \mathrm{rad}(\{0\})$ or $c \in \mathrm{rad}(\{0\})$, contradicting $b, c \notin \mathrm{rad}(A)$. Hence, $A$ is weakly primary.

## 3 Discussion and Conclusion

In this paper, we have introduced and studied weakly prime and weakly primary $k$-ideals in commutative ternary semirings. Our main results extend several classical theorems from semiring theory to the ternary setting, providing characterizations via ideal quotients, structural constraints on non-primary ideals, and preservation of weak primariness under intersections. These results show that the theory of weakly prime and weakly primary ideals in commutative rings, initiated in previous work, [1], and subsequently developed further, [2], can be naturally extended to ternary algebraic structures.

The extension from binary to ternary multiplication is not merely a formal exercise. The proof of our main structural theorem reveals that the ternary structure allows for richer flexibility in disturbing zero products: unlike the binary case, where one can only add a single element to one factor, the ternary setting permits simultaneous perturbations of multiple factors. This additional flexibility may lead to new phenomena not present in binary semirings. Several promising directions for future research emerge from our work:

**Graded ternary semirings.** Following the framework on graded 1-absorbing primary ideals presented, [7], one could define graded weakly primary $k$-ideals in graded commutative ternary semirings. A key challenge is that the expansion $(a + b)^m$ used in our subtractive radical lemma may not preserve homogeneity, requiring careful consideration of homogeneous components. This could lead to new classes of graded ideals with no direct analogs in the binary setting.

**Fuzzy ternary semirings.** The framework developed here could be adapted to the fuzzy setting based on the results, [6]. A fuzzy $k$-ideal $\mu : R \to [0, 1]$ could be defined to be weakly primary if for all $a, b, c \in R$, $\mu(abc) > 0$ implies that at least one of $\mu(a), \mu(b), \mu(c)$ satisfies the corresponding condition with respect to the fuzzy radical $\mu_{\mathrm{rad}}$. This would connect the classical and fuzzy theories and may yield new characterizations via level cuts.

**Classification of non-primary weakly primary $k$-ideals.** One of our results shows that any non-primary weakly primary $k$-ideal satisfies $A^3 = \{0\}$. We conjecture that such ideals are precisely the non-zero ideals $A \subseteq \mathrm{rad}(\{0\})$ with $A^3 = \{0\}$. Proving this would provide a complete classification and reveal the structure of the nilradical in ternary semirings.

**Generalizations to higher arities.** Our work raises the question of whether analogous results hold for $n$-ary semirings for arbitrary $n \geq 3$. The proof of our main theorem relies on the ability to modify multiple factors simultaneously, suggesting that similar arguments may extend to higher arities, although the complexity of the proof increases with $n$.

**Non-commutative ternary semirings.** While our work assumes commutativity of the ternary multiplication, non-commutative ternary semirings arise naturally in many contexts, particularly in the study of ternary algebras over non-commutative base rings.

Extending our results to the non-commutative case would require different types of ideals, such as left, right, and middle ideals, and may reveal new structural insights.

In conclusion, this paper establishes the foundations of weakly prime and weakly primary $k$-ideals in commutative ternary semirings and opens several promising avenues for future research. The interplay between the ternary structure and the $k$-ideal condition yields results that are both analogous to and distinct from those in the binary case, suggesting that ternary semirings provide opportunities for further development in ideal theory.

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**Declaration of Generative AI and AI-assisted Technologies in the Writing Process**
During the preparation of this work, the author(s) utilized artificial intelligence (AI) technologies solely for the purpose of linguistic refinement and grammatical correction. Following the application of these tools, the author(s) comprehensively reviewed, verified, and edited the resulting text. The author(s) maintain full accountability and responsibility for the accuracy, integrity, and final content of the publication.

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**Conflict of Interest**

The authors have no conflicts of interest to declare that are relevant to the content of this article.

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