

# A Novel Explicit ARL Formula via Integral Equation on DEWMA for Detecting Mean Shifts in the Autoregressive Process with Exogenous Variables

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**Abstract:** - This study develops explicit formulas for the Average Run Length (ARL) of the Double Exponentially Weighted Moving Average (DEWMA) control chart under  $ARX(p,r)$  processes with exponential white noise. The proposed approach provides an efficient analytical framework for monitoring autocorrelated and non-normally distributed processes. The existence and uniqueness of the ARL solution are established using Banach's fixed-point theorem, while the accuracy of the derived formulas is validated against Numerical Integral Equation (NIE) solutions obtained via the Midpoint Rule, Gauss-Legendre Quadrature Rule, Trapezoidal Rule, and Simpson's Rule. The numerical results show that the explicit formulas produce ARL values that are in close agreement with those obtained from the NIE methods, while substantially reducing computational burden. Furthermore, performance comparisons based on the Expected Average Run Length (EARL), Average Extra Quadratic Loss (AEQL), and Performance Comparison Index (PCI) reveal that the DEWMA control chart is more effective than the EWMA control chart in detecting small and moderate shifts. A real-data application to natural gas prices, incorporating West Texas Intermediate (WTI) crude oil prices as an exogenous variable, further demonstrates the practical usefulness and effectiveness of the proposed methodology for process monitoring in the presence of autocorrelation and external influences.

**Key-Words:** - Explicit formulas, Numerical Integral Equation Method, Average Run Length, Autoregressive model with exogenous variables, Double Exponentially Weighted Moving Average, Expected Average Run Length, Average Extra Quadratic Loss, Performance Comparison Index.

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## 1 Introduction

The production of a product typically involves assembling numerous components, each of which may be manufactured using different machines. For this reason, it is essential that every component attains a high level of quality and reliability. Important attributes such as length, hardness, and density are specified and controlled throughout the production process. Nevertheless, variations or defects may arise during manufacturing due to factors such as human error, machine conditions, or natural influences. These irregularities are not always immediately observable. As a result, it is necessary to define acceptable tolerance limits, which are generally based on customer requirements, product design specifications, or intended product functionality. To manage and control such

variability, Statistical Process Control (SPC) is widely implemented as a systematic approach for monitoring and assessing process performance. SPC enhances process precision and consistency, minimizes variability, and ensures that products conform to established standards and specifications. Among the various SPC techniques, control charts are especially prevalent in industrial applications because of their effectiveness in tracking process behavior and detecting abnormalities in a timely manner. This enables manufacturers to sustain and improve product quality efficiently. In particular, the Cumulative Sum (CUSUM) chart and the Exponentially Weighted Moving Average (EWMA) chart are commonly employed to identify small shifts in process performance. Study [1] introduced the EWMA chart for detecting slight variations in

process means within the SPC framework, whereas study [2] highlighted the CUSUM chart as a valuable method for identifying accumulated changes. Furthermore, study [3] utilized the EWMA chart to monitor the severity of events in network security systems, while study [4] applied both EWMA and CUSUM charts in economic and financial contexts to detect turning points in IBM stock prices effectively.

Nevertheless, cumulative and weighted control charts continue to play a central role in statistical process monitoring, particularly the CUSUM (Cumulative Sum) and EWMA (Exponentially Weighted Moving Average) charts, which have been widely adopted for detecting process variations due to their effectiveness in identifying even small shifts in process parameters. In recent years, the EWMA framework has been extensively developed and refined to further enhance its sensitivity to minor changes. Notable extensions include the Modified EWMA [5] and Extended EWMA [6], both of which have been demonstrated to significantly improve detection capability across a wide range of applications. Among these advancements, the Double EWMA (DEWMA) control chart, first introduced in [7], has attracted considerable attention as an alternative approach. Subsequent research in [8] extended this concept and provided evidence that DEWMA outperforms traditional control charts, particularly in detecting small shifts in process parameters. Furthermore, [9] formally characterized DEWMA as an extension of the EWMA scheme based on double exponential smoothing, thereby enhancing its responsiveness to gradual changes. In addition, [10] conducted a systematic review and bibliometric analysis, highlighting the widespread application of control charts, especially EWMA, in healthcare quality monitoring systems. Empirical findings reported in [11] further indicate that DEWMA exhibits superior sensitivity and accuracy compared to traditional methods when applied to censored data, such as cancer patient recovery times. Building upon these developments, studies in [12] and [13] integrated multivariate Partial Least Squares (PLS) models with the DEWMA framework to address high-dimensional monitoring problems, while [14] evaluated the performance of DEWMA-based variants, including squared and maximum statistics, under skewed distributional settings. From a distributional perspective, [15] investigated the performance of DEWMA and EWMA control charts under non-normal distributions, such as the gamma and  $t$  distributions, and reported that DEWMA provides superior detection performance. This line of research was further extended in [16], where both charts were examined under a gamma distribution

using a conditional expectation approach, and their performance was evaluated through simulation based on Average Run Length (ARL) metrics. Finally, [17] assessed the effectiveness of DEWMA and EWMA control charts for autocorrelated data, confirming their applicability in monitoring dependent process structures.

Time series models take various forms, including the autoregressive (AR), moving average (MA), and autoregressive moving average (ARMA) models. In general, time series models are often assumed to follow a normal distribution. However, although most control charts are based on the assumptions of independence and normality, real-world data in practice are often time series in nature, which may exhibit trends, seasonality, and autocorrelation. Models such as autoregressive (AR) and moving average (MA) can be collectively referred to as ARIMA models, which are widely used to handle such characteristics. Typically, the noise term (white noise) is assumed to follow a normal distribution; however, in some situations, it may follow an exponential distribution. This study focuses on the case where the noise follows an exponential distribution [18]. The use of models with exponentially distributed white noise has been widely explored. In [19], a numerical integration approach based on Gauss's rule was proposed to approximate the Average Run Length (ARL) for DEWMA control charts. This method was applied to data generated from an  $ARX(p,r)$  process with a trend and exponentially distributed white noise. The results showed that the method is computationally efficient, requiring less than one minute. In [20], a closed-form solution for ARL was presented using differential equations and numerical integration for EWMA control charts when the process follows a seasonal autoregressive model  $SAR(p)_L$ . The results were also compared with the CUSUM control chart, and it was found that the EWMA chart outperforms the CUSUM chart in detecting shifts in the process mean. Subsequently, in [21], a closed-form solution for ARL was developed for data following a  $MAX(1, r)$  model, and the performance of the EWMA chart was evaluated by estimating ARL using two numerical integration methods: Gauss's rule and the midpoint rule. In terms of computational time, the closed-form solution required the least time, followed by the midpoint rule, while Gauss's rule required the longest time. Nevertheless, all three methods yielded very similar ARL values. In [22], a seasonal autoregressive model  $SAR(p)_L$  was applied in conjunction with a Double EWMA scheme to analyze air pollution data in Thailand. Furthermore, numerous studies have applied ARIMAX models to

forecast various types of data. For example, [23] used ARIMAX to forecast and monitor malaria outbreaks in high-risk areas of Bhutan, while [24] applied it to forecast motorcycle sales by category in the Banyuwangi region. In addition, ARIMAX has been integrated with Radial Basis Function neural networks to forecast net cash flow.

The Average Run Length (ARL) is a key measure used to evaluate the performance of control charts. It can be classified into two types: the in-control ARL ( $ARL_0$ ) and the out-of-control ARL ( $ARL_1$ ). The in-control ARL ( $ARL_0$ ) represents the expected number of samples taken from a process operating under control before a false alarm occurs; therefore, it should be large to minimize false alarms. In contrast, the out-of-control ARL ( $ARL_1$ ) represents the expected number of observations from a process operating out of control before a true alarm is triggered; thus, it should be small to enable rapid detection of process shifts. The ARL can be computed using various approaches, such as the Markov approach proposed in [25], and comparative studies between the Markov chain method and the Numerical Integral Equation (NIE) method in [26]. Furthermore, [27] developed an NIE-based approach that is well suited for real-world data exhibiting both temporal dependence and non-normal distributions. However, these methods differ in computational time depending on the approximation technique employed. As an alternative, explicit formulas have been proposed to compute ARL more efficiently and conveniently. In [28], an ARL formula was developed for the Double EWMA (DEWMA) control chart under an MA(q) model with exponential white noise. Subsequently, [29] derived explicit formulas for the Average Run Length (ARL) of the DEWMA control chart under an autoregressive model with a trend, demonstrating a significant reduction in computational time compared with the Numerical Integral Equation (NIE) method. Furthermore, [30] presented ARL formulas for the LSFIMAX model by applying Banach's fixed-point theorem to derive explicit ARL formulas for the EWMA control chart. Finally, [31] and [32] extended this methodology to seasonal autoregressive (SAR) models and demonstrated its applicability using real medical data.

From the literature review discussed above, it can be observed that the derivation of Average Run Length (ARL) using explicit formulas for the Double EWMA (DEWMA) control chart under an Autoregressive Process with Exogenous Variables, ARX(p, r), remains limited. Therefore, this study proposes explicit formulas for the DEWMA control chart under the aforementioned model and compares

the results with the Numerical Integral Equation (NIE) method. Several numerical integration techniques are employed for comparison, including the Midpoint Rule, Gaussian Rule, Trapezoidal Rule, and Simpson's Rule. The performance of the control chart is evaluated using both simulated data and real-world data. In addition, real economic data, particularly natural gas trading data influenced by the external factor of West Texas Intermediate (WTI) crude oil prices, are used, with data collected daily from December 1, 2025, to February 28, 2026. are utilized to demonstrate the effectiveness of the proposed method in detecting process shifts. All results obtained from the explicit formulas and the NIE method in this study were computed using Wolfram Mathematica. The computations were performed on a system with the following specifications: Windows 10 Home operating system, Intel Core i7-11370H processor with a clock speed of 3.3–4.8 GHz, and 16.0 GB of RAM (15.7 GB usable).

## 2 Materials and Methods

### 2.1 Exponentially Weighted Moving Average (EWMA) Control Chart

The EWMA control chart, which was initially created in [2], is capable of identifying even the smallest fluctuations in the process mean. The equation in which the EWMA statistic is expressed is as follows:

$$Z_t = (1 - \lambda_1)Z_{t-1} + \lambda_1 Y_t \quad (1)$$

where  $Y_t$  is a sequence of the ARX(p,r) with exponential white noise, the starting value  $Y = u$  is an initial value,  $\lambda_1$  is the weighting parameter for past data in the EWMA control chart, where  $0 < \lambda_1 \leq 1$

The upper control limit is the component of the EWMA control chart's control limits. Lower control limit:

$$\begin{aligned} UCL &= \mu + F_1 \sqrt{\frac{\lambda_1}{2 - \lambda_1}} \\ CL &= \mu \\ LCL &= \mu - F_1 \sqrt{\frac{\lambda_1}{2 - \lambda_1}} \end{aligned} \quad (2)$$

where  $F_1$  is the width of the control limits.

The EWMA stopping time  $\tau_h$  is defined as:

$$\tau_h = \inf\{t > 0 : Z_t \geq UCL\} \text{ for } h > u.$$

where  $\tau_h$  is the stopping time, and  $h$  is the UCL.

## 2.2 Double EWMA Control Chart

The Double EWMA control chart was introduced in [7], [8] when the parameters  $\lambda_1$  and  $\lambda_2$  are specified to be unequal, and with modifications as shown in Equation (3)

$$DE_t = (1 - \lambda_2)DE_{t-1} + \lambda_2 Z_t \quad (3)$$

where  $\lambda_1$  and  $\lambda_2$  are exponential smoothing parameters with  $0 < \lambda_1 \leq 1$  and  $0 < \lambda_1 \leq \lambda_2$  and the initial value is a constant,  $DE_0 = u$ . From the DEWMA control chart, the upper control limit (UCL) and lower control limit (LCL) are as follows

$$UCL = \mu + F_2 \sigma \sqrt{\frac{\lambda_1^2 \lambda_2^2}{(\lambda_1 - \lambda_2)^2} \left[ \frac{(1 - \lambda_2)^2}{1 - (1 - \lambda_2)^2} + \frac{(1 - \lambda_1)^2}{1 - (1 - \lambda_1)^2} - 2 \frac{(1 - \lambda_1)(1 - \lambda_2)}{1 - (1 - \lambda_1)(1 - \lambda_2)} \right]}$$

$$LCL = \mu - F_2 \sigma \sqrt{\frac{\lambda_1^2 \lambda_2^2}{(\lambda_1 - \lambda_2)^2} \left[ \frac{(1 - \lambda_2)^2}{1 - (1 - \lambda_2)^2} + \frac{(1 - \lambda_1)^2}{1 - (1 - \lambda_1)^2} - 2 \frac{(1 - \lambda_1)(1 - \lambda_2)}{1 - (1 - \lambda_1)(1 - \lambda_2)} \right]} \quad (4)$$

where  $\mu$  is the target mean,  $\sigma$  is the process standard deviation, and  $F_2$  is the breadth of the control limits.

The stopping time of the DEWMA control chart  $\tau_b$  is determined by the following:

$$\tau_b = \inf\{t > 0 : D_t \geq UCL\} \quad (5)$$

where  $\tau_b$  is the stopping time.

## 3 The ARL of the DEWMA Control Chart

### 3.1 Autoregressive Model with Exogenous Variables (ARX(p,r))

ARX(p,r) with exogenous variables can be derived as

$$y_t = \phi_0 + \phi_1 y_{t-1} + \dots + \phi_p y_{t-p} + \sum_{i=1}^r \alpha_i x_i + \varepsilon_i \quad (6)$$

where  $Y_t$  is a sequence of the ARX(p,r) with exogenous variables with exponential white noise  $\text{Exp}(\beta)$ ,  $\alpha_i$  is the regression coefficient of  $x$ ,  $\phi_i$  is the autoregressive parameter.

### 3.2 Explicit for the ARL of ARX(p,r) with Exogenous Variables

Explicit form for the ARL refers to deriving a closed-form expression for the Average Run Length (ARL) without relying on numerical computation methods such as the Numerical Integral Equation (NIE) approach. In this study, explicit ARL formulas for the DEWMA control chart are derived under the Autoregressive Process with Exogenous Variables (ARX(p,r)) model. Define the integral equation  $L(u) = \text{ARL}$  of the DEWMA control chart. The

equation  $u \in [0, b]$  can be used to express the integral equation given in Equation (7):

$$\text{ARL} = L(u) = E_\infty(\tau_b) \geq T \quad (7)$$

where  $\tau_b$  denotes the time at which the control chart signals that the process is out of control,  $E_\infty(\tau_b)$  represents the expectation associated with the distribution of the observed value at the change point occurring at time  $t$ , and  $T$  is a constant. The integral equation of the DEWMA control chart, in the form of a Fredholm integral equation of the second kind, can be expressed as Equation (8):

$$L(u) = 1 + \frac{1}{\lambda_1 \lambda_2} \int_0^b L(w) f\left(\frac{w - (1 - \lambda_2)u - M}{\lambda_1 \lambda_2}\right) dw \quad (8)$$

where

$$M = \lambda_2 (1 - \lambda_1) z_0 + \lambda_1 \lambda_2 \left( \phi_0 + \phi_1 y_{t-1} + \dots + \phi_p y_{t-p} + \sum_{i=1}^r \alpha_i x_i \right)$$

By the Numerical Integral Equations method (NIE), four approaches are employed in this research as follows:

Let  $L(u)$  denote the ARL of a Two-sided EWMA control chart when the initial value is  $u$ ,  $DE_0 = u$

Since:  $DE_1 = (1 - \lambda_2)u + \lambda_1 \lambda_2 + \varepsilon_i$

$$M = \lambda_2 (1 - \lambda_1) z_0 + \lambda_1 \lambda_2 \left( \phi_0 + \phi_1 y_{t-1} + \dots + \phi_p y_{t-p} + \sum_{i=1}^r \alpha_i x_i \right)$$

and Lower and upper control limits  $\beta_L = a$  and  $\beta_U = b$  respectively, for the DEWMA statistics  $DE_1$  in an in-control state:

$$a \leq DE_1 \leq b$$

$$a \leq (1 - \lambda_2)u + \lambda_1 \lambda_2 + \varepsilon_i \leq b$$

$$\frac{a - (1 - \lambda_2)u + M}{\lambda_1 \lambda_2} \leq \varepsilon_i \leq \frac{b - (1 - \lambda_2)u + M}{\lambda_1 \lambda_2}$$

So that

$$a \leq DE_1 \leq b$$

$$L(u) = 1 + \frac{1}{\lambda_1 \lambda_2} \int_a^b L(w) f\left(\frac{w - (1 - \lambda_2)u - M}{\lambda_1 \lambda_2}\right) dw.$$

$$L(u) = \frac{\int_{\frac{a - (1 - \lambda_2)u + M}{\lambda_1 \lambda_2}}^{\frac{b - (1 - \lambda_2)u + M}{\lambda_1 \lambda_2}} L\left[(1 - \lambda_2)u + M + \lambda_1 \lambda_2 y\right] f(y) dy}{\lambda_1 \lambda_2} \quad (9)$$

Since

$$W = (1 - \lambda_2)u + M + \lambda_1 \lambda_2 y$$

$$\frac{W - (1 - \lambda_2)u - M}{\lambda_1 \lambda_2} = y, \quad dy = \frac{1}{\lambda_1 \lambda_2} dw$$

So that we use a Fredholm integral equation of the second, which can be rewritten by the integral variable; we derive the subsequent integral equation:

$$L(u) = 1 + \frac{1}{\lambda_1 \lambda_2} \int_a^b L(w) f\left(\frac{w - (1 - \lambda_2)u - M}{\lambda_1 \lambda_2}\right) dw \quad (10)$$

we can solve

$$L(u) = 1 + \frac{1}{\beta \lambda_1 \lambda_2} \int_a^b L(w) e^{-\frac{w}{\beta \lambda_1 \lambda_2}} e^{\frac{(1 - \lambda_2)u + M}{\beta \lambda_1 \lambda_2}} dw$$

Therefore, we obtain

$$L(u) = 1 + \frac{1}{\beta \lambda_1 \lambda_2} e^{\frac{(1 - \lambda_2)u + M}{\beta \lambda_1 \lambda_2}} \int_a^b L(w) e^{-\frac{w}{\beta \lambda_1 \lambda_2}} dw$$

And we Set

$$Q(u) = \frac{1}{\beta \lambda_1 \lambda_2} e^{\frac{(1 - \lambda_2)u + M}{\beta \lambda_1 \lambda_2}}$$

$$\text{and } R = \int_a^b L(w) e^{-\frac{w}{\beta \lambda_1 \lambda_2}} dw$$

where  $a$  and  $b$  are the lower and upper control limits

Let  $L(u)$  denote the ARL of a Two-sided DEWMA control chart when the initial value is  $u$ ,  $DE_0 = u$

$$\text{from } R = \int_a^b L(w) e^{-\frac{w}{\beta \lambda_1 \lambda_2}} dw$$

we can written

$$R = \int_a^b (1 + Q(w)R) e^{-\frac{w}{\beta \lambda_1 \lambda_2}} dw$$

Therefore, we obtain

$$R = \int_a^b e^{-\frac{w}{\beta \lambda_1 \lambda_2}} dw + \int_a^b Q(w)R e^{-\frac{w}{\beta \lambda_1 \lambda_2}} dw$$

$$R = \int_a^b e^{-\frac{w}{\beta \lambda_1 \lambda_2}} dw + \int_a^b \left( \frac{1}{\beta \lambda_1 \lambda_2} \right) e^{\frac{(1 - \lambda_2)u + M}{\beta \lambda_1 \lambda_2}} R e^{-\frac{w}{\beta \lambda_1 \lambda_2}} dw$$

$$R = \left[ -\beta \lambda_1 \lambda_2 e^{-\frac{w}{\beta \lambda_1 \lambda_2}} \right]_a^b + \left( \frac{R}{\beta \lambda_1 \lambda_2} \right) e^{\frac{M}{\beta \lambda_1 \lambda_2}} \left[ -\beta \lambda_1 e^{-\frac{w}{\beta \lambda_2}} \right]_a^b$$

$$R = -\beta \lambda_1 \lambda_2 \left[ e^{\frac{-b}{\beta \lambda_1 \lambda_2}} - e^{\frac{-a}{\beta \lambda_1 \lambda_2}} \right] - \frac{\beta}{\lambda_2} e^{\frac{-M}{\beta \lambda_1 \lambda_2}} \left[ e^{\frac{-b}{\beta \lambda_2}} - e^{\frac{-a}{\beta \lambda_2}} \right]$$

Solving for a constant R

$$1 + \frac{e^{\frac{-M}{\beta \lambda_1 \lambda_2}}}{\lambda_2} \left[ e^{\frac{-b}{\beta \lambda_2}} - e^{\frac{-a}{\beta \lambda_2}} \right] R = -\beta \lambda_1 \lambda_2 \left[ e^{\frac{-b}{\beta \lambda_1 \lambda_2}} - e^{\frac{-a}{\beta \lambda_1 \lambda_2}} \right]$$

$$R = \frac{-\beta \lambda_1 \lambda_2 \left[ e^{\frac{-b}{\beta \lambda_1 \lambda_2}} - e^{\frac{-a}{\beta \lambda_1 \lambda_2}} \right]}{1 + \frac{e^{\frac{-M}{\beta \lambda_1 \lambda_2}}}{\lambda_2} \left[ e^{\frac{-b}{\beta \lambda_2}} - e^{\frac{-a}{\beta \lambda_2}} \right]}$$

Substituting R into Eq. 10 thus the function  $L(u)$  can be written as  $L(u) = 1 + Q(w) + R$

$$L(u) = 1 + \frac{1}{\beta \lambda_1 \lambda_2} e^{\frac{(1 - \lambda_2)u + M}{\beta \lambda_1 \lambda_2}} \left[ \frac{-\beta \lambda_1 \lambda_2 \left[ e^{\frac{-b}{\beta \lambda_1 \lambda_2}} - e^{\frac{-a}{\beta \lambda_1 \lambda_2}} \right]}{1 + \frac{e^{\frac{-M}{\beta \lambda_1 \lambda_2}}}{\lambda_2} \left[ e^{\frac{-b}{\beta \lambda_2}} - e^{\frac{-a}{\beta \lambda_2}} \right]} \right]$$

Where the ARX(p,r) model with exogenous variables with exponential white noise

$\text{Exp}(\beta)$ ,  $\alpha_i$  is the regression coefficient of  $x$  and

$$M = \lambda_2 (1 - \lambda_1) z_0 + \lambda_1 \lambda_2 \left( \phi_0 + \phi_1 y_{t-1} + \dots + \phi_p y_{t-p} + \sum_{i=1}^r \alpha_i x_i \right)$$

solving the equation  $L(u)$  can be written as

$$L(u) = 1 - \frac{e^{\frac{(1 - \lambda_2)u + M}{\beta_0 \lambda_1 \lambda_2}} \left[ e^{\frac{-b}{\beta_0 \lambda_1 \lambda_2}} - e^{\frac{-a}{\beta_0 \lambda_1 \lambda_2}} \right]}{1 + \frac{e^{\frac{-M}{\beta_0 \lambda_1 \lambda_2}}}{\lambda_2} \left[ e^{\frac{-b}{\beta_0 \lambda_2}} - e^{\frac{-a}{\beta_0 \lambda_2}} \right]} \quad (11)$$

As mentioned above, the value of the parameter  $\beta = \beta_0$  when the process is in control gives the

formula for the  $ARL_0$  as:

$$ARL_0 = 1 - \frac{e^{\frac{(1 - \lambda_2)u + M}{\beta_0 \lambda_1 \lambda_2}} \left[ e^{\frac{-b}{\beta_0 \lambda_1 \lambda_2}} - e^{\frac{-a}{\beta_0 \lambda_1 \lambda_2}} \right]}{1 + \frac{e^{\frac{-M}{\beta_0 \lambda_1 \lambda_2}}}{\lambda_2} \left[ e^{\frac{-b}{\beta_0 \lambda_2}} - e^{\frac{-a}{\beta_0 \lambda_2}} \right]} \quad (12)$$

The formula for  $ARL_1$  can be obtained in a similar manner. When the process is out of control, the value of the parameter  $\beta = \beta_1$  gives the formula for the

$ARL_1$  as:

$$ARL_1 = 1 - \frac{e^{\frac{(1 - \lambda_2)u + M}{\beta_1 \lambda_1 \lambda_2}} \left[ e^{\frac{-b}{\beta_1 \lambda_1 \lambda_2}} - e^{\frac{-a}{\beta_1 \lambda_1 \lambda_2}} \right]}{1 + \frac{e^{\frac{-M}{\beta_1 \lambda_1 \lambda_2}}}{\lambda_2} \left[ e^{\frac{-b}{\beta_1 \lambda_2}} - e^{\frac{-a}{\beta_1 \lambda_2}} \right]} \quad (13)$$

### 3.3 The Uniqueness and Existence of the Explicit ARL Formula

In this section, the Banach Fixed-point theorem is used to prove that the explicit formula of the ARL has an existence and gives a unique solution uses

Banach's fixed-point theorem to demonstrate the existence and uniqueness of the ARL solution, as the explicit ARL formula necessitates such proof from (10):

$$L(u) = 1 + \frac{1}{\lambda_1 \lambda_2} \int_a^b L(w) f\left(\frac{w - (1 - \lambda_2)u - M}{\lambda_1 \lambda_2}\right) dw$$

Proposition 1. Suppose  $L$  defined on a complete metric space and  $G : L_1 \rightarrow L_2$  be a contraction mapping on  $H$  such that:  $\|G(L_1) - G(L_2)\| \leq \delta \|L_1 - L_2\|$  For  $L_1, L_2 \in L$ ,

Proof: Let  $G$  be a contraction mapping operator defined as:

$$G(L(u)) = 1 + \frac{1}{\lambda_1 \lambda_2} \int_a^b L(w) f\left(\frac{w - (1 - \lambda_2)u - M}{\lambda_1 \lambda_2}\right) dw$$

for any  $L_1, L_2 \in C[a, b]$ , with  $\|H\|_\infty = \sup_{u \in [a, b]} \|L(u)\|$ .

First consider:  $\|G(L_1) - G(L_2)\|_\infty =$

$$\begin{aligned} & \sup_{u \in [a, b]} \left\| \left( 1 + \frac{1}{\lambda_1 \lambda_2} \int_a^b L_1(w) f\left(\frac{w - (1 - \lambda_2)u - M}{\lambda_1 \lambda_2}\right) dw \right) - \left( 1 + \frac{1}{\lambda_1 \lambda_2} \int_a^b L_2(w) f\left(\frac{w - (1 - \lambda_2)u - M}{\lambda_1 \lambda_2}\right) dw \right) \right\| \\ &= \left\| \left( 1 + \frac{1}{\lambda_1 \lambda_2} \int_a^b L_1(w) f\left(\frac{w - (1 - \lambda_2)u - M}{\lambda_1 \lambda_2}\right) dw \right) - \left( 1 + \frac{1}{\lambda_1 \lambda_2} \int_a^b L_2(w) f\left(\frac{w - (1 - \lambda_2)u - M}{\lambda_1 \lambda_2}\right) dw \right) \right\| \\ &\leq \frac{1}{\lambda_1 \lambda_2} \int_a^b \|L_1(w) - L_2(w)\|_\infty f\left(\frac{w - (1 - \lambda_2)u - M}{\lambda_1 \lambda_2}\right) dw \end{aligned}$$

Use Hölder's Inequality, can be written as follows

$$\begin{aligned} \|G(L_1(u)) - G(L_2(u))\|_\infty &\leq \frac{1}{\lambda_1 \lambda_2} \int_0^b \|L_1 - L_2\| f\left(\frac{w - (1 - \lambda_2)u - M}{\lambda_1 \lambda_2}\right) dw \\ &\leq \frac{1}{\lambda_1 \lambda_2} \|L_1 - L_2\|_\infty \int_0^b f\left(\frac{w - (1 - \lambda_2)u - M}{\lambda_1 \lambda_2}\right) dw \\ &\leq \frac{b^\delta}{\lambda_1 \lambda_2} \|L_1 - L_2\|_\infty \left( \frac{e^{\frac{(1 - \lambda_2)u + M}{\beta \lambda_1 \lambda_2}}}{\beta \lambda_1 \lambda_2} \right)^{1 - \delta} \left( \beta \lambda_1 \lambda_2 (1 - \delta) \left( 1 - e^{\frac{-b}{\beta \lambda_1 \lambda_2 (1 - \delta)}} \right) \right)^{\delta} \\ &\leq \frac{b^\delta}{\lambda_1 \lambda_2} (1 - \delta) e^{\frac{(1 - \lambda_2)u + M}{\beta \lambda_1 \lambda_2}} \left( \frac{-b}{1 - e^{\frac{-b}{\beta \lambda_1 \lambda_2 (1 - \delta)}}} \right)^{1 - \delta} \|L_1 - L_2\|_\infty \\ &\leq \delta \|L_1 - L_2\|_\infty \end{aligned}$$

$$\text{where } \delta = \frac{b^\delta}{\lambda_1 \lambda_2} (1 - \delta) e^{\frac{(1 - \lambda_2)u + M}{\beta \lambda_1 \lambda_2}} \left( \frac{-b}{1 - e^{\frac{-b}{\beta \lambda_1 \lambda_2 (1 - \delta)}}} \right)^{1 - \delta}$$

Therefore  $\|G(L_1) - G(L_2)\| \leq \delta \|L_1 - L_2\|$ , then  $G$  is the contraction mapping on a complete metric space, and by the Banach fixed point theorem,  $G$  has a unique solution in  $L$ .

### 3.4 The NIE for the ARL of ARX(p,r) with Exogenous Variables

Define the integral equation  $L(u) = ARL$  of the DEWMA control chart. The equation  $u \in [0, b]$  can be used to express the integral equation given in Equation (14):

$$ARL = L(u) = E_\infty(\tau_b) \geq T \quad (14)$$

where  $\tau_b$  denotes the time at which the control chart signals that the process is out of control,  $E_\infty(\tau_b)$  represents the expectation associated with the distribution of the observed value at the change point occurring at time  $t$ , and  $T$  is a constant. The integral equation of the DEWMA control chart, in the form of a second-kind Fredholm integral equation, can be expressed as Equation (15):

$$L(u) = 1 + \frac{1}{\lambda_1 \lambda_2} \int_0^b L(w) f\left(\frac{w - (1 - \lambda_2)u - M}{\lambda_1 \lambda_2}\right) dw \quad (15)$$

where

$$M = \lambda_2 (1 - \lambda_1) z_0 + \lambda_1 \lambda_2 \left( \phi_0 + \phi_1 y_{t-1} + \dots + \phi_p y_{t-p} + \sum_{i=1}^r \alpha_i x_i \right)$$

By the Numerical Integral Equations method (NIE), four approaches are employed in this research as follows:

#### 3.4.1 Midpoint Rule

Let  $y = L(u)$  in the interval  $[0, b]$  be divided into  $m$  subintervals, each with a width of  $h = \frac{b}{m}$ .

Based on the midpoint rule with  $m$  subintervals, it can be expressed as Equation (16).

$$L(u) = 1 + \frac{1}{\lambda_1 \lambda_2} \sum_{i=1}^m w_i j(a_i) f\left(\frac{a_i - (1 - \lambda_2)u - M}{\lambda_1 \lambda_2}\right) \quad (16)$$

where

$$M = \lambda_2 (1 - \lambda_1) z_0 + \lambda_1 \lambda_2 \left( \phi_0 + \phi_1 y_{t-1} + \dots + \phi_p y_{t-p} + \sum_{i=1}^r \alpha_i x_i \right)$$

#### 3.4.2 Trapezoidal Rule

Let  $y = L(u)$  in the interval  $[0, b]$  be divided into  $m$  subintervals, each with a width of  $h = \frac{b}{m}$ .

the midpoint rule with  $m$  subintervals, it can be expressed as Equation (17).

$$T(L, b) = \frac{b}{2} (L(0) - L(b)) + b \sum_{i=1}^m L(u_i) \quad (17)$$

where  $k = 0, 1, 2, \dots, m$  the approximation of the integral of the equation over the interval  $[0, b]$  is

given in the form of  $\int_0^b L(u) du \approx T(L, h)$  the numerical

integral equation approximation, based on the trapezoidal rule, of the ARX (p,r) with exogenous variables on the DEWMA control chart can be expressed as Equation (18)

$$L(u) = 1 + \frac{1}{\lambda_1 \lambda_2} \sum_{i=1}^m w_i j(a_i) f\left(\frac{a_i - (1 - \lambda_2)u - M}{\lambda_1 \lambda_2}\right) \quad (18)$$

where

$$M = \lambda_2 (1 - \lambda_1) z_0 + \lambda_1 \lambda_2 \left( \phi_0 + \phi_1 y_{t-1} + \dots + \phi_p y_{t-p} + \sum_{i=1}^r \alpha_i x_i \right)$$

### 3.4.3 Gauss-Legendre Quadrature Rule

In this research, the Gauss-Legendre Quadrature rule is abbreviated as the Gauss rule. It is defined by the set of  $a_k$  where  $0 \leq a_1 \leq \dots \leq a_m \leq b$ , and the set of weights  $w_k = b/m \geq 0; k = 1, 2, \dots, m$  are constant. The numerical integral approximation takes the form shown in Equation (19).

$$\int_0^b W(y) f(y) dy \approx \sum_{k=1}^m w_k f(a_k) \quad (19)$$

where  $a_k = \frac{b}{m} \left( k - \frac{1}{2} \right); k = 1, 2, \dots, m$ .

The numerical integral equation approximation, based on the trapezoidal rule, of the ARX (p,r) with exogenous variables on the DEWMA control chart can be expressed as Equation (20)

$$L(u) = 1 + \frac{1}{\lambda_1 \lambda_2} \sum_{i=1}^m w_i j(a_i) f\left(\frac{a_i - (1 - \lambda_2)u - M}{\lambda_1 \lambda_2}\right) \quad (20)$$

where

$$M = \lambda_2 (1 - \lambda_1) z_0 + \lambda_1 \lambda_2 \left( \phi_0 + \phi_1 y_{t-1} + \dots + \phi_p y_{t-p} + \sum_{i=1}^r \alpha_i x_i \right)$$

### 3.2.4 Simpson's Rule

Simpson's Rule, in which the interval is divided into 2m subintervals, can be expressed as follows. Let  $y = L(u)$  in the interval  $[0, b]$  be divided into

2m subintervals with a width of  $h = \frac{b}{2m}$ , then

it can be represented as Equation (21).

$$S(L, h) = \frac{h}{3} (L(a) - L(H_u)) + \frac{2h}{3} \quad (21)$$

$$\sum_{k=1}^m f(y_{2k}) + \frac{4h}{3} \sum_{k=1}^m f(y_{2k-1}); k = 0, 1, 2, \dots, m$$

The approximation of the integral of the equation  $L(u)$  over the interval is in the form of

$\int_0^b f(u) du \approx S(L, h)$  the numerical integral equation

approximation, based on the trapezoidal rule, of the ARX (p,r) with exogenous variables on the DEWMA control chart, can be expressed as Equation(22)

$$L(u) = 1 + \frac{1}{\lambda_1 \lambda_2} \sum_{i=1}^m w_i j(a_i) f\left(\frac{a_i - (1 - \lambda_2)u - M}{\lambda_1 \lambda_2}\right) \quad (22)$$

where

$$M = \lambda_2 (1 - \lambda_1) z_0 + \lambda_1 \lambda_2 \left( \phi_0 + \phi_1 y_{t-1} + \dots + \phi_p y_{t-p} + \sum_{i=1}^r \alpha_i x_i \right)$$

This study evaluates the ARL results obtained from the explicit formulations and the Numerical Integral Equation (NIE) method for the ARX (p,r) process implemented on the DEWMA control chart. The accuracy of the ARL is compared using the percentage accuracy, which can be calculated as follows:

$$\%Accuracy = 100 - \left| \frac{L(u) - L(u)}{L(u)} \right| \times 100\% \quad (23)$$

In addition, the performance of the control charts in this study is evaluated using several performance metrics. One of these measures is the Expected Average Run Length (EARL) [33], [34]. The EARL is employed to assess the performance of various control charts. A control chart with a lower EARL value is considered to have better performance. The EARL can be expressed as follows:

$$EARL = \frac{1}{\Delta} \sum_{\delta_i = \delta_{\min}}^{\delta_{\max}} (ARL(\delta_i)) \quad (24)$$

where  $\delta_{\max}$  and  $\delta_{\min}$  represent the upper and lower limits of the shift parameter  $\delta$ ,  $ARL(\delta)$  is the ARL that corresponds to the specific shift, and  $\Delta$  is the number of ranges of  $\delta$  between  $\delta_{\max}$  and  $\delta_{\min}$ . In addition to the EARL performance metric, this study also employs the Average Extra Quadratic Loss (AEQL) and the Performance Comparison Index (PCI) to evaluate the performance of the control charts. The AEQL can be calculated as follows [34]:

$$AEQL = \frac{1}{\Delta} \sum_{\delta_i = \delta_{\min}}^{\delta_{\max}} ((\delta_i)^2 \times ARL(\delta_i)) \quad (25)$$

where  $\delta$  represents the magnitude of the process shift parameter, and  $\Delta$  denotes the number of

intervals of  $\delta$  ranging from  $\delta_{\min}$  to  $\delta_{\max}$ . The control chart with the lowest AEQL value is considered the most efficient. Meanwhile, the PCI value is determined by dividing the AEQL of a given control chart by the AEQL of the control chart with the minimum AEQL value, which represents the most efficient control chart. The PCI can be expressed as follows:

$$PCI = \frac{AEQL}{AEQL_{\text{lowest}}} \quad (26)$$

## 4 Numerical Results

Table 1. (Appendix) presents the accuracy measurements of the ARL obtained from the explicit formulas and the four Numerical Integral Equation (NIE) methods for the ARX(1,2) model with respect to the specified parameters „ for the in-control process while the out-of-control process is defined as The shift sizes are set to 0.0001, 0.0005, 0.001, 0.005, 0.01, 0.05, 0.1, 0.5, and 1, with the target value defined as  $\lambda_1 = 0.05$   $\phi_0 = 0.1$   $\phi_1 = 0.2$ ,  $\phi_2 = 0.1$ ,  $\alpha_1 = 1$   $\beta_0 = 1$   $\beta_1 = (1 + \delta)\beta_0$ .  $\delta$  ARL<sub>0</sub> = 370.

Table 2. (Appendix) presents the accuracy measurements of the ARL obtained from the explicit formulas and the four Numerical Integral Equation (NIE) methods for the ARX(2,3) model with respect to the specified parameters .for the in-control process while the out-of-control process is defined as The shift sizes are set to 0.0001, 0.0005, 0.001, 0.005, 0.01, 0.05, 0.1, 0.5, and 1, with the target value defined as Table 3 (Appendix) presents the accuracy measurements of the ARL obtained from the explicit formulas and the four Numerical Integral Equation (NIE) methods for the ARX(3,2) model with respect to the parameter for the in-control process while the out-of-control process is defined as The shift sizes are set to 0.0001, 0.0005, 0.001, 0.005, 0.01, 0.05, 0.1, 0.5, and 1, with the target value defined as  $\lambda_1 = 0.05$ ,  $\phi_0 = 0.1$ ,  $\phi_1 = 0.1$ ,  $\phi_2 = 0.2$ ,  $\alpha_1 = 1$ ,  $\alpha_2 = 1$ ,  $\alpha_3 = 1$ ,  $\beta_0 = 1$   $\beta_1 = (1 + \delta)\beta_0$   $\delta$  ARL<sub>0</sub> = 370.

$\lambda_1 = 0.05$ ,  $\phi_0 = 1$ ,  $\phi_1 = 0.1$ ,  $\phi_2 = 0.2$ ,  $\phi_3 = 0.1$ ,  $\alpha_1 = 1$ ,

$\alpha_2 = 1$ ,  $\beta_0 = 1$   $\beta_1 = (1 + \delta)\beta_0$ .  $\delta$  ARL<sub>0</sub> = 370.

The results indicate that the ARL values obtained from the explicit formulas and the four Numerical Integral Equation (NIE) methods are highly consistent. The ARL approximations derived from the NIE approaches do not differ significantly in terms of accuracy. In particular, Simpson's Rule achieves an accuracy rate of 100%, while the Midpoint Rule, Gauss Rule, and Trapezoidal Rule

exhibit comparable levels of accuracy and require computation times of no more than 10 seconds. Meanwhile, the explicit formula approach, as presented in Tables 1, 2, and 3 in the Appendix, requires a CPU time of approximately 0.000 seconds, representing a substantial reduction in computational time when compared with the NIE methods. Furthermore, Table 4 (Appendix) presents an analysis of the performance of the DEWMA and EWMA control charts using the explicit formulas to evaluate the ARL results under the target ARL values of 370 and 500 for the ARX(1,1) model with respect to the specified parameters  $\lambda_1 = 0.05$ ,  $\lambda_2 = 0.1$ ,  $\phi_0 = 0.1$ ,  $\phi_1 = 0.2$ , and  $\alpha_1 = 1$ .

As illustrated in Figure 1 (Appendix), the comparison of the Average Run Length (ARL) for the ARX(1,1) model between the DEWMA and EWMA control charts under the target ARL values of 370 and 500 shows that both control charts exhibit a decreasing trend in ARL as the parameter shift levels increase. However, the DEWMA control chart demonstrates greater sensitivity in detecting process shifts than the EWMA control chart, indicating that the DEWMA control chart provides superior detection performance across all parameter levels. In addition, the DEWMA control chart consistently exhibits superior performance at all levels of parameter changes.

Table 5. (Appendix) further presents an analysis of the performance of the DEWMA and EWMA control charts using the explicit formulas to evaluate the ARL results under the target ARL values of 370 and 500 for the ARX(2,1) model with respect to the specified parameters As shown in Figure 2 (Appendix), the comparison of the Average Run Length (ARL) for the ARX(2,1) model between the DEWMA and EWMA control charts under the target ARL values of 370 and 500 reveals that both control charts display a decreasing ARL trend as the parameter shift levels increase. Nevertheless, the DEWMA control chart demonstrates higher sensitivity in detecting process changes than the EWMA control chart, indicating that the DEWMA control chart achieves better detection performance across all parameter levels. Moreover, the DEWMA control chart consistently provides superior performance at all levels of parameter changes  $\lambda_1 = 0.05$ ,  $\lambda_2 = 0.1$ ,  $\phi_0 = 1$ ,  $\phi_1 = 0.2$ ,  $\phi_2 = 0.1$ ,  $\alpha_1 = 1$ .

The results indicate that, in all cases, the ARL values of the DEWMA control chart are significantly lower than those of the EWMA control chart. In addition, the EARL, AEQL, and PCI measures were employed to evaluate the performance of both control charts at all levels of parameter changes. A control

chart with lower EARL and AEQL values is considered more effective, while the PCI criterion shows that the DEWMA control chart attains a value equal to 1. Therefore, the findings demonstrate that although both control charts can accurately determine ARL values using the explicit formula approach, the DEWMA control chart consistently outperforms the EWMA control chart in detecting process shifts across all levels of parameter changes.

#### 4.1 Application

This study presents the Average Run Length (ARL) obtained from the proposed explicit formulas and four Numerical Integral Equation (NIE) methods using real economic data, specifically natural gas trading data influenced by the exogenous variable of West Texas Intermediate (WTI) crude oil prices. The daily data were collected from December 1, 2025, to February 28, 2026, in order to demonstrate the effectiveness of the proposed methods in detecting process shifts under real-world conditions. Initially, an appropriate time-series model was identified, and the model parameters were estimated using the Maximum Likelihood Estimation (MLE) method. The results indicated that the ARX(1,1) model provided the best fit to the data. The estimated model parameters are  $\phi_1 = 0.847$ ,  $\alpha_1 = 0.062$  presented in Table 7 (Appendix). Based on these estimated parameters, the corresponding forecasting equation is given in Equation  $\hat{Y}_t = 0.847Y_{t-1} + 0.062X_t$  (27)

After estimating the model parameters, the ARL results presented in Table 6 (Appendix) compare the explicit formula and the four Numerical Integral Equation (NIE) methods under the target ARL values of 370 and 500, using real natural gas trading data influenced by WTI crude oil prices. The results demonstrate that the DEWMA control chart consistently produces lower ARL values than the EWMA control chart, indicating superior sensitivity in detecting process shifts. Furthermore, the Expected Average Run Length (EARL), Average Extra Quadratic Loss (AEQL), and Performance Comparison Index (PCI) were employed to evaluate the performance of both control charts across all levels of parameter shifts. In general, a control chart with lower EARL and AEQL values is considered more efficient in detecting process changes. Moreover, according to the PCI criterion, the DEWMA control chart achieved a PCI value of 1, indicating the best overall performance. Therefore, the results suggest that although both the DEWMA and EWMA control charts can accurately estimate ARL values using the proposed explicit formulas, the

DEWMA control chart consistently outperforms the EWMA control chart in detecting process shifts across all levels of parameter changes. As illustrated in Figure 3 (Appendix), the comparison of the Average Run Length (ARL) for the ARX(1,1) model using real natural gas trading data influenced by WTI crude oil prices under the target ARL values of 370 and 500 shows that the ARL values of both control charts decrease as the magnitude of the parameter shift increases. However, the DEWMA control chart exhibits greater sensitivity to process shifts than the EWMA control chart, resulting in faster detection of out-of-control conditions. Consequently, the DEWMA control chart consistently demonstrates superior detection performance over the EWMA control chart across all levels of parameter changes.

#### 5 Conclusion

This study presents explicit formulas for the Average Run Length (ARL) of the ARX(p,r) model on the DEWMA control chart. The results demonstrate that the explicit formulas provide high accuracy while substantially reducing computational processing time (CPU time). Furthermore, when comparing the ARL values obtained from the explicit formulas with those derived from the Numerical Integral Equation (NIE) method using percentage accuracy criteria, the findings reveal no significant differences in the ARL values between the two approaches. In addition, the performance of the DEWMA and EWMA control charts in detecting changes in the process mean was investigated across all levels of parameter shifts. The results indicate that the DEWMA control chart consistently outperforms the EWMA control chart. Moreover, when considering the EARL, AEQL, and PCI performance criteria, the DEWMA control chart produces lower EARL and AEQL values than the EWMA control chart, while the PCI value is equal to 1. the study also found that the results obtained from models with known parameters and those obtained from real data with estimated parameters exhibit similar patterns, indicating that the DEWMA control chart provides the best overall performance. for future research, the explicit ARL formulas on the DEWMA control chart can be further developed for other interesting models. In addition, the integral equation approach may also be extended to derive formulas for Average Run Length values on other types of control charts.

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### **Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)**

- Chinnawat Muangkeaw has organized the writing original draft and software.
- Yupaporn Areepong has conceptualization and validation
- Saowanit Sukparungsee has implemented the methodology and simulation.

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The authors declare no.

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## APPENDIX

Table 1. ARL of explicit formulas and NIE methods of ARL for the ARX(1,2) model with  $\lambda_1 = 0.05$ ,  $\phi_1 = 0.2$ ,  $\alpha_1 = 1$  and  $\alpha_2 = 1.5$

| $\lambda_2$<br>$\delta$ |        | DEWMA for ARX (1,2) |       |            |       |            |       |            |       |             |       |
|-------------------------|--------|---------------------|-------|------------|-------|------------|-------|------------|-------|-------------|-------|
|                         |        | Explicit            |       | Midpoint   |       | Gaussian   |       | Simpson    |       | Trapezoidal |       |
|                         |        | ARL                 | TIME  | ARL        | TIME  | ARL        | TIME  | ARL        | TIME  | ARL         | TIME  |
| 0.1                     | 0      | 370.078             | 0.000 | 369.077    | 1.703 | 369.077    | 1.672 | 370.078    | 6.673 | 369.731     | 1.688 |
|                         | 0.0001 | 362.530             | 0.000 | 362.530    | 1.704 | 362.530    | 1.703 | 362.531    | 6.750 | 362.190     | 1.703 |
|                         | 0.0005 | 335.187             | 0.000 | 335.186    | 1.718 | 335.186    | 1.656 | 335.188    | 6.672 | 334.873     | 1.734 |
|                         | 0.001  | 306.311             | 0.000 | 306.310    | 1.703 | 306.310    | 1.735 | 306.311    | 6.765 | 306.023     | 1.719 |
|                         | 0.005  | 181.364             | 0.000 | 181.364    | 1.703 | 181.364    | 1.719 | 181.365    | 6.750 | 181.195     | 1.719 |
|                         | 0.01   | 120.156             | 0.000 | 120.156    | 1.719 | 120.156    | 1.734 | 120.157    | 6.797 | 120.045     | 1.703 |
|                         | 0.05   | 32.663              | 0.000 | 32.663     | 1.687 | 32.663     | 1.687 | 32.663     | 6.750 | 32.633      | 1.735 |
|                         | 0.10   | 17.306              | 0.000 | 17.306     | 1.672 | 17.306     | 1.719 | 17.306     | 6.797 | 17.291      | 1.703 |
|                         | 0.50   | 4.286               | 0.000 | 4.286      | 1.734 | 4.286      | 1.704 | 4.286      | 6.734 | 4.283       | 1.688 |
|                         | 1.00   | 2.689               | 0.000 | 2.689      | 1.734 | 2.689      | 1.687 | 2.689      | 6.828 | 2.688       | 1.703 |
| %accuracy               |        |                     |       | 99.730     |       | 99.730     |       | 100        |       | 99.906      |       |
| $\lambda_2$<br>$\delta$ |        | DEWMA for ARX (1,2) |       |            |       |            |       |            |       |             |       |
|                         |        | Explicit            |       | Midpoint   |       | Gaussian   |       | Simpson    |       | Trapezoidal |       |
|                         |        | ARL                 | TIME  | ARL        | TIME  | ARL        | TIME  | ARL        | TIME  | ARL         | TIME  |
| 0.2                     | 0      | 370.010             | 0.000 | 369.007    | 1.672 | 369.007    | 1.687 | 370.011    | 6.813 | 369.681     | 1.703 |
|                         | 0.0001 | 357.073             | 0.000 | 357.070    | 1.703 | 357.070    | 1.719 | 357.074    | 7.031 | 356.755     | 1.734 |
|                         | 0.0005 | 313.229             | 0.000 | 313.227    | 1.687 | 313.227    | 1.703 | 313.230    | 6.845 | 312.950     | 1.688 |
|                         | 0.001  | 271.495             | 0.000 | 271.493    | 1.687 | 271.493    | 1.735 | 271.496    | 6.844 | 271.253     | 1.718 |
|                         | 0.005  | 130.908             | 0.000 | 130.908    | 1.688 | 130.908    | 1.703 | 130.909    | 6.781 | 130.792     | 1.719 |
|                         | 0.01   | 79.014              | 0.000 | 79.014     | 1.672 | 79.014     | 1.719 | 79.014     | 6.828 | 78.944      | 1.688 |
|                         | 0.05   | 18.053              | 0.000 | 18.052     | 1.687 | 18.052     | 1.688 | 18.053     | 6.782 | 18.037      | 1.750 |
|                         | 0.10   | 8.783               | 0.000 | 8.783      | 1.687 | 8.783      | 1.719 | 8.783      | 6.766 | 8.776       | 1.718 |
|                         | 0.50   | 1.774               | 0.000 | 1.774      | 1.750 | 1.774      | 1.688 | 1.774      | 6.766 | 1.773       | 1.750 |
|                         | 1.00   | 1.202               | 0.000 | 1.202      | 1.703 | 1.202      | 1.718 | 1.202      | 6.844 | 1.202       | 1.735 |
| %accuracy               |        |                     |       | 99.729     |       | 99.729     |       | 99.999     |       | 99.911      |       |
| $\lambda_2$<br>$\delta$ |        | DEWMA for ARX (1,2) |       |            |       |            |       |            |       |             |       |
|                         |        | Explicit            |       | Midpoint   |       | Gaussian   |       | Simpson    |       | Trapezoidal |       |
|                         |        | ARL (TIME)          |       | ARL (TIME) |       | ARL (TIME) |       | ARL (TIME) |       | ARL (TIME)  |       |
| 0.3                     | 0      | 370.129             | 0.000 | 369.123    | 1.750 | 369.123    | 1.641 | 370.129    | 6.672 | 369.81      | 1.719 |
|                         | 0.0001 | 346.211             | 0.000 | 346.206    | 1.703 | 346.206    | 1.687 | 346.212    | 6.672 | 356.755     | 1.734 |
|                         | 0.0005 | 275.096             | 0.000 | 275.092    | 1.719 | 275.092    | 1.688 | 275.096    | 6.781 | 312.950     | 1.688 |
|                         | 0.001  | 218.882             | 0.000 | 218.88     | 1.687 | 218.880    | 1.688 | 218.882    | 6.704 | 271.253     | 1.718 |
|                         | 0.005  | 83.021              | 0.000 | 83.021     | 1.672 | 83.021     | 1.703 | 83.021     | 6.796 | 130.792     | 1.719 |
|                         | 0.01   | 46.713              | 0.000 | 46.712     | 1.703 | 46.712     | 1.704 | 46.713     | 6.735 | 78.944      | 1.688 |
|                         | 0.05   | 10.369              | 0.000 | 10.369     | 1.703 | 10.369     | 1.703 | 10.369     | 6.734 | 18.037      | 1.750 |
|                         | 0.10   | 5.309               | 0.000 | 5.309      | 1.703 | 5.309      | 1.703 | 5.309      | 6.766 | 8.776       | 1.718 |
|                         | 0.50   | 1.456               | 0.000 | 1.456      | 1.719 | 1.456      | 1.704 | 1.456      | 6.766 | 1.773       | 1.750 |
|                         | 1.00   | 1.12                | 0.000 | 1.120      | 1.750 | 1.120      | 1.703 | 1.120      | 6.719 | 1.202       | 1.735 |
| %accuracy               |        |                     |       | 99.728     |       | 99.728     |       | 100        |       | 99.914      |       |

The CPU times are provided in parentheses (unit: second)

Source: created by the authors

Table 2. ARL of explicit formulas and NIE methods of ARL for the ARX(2, 3) model with  $\lambda_1 = 0.05$ ,  $\phi_0 = 1$ ,  $\phi_1 = 0.1$ ,  $\phi_2 = 0.2$ ,  $\alpha_1 = 1$ ,  $\alpha_2 = 1$  and  $\alpha_3 = 1$

| $\lambda_2$ | $\delta$ | DEWMA for ARX (2,3) |       |            |       |            |       |            |       |             |       |
|-------------|----------|---------------------|-------|------------|-------|------------|-------|------------|-------|-------------|-------|
|             |          | Explicit            |       | Midpoint   |       | Gaussian   |       | Simpson    |       | Trapezoidal |       |
|             |          | ARL                 | TIME  | ARL        | TIME  | ARL        | TIME  | ARL        | TIME  | ARL         | TIME  |
| 0.1         | 0        | 370.056             | 0.000 | 369.041    | 1.781 | 369.041    | 1.796 | 370.057    | 7.218 | 369.774     | 1.906 |
|             | 0.0001   | 315.791             | 0.000 | 315.780    | 1.812 | 315.780    | 1.813 | 315.792    | 7.219 | 315.547     | 2.297 |
|             | 0.0005   | 199.145             | 0.000 | 199.140    | 1.813 | 199.140    | 1.828 | 199.145    | 7.453 | 198.987     | 2.031 |
|             | 0.001    | 136.348             | 0.000 | 136.346    | 1.812 | 136.346    | 1.843 | 136.349    | 8.219 | 136.24      | 1.969 |
|             | 0.005    | 39.057              | 0.000 | 39.057     | 1.828 | 39.057     | 1.813 | 39.057     | 8.110 | 39.026      | 1.937 |
|             | 0.01     | 20.889              | 0.000 | 20.889     | 1.859 | 20.889     | 1.812 | 20.890     | 7.812 | 20.873      | 1.938 |
|             | 0.05     | 4.871               | 0.000 | 4.871      | 1.813 | 4.871      | 1.829 | 4.871      | 7.485 | 4.868       | 1.937 |
|             | 0.10     | 2.783               | 0.000 | 2.783      | 1.843 | 2.783      | 1.843 | 2.783      | 7.687 | 2.782       | 1.875 |
|             | 0.50     | 1.192               | 0.000 | 1.192      | 1.860 | 1.192      | 1.844 | 1.192      | 7.266 | 1.192       | 1.953 |
|             | 1.00     | 1.048               | 0.000 | 1.048      | 1.844 | 1.048      | 1.828 | 1.048      | 7.312 | 1.048       | 1.969 |
| %accuracy   |          |                     |       | 99.725     |       | 99.725     |       | 99.999     |       | 99.923      |       |
| $\lambda_2$ | $\delta$ | DEWMA for ARX (2,3) |       |            |       |            |       |            |       |             |       |
|             |          | Explicit            |       | Midpoint   |       | Gaussian   |       | Simpson    |       | Trapezoidal |       |
|             |          | ARL                 | TIME  | ARL        | TIME  | ARL        | TIME  | ARL        | TIME  | ARL         | TIME  |
| 0.2         | 0        | 370.001             | 0.000 | 368.918    | 1.844 | 368.918    | 1.797 | 370.001    | 6.813 | 369.897     | 1.813 |
|             | 0.0001   | 248.693             | 0.000 | 248.654    | 1.859 | 248.654    | 1.859 | 248.693    | 7.031 | 248.591     | 1.844 |
|             | 0.0005   | 107.941             | 0.000 | 107.932    | 1.860 | 107.932    | 1.828 | 107.941    | 6.845 | 107.881     | 1.859 |
|             | 0.001    | 63.485              | 0.000 | 63.481     | 1.859 | 63.481     | 1.860 | 63.485     | 6.844 | 63.446      | 1.875 |
|             | 0.005    | 15.356              | 0.000 | 15.355     | 1.812 | 15.355     | 1.828 | 15.356     | 6.781 | 15.346      | 1.859 |
|             | 0.01     | 8.256               | 0.000 | 8.256      | 1.875 | 8.256      | 1.859 | 8.256      | 6.828 | 8.251       | 1.844 |
|             | 0.05     | 2.374               | 0.000 | 2.374      | 1.875 | 2.374      | 1.844 | 2.374      | 6.782 | 2.373       | 1.812 |
|             | 0.10     | 1.631               | 0.000 | 1.631      | 1.828 | 1.631      | 1.859 | 1.631      | 6.766 | 1.630       | 1.875 |
|             | 0.50     | 1.065               | 0.000 | 1.065      | 1.875 | 1.065      | 1.875 | 1.065      | 6.766 | 1.065       | 1.828 |
|             | 1.00     | 1.014               | 0.000 | 1.014      | 1.843 | 1.014      | 1.844 | 1.014      | 6.844 | 1.014       | 1.828 |
| %accuracy   |          |                     |       | 99.707     |       | 99.707     |       | 100        |       | 99.971      |       |
| $\lambda_2$ | $\delta$ | DEWMA for ARX (2,3) |       |            |       |            |       |            |       |             |       |
|             |          | Explicit            |       | Midpoint   |       | Gaussian   |       | Simpson    |       | Trapezoidal |       |
|             |          | ARL (TIME)          |       | ARL (TIME) |       | ARL (TIME) |       | ARL (TIME) |       | ARL (TIME)  |       |
| 0.3         | 0        | 370.087             | 0.000 | 368.933    | 1.844 | 368.933    | 1.907 | 370.087    | 7.578 | 370.150     | 1.906 |
|             | 0.0001   | 222.148             | 0.000 | 222.089    | 1.860 | 222.089    | 2.000 | 222.148    | 7.329 | 222.119     | 1.813 |
|             | 0.0005   | 85.902              | 0.000 | 85.891     | 1.937 | 85.891     | 2.015 | 85.902     | 7.516 | 85.867      | 1.922 |
|             | 0.001    | 48.930              | 0.000 | 48.926     | 1.984 | 48.926     | 2.031 | 48.930     | 7.406 | 48.907      | 2.156 |
|             | 0.005    | 11.623              | 0.000 | 11.622     | 2.000 | 11.622     | 1.985 | 11.623     | 7.281 | 11.617      | 1.906 |
|             | 0.01     | 6.341               | 0.000 | 6.341      | 1.875 | 6.341      | 1.937 | 6.341      | 7.219 | 6.338       | 1.891 |
|             | 0.05     | 2.006               | 0.000 | 2.006      | 1.938 | 2.006      | 1.969 | 2.006      | 7.391 | 2.006       | 1.875 |
|             | 0.10     | 1.461               | 0.000 | 1.461      | 1.859 | 1.461      | 1.953 | 1.461      | 7.594 | 1.461       | 2.000 |
|             | 0.50     | 1.047               | 0.000 | 1.047      | 1.922 | 1.047      | 1.938 | 1.047      | 7.438 | 1.047       | 1.890 |
|             | 1.00     | 1.010               | 0.000 | 1.010      | 1.860 | 1.010      | 1.906 | 1.010      | 7.453 | 1.010       | 1.860 |
| %accuracy   |          |                     |       | 99.688     |       | 99.688     |       | 100        |       | 99.982      |       |

The CPU times are provided in parentheses (unit: second)

Source: created by the authors

Table 3. ARL of explicit formulas and NIE methods of ARL for the ARX(3,2) model with  $\lambda_1 = 0.05$ ,  $\phi_0 = 1$ ,  $\phi_1 = 0.1$ ,  $\phi_2 = 0.2$ ,  $\phi_3 = 0.1$ ,  $\alpha_1 = 1$  and  $\alpha_2 = 1$

|             |          | DEWMA for ARX (3,2) |       |            |       |            |       |            |        |             |       |
|-------------|----------|---------------------|-------|------------|-------|------------|-------|------------|--------|-------------|-------|
| $\lambda_2$ | $\delta$ | Explicit            |       | Midpoint   |       | Gaussian   |       | Simpson    |        | Trapezoidal |       |
|             |          | ARL                 | TIME  | ARL        | TIME  | ARL        | TIME  | ARL        | TIME   | ARL         | TIME  |
| 0.1         | 0        | 370.009             | 0.000 | 369.008    | 1.812 | 369.008    | 1.797 | 370.009    | 7.218  | 369.660     | 1.844 |
|             | 0.0001   | 361.196             | 0.000 | 361.196    | 1.813 | 361.196    | 1.985 | 361.196    | 7.219  | 360.856     | 1.922 |
|             | 0.0005   | 329.786             | 0.000 | 329.786    | 1.859 | 329.786    | 2.000 | 329.786    | 7.453  | 329.476     | 1.922 |
|             | 0.001    | 297.465             | 0.000 | 297.465    | 1.907 | 297.465    | 2.250 | 297.465    | 8.219  | 297.185     | 1.906 |
|             | 0.005    | 166.898             | 0.000 | 166.898    | 1.921 | 166.898    | 1.907 | 166.898    | 8.110  | 166.742     | 1.953 |
|             | 0.01     | 107.940             | 0.000 | 107.940    | 1.860 | 107.940    | 1.890 | 107.940    | 7.812  | 107.839     | 2.000 |
|             | 0.05     | 28.708              | 0.000 | 28.708     | 1.828 | 28.708     | 2.078 | 28.708     | 7.485  | 28.682      | 2.109 |
|             | 0.10     | 15.341              | 0.000 | 15.341     | 1.906 | 15.341     | 2.032 | 15.341     | 7.687  | 15.328      | 2.016 |
|             | 0.50     | 4.005               | 0.000 | 4.005      | 1.984 | 4.005      | 1.937 | 4.005      | 7.266  | 4.002       | 1.938 |
|             | 1.00     | 2.572               | 0.000 | 2.572      | 1.890 | 2.572      | 1.922 | 2.572      | 7.312  | 2.570       | 1.985 |
| %accuracy   |          |                     |       | 99.729     |       | 99.729     |       | 100        |        | 99.905      |       |
|             |          | DEWMA for ARX (3,2) |       |            |       |            |       |            |        |             |       |
| $\lambda_2$ | $\delta$ | Explicit            |       | Midpoint   |       | Gaussian   |       | Simpson    |        | Trapezoidal |       |
|             |          | ARL                 | TIME  | ARL        | TIME  | ARL        | TIME  | ARL        | TIME   | ARL         | TIME  |
| 0.2         | 0        | 370.114             | 0.000 | 369.111    | 1.907 | 369.111    | 1.813 | 370.114    | 6.813  | 369.782     | 2.031 |
|             | 0.0001   | 359.989             | 0.000 | 359.986    | 1.984 | 359.986    | 1.875 | 359.989    | 7.031  | 359.666     | 2.000 |
|             | 0.0005   | 324.446             | 0.000 | 324.444    | 1.953 | 324.444    | 1.860 | 324.446    | 6.845  | 324.155     | 1.859 |
|             | 0.001    | 288.738             | 0.000 | 288.736    | 1.906 | 288.736    | 1.875 | 288.738    | 6.844  | 288.479     | 2.063 |
|             | 0.005    | 152.819             | 0.000 | 152.818    | 1.938 | 152.818    | 1.969 | 152.819    | 6.781  | 152.682     | 1.890 |
|             | 0.01     | 95.506              | 0.000 | 95.506     | 1.922 | 95.506     | 1.938 | 95.506     | 6.828  | 95.421      | 1.954 |
|             | 0.05     | 22.362              | 0.000 | 22.362     | 1.890 | 22.362     | 1.906 | 22.362     | 6.782  | 22.343      | 1.922 |
|             | 0.10     | 10.698              | 0.000 | 10.698     | 1.891 | 10.698     | 1.985 | 10.698     | 6.766  | 10.689      | 1.953 |
|             | 0.50     | 1.928               | 0.000 | 1.928      | 1.891 | 1.928      | 1.984 | 1.928      | 6.766  | 1.927       | 1.984 |
|             | 1.00     | 1.240               | 0.000 | 1.240      | 1.875 | 1.240      | 1.922 | 1.240      | 6.844  | 1.239       | 2.015 |
| %accuracy   |          |                     |       | 99.729     |       | 99.729     |       | 100        |        | 99.910      |       |
|             |          | DEWMA for ARX (3,2) |       |            |       |            |       |            |        |             |       |
| $\lambda_2$ | $\delta$ | Explicit            |       | Midpoint   |       | Gaussian   |       | Simpson    |        | Trapezoidal |       |
|             |          | ARL (TIME)          |       | ARL (TIME) |       | ARL (TIME) |       | ARL (TIME) |        | ARL (TIME)  |       |
| 0.3         | 0        | 370.006             | 0.000 | 369.001    | 3.719 | 369.001    | 3.047 | 370.006    | 10.282 | 370.150     | 2.703 |
|             | 0.0001   | 349.662             | 0.000 | 349.658    | 3.157 | 349.658    | 3.328 | 349.662    | 10.296 | 222.119     | 2.703 |
|             | 0.0005   | 286.608             | 0.000 | 286.605    | 3.171 | 286.605    | 3.296 | 286.608    | 10.282 | 85.867      | 2.860 |
|             | 0.001    | 233.863             | 0.000 | 233.861    | 3.079 | 233.861    | 3.234 | 233.863    | 10.406 | 48.907      | 2.578 |
|             | 0.005    | 94.434              | 0.000 | 94.433     | 3.125 | 94.433     | 3.375 | 94.434     | 10.266 | 11.617      | 2.657 |
|             | 0.01     | 53.985              | 0.000 | 53.985     | 3.016 | 53.985     | 3.204 | 53.985     | 10.578 | 6.338       | 2.640 |
|             | 0.05     | 12.017              | 0.000 | 12.017     | 3.125 | 12.017     | 3.297 | 12.017     | 10.687 | 2.006       | 2.563 |
|             | 0.1      | 6.061               | 0.000 | 6.061      | 3.047 | 6.061      | 3.109 | 6.061      | 10.219 | 1.461       | 2.578 |
|             | 0.5      | 1.529               | 0.000 | 1.529      | 3.047 | 1.529      | 3.046 | 1.529      | 10.203 | 1.047       | 2.577 |
|             | 1        | 1.139               | 0.000 | 1.139      | 3.031 | 1.139      | 3.375 | 1.139      | 10.546 | 1.010       | 2.625 |
| %accuracy   |          |                     |       | 99.728     |       | 99.728     |       | 100        |        | 99.961      |       |

The CPU times are provided in parentheses (unit: second)

Source: created by the authors

Table 4. The ARL of DEWMA control chart using explicit formula against EWMA control chart for the ARX(1,1) model with  $\lambda_1 = 0.05$ ,  $\lambda_2 = 0.1$ ,  $\phi_0 = 0.1$ ,  $\phi_1 = 0.2$ , and  $\alpha_1 = 1$

| $\delta$    | ARL = 370           |                      | ARL = 500              |                    |
|-------------|---------------------|----------------------|------------------------|--------------------|
|             | DEWMA               | EWMA                 | DEWMA                  | EWMA               |
|             | <b>b=0.00417182</b> | <b>h=0.105992317</b> | <b>b=0. 0.00417389</b> | <b>h=0.1060268</b> |
| 0           | 370.518             | 370.126              | 500.039                | 500.002            |
| 0.0001      | 348.613             | 362.576              | 460.907                | 486.295            |
| 0.0005      | 281.994             | 335.227              | 351.109                | 438.242            |
| 0.0007      | 257.427             | 323.043              | 313.780                | 417.610            |
| 0.001       | 227.702             | 306.344              | 270.660                | 390.065            |
| 0.005       | 90.131              | 181.376              | 96.098                 | 207.581            |
| 0.007       | 69.413              | 150.664              | 72.878                 | 168.249            |
| 0.01        | 51.747              | 120.162              | 53.632                 | 131.030            |
| 0.05        | 12.539              | 32.664               | 12.635                 | 33.369             |
| 0.07        | 9.364               | 24.043               | 9.415                  | 24.410             |
| 0.10        | 6.943               | 17.307               | 6.969                  | 17.487             |
| 0.50        | 2.306               | 4.287                | 2.307                  | 4.293              |
| 0.70        | 1.959               | 3.371                | 1.960                  | 3.374              |
| 1.00        | 1.693               | 2.546                | 1.693                  | 2.548              |
| <b>EARL</b> | 123.739             | 159.552              | 153.863                | 201.753            |
| <b>AEQL</b> | 0.241               | 0.404                | 0.242                  | 0.405              |
| <b>PCI</b>  | 1                   | 1.672                | 1                      | 1.675              |

Source: created by the authors

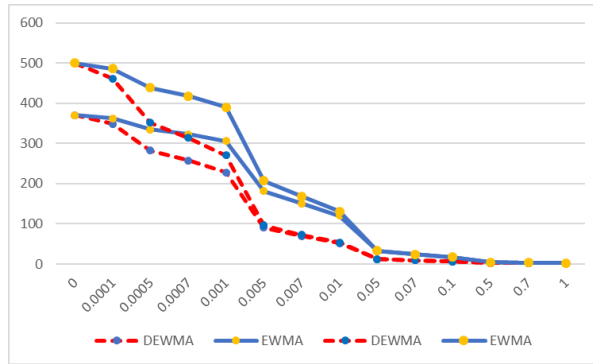


Fig. 1: ARL of DEWMA and EWMA control charts for ARX (1,1) model with  $\lambda_1 = 0.05$ ,  $\lambda_2 = 0.1$ ,  $\phi_0 = 0.1$ ,  $\phi_1 = 0.2$ , and  $\alpha_1 = 1$ ,  
Source: created by the authors

Table 5. The ARL of DEWMA control chart using explicit formula against EWMA control chart for the ARX(2,1) model with  $\lambda_1 = 0.05$ ,  $\lambda_2 = 0.1$ ,  $\phi_0 = 1$ ,  $\phi_1 = 0.2$ ,  $\phi_2 = 0.1$  and  $\alpha_1 = 1$

| $\delta$    | ARL = 370             |                        | ARL = 500           |                    |
|-------------|-----------------------|------------------------|---------------------|--------------------|
|             | DEWMA                 | EWMA                   | DEWMA               | EWMA               |
|             | <b>b=0.0037669925</b> | <b>h=0.09539992317</b> | <b>b=0.00376893</b> | <b>h=0.0954329</b> |
| 0           | 370.207               | 370.41                 | 500.325             | 500.296            |
| 0.0001      | 347.288               | 360.403                | 459.311             | 482.18             |
| 0.0005      | 278.414               | 325.268                | 345.981             | 421.198            |
| 0.0007      | 253.325               | 310.156                | 308.026             | 421.198            |
| 0.001       | 223.188               | 289.958                | 264.541             | 363.735            |
| 0.005       | 86.767                | 155.439                | 92.311              | 174.237            |
| 0.007       | 66.647                | 126.283                | 69.85               | 138.364            |
| 0.01        | 49.577                | 98.6382                | 51.312              | 105.813            |
| 0.05        | 11.97                 | 25.8083                | 12.058              | 26.2401            |
| 0.07        | 8.942                 | 19.0895                | 8.988               | 19.3163            |
| 0.10        | 6.635                 | 13.8803                | 6.659               | 13.9937            |
| 0.50        | 2.223                 | 3.77426                | 2.224               | 3.77915            |
| 0.70        | 1.895                 | 3.03277                | 1.896               | 3.0354             |
| 1.00        | 1.643                 | 2.46992                | 1.644               | 2.47133            |
| <b>EARL</b> | 122.051               | 150.329                | 151.794             | 191.132            |
| <b>AEQL</b> | 0.234                 | 0.372                  | 0.234               | 0.373              |
| <b>PCI</b>  | 1                     | 1.591                  | 1                   | 1.592              |

Source: created by the authors

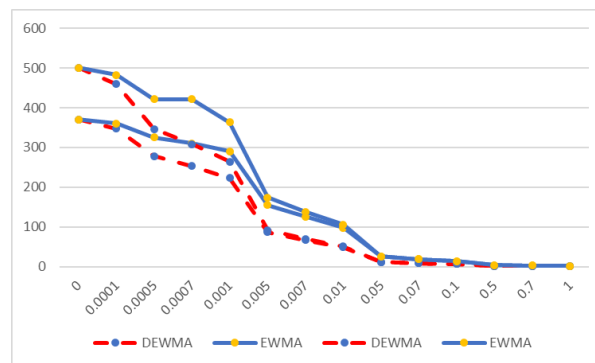


Fig. 2: ARL of DEWMA and EWMA control charts for ARX (2,1) model with  $\lambda_1 = 0.05$ ,  $\lambda_2 = 0.1$ ,  $\phi_0 = 0.1$ ,  $\phi_1 = 0.15$  and  $\alpha_1 = 1$

Source: created by the authors

Table 6. ARL comparison for the ARX(1,1) model on the DEWMA and EWMA control charts for data Natural Gas data with  $\lambda_1 = 0.05$ ,  $\lambda_2 = 0.1$ ,  $\phi_1 = 0.847$ , and  $\alpha_1 = 0.062$

| $\delta$    | ARL = 370           |                     | ARL = 500           |                    |
|-------------|---------------------|---------------------|---------------------|--------------------|
|             | DEWMA               | EWMA                | DEWMA               | EWMA               |
|             | <b>b=0.00152354</b> | <b>h=0.02081768</b> | <b>b=.001524326</b> | <b>h=0.0208274</b> |
| 0           | 370.182             | 370.087             | 500.047             | 500.349            |
| 0.0001      | 318.626             | 325.491             | 410.268             | 422.064            |
| 0.0005      | 204.813             | 219.795             | 239.035             | 259.832            |
| 0.0007      | 173.856             | 189.168             | 197.867             | 218.043            |
| 0.001       | 141.79              | 156.526             | 157.327             | 175.745            |
| 0.005       | 41.615              | 48.080              | 42.821              | 49.705             |
| 0.007       | 30.988              | 35.960              | 31.642              | 36.849             |
| 0.01        | 22.555              | 26.254              | 22.892              | 26.715             |
| 0.05        | 5.627               | 6.526               | 5.644               | 6.548              |
| 0.07        | 4.351               | 5.024               | 4.360               | 5.036              |
| 0.10        | 3.385               | 3.884               | 3.390               | 3.891              |
| 0.50        | 1.531               | 1.675               | 1.531               | 1.676              |
| 0.70        | 1.388               | 1.500               | 1.389               | 1.500              |
| 1.00        | 1.278               | 1.363               | 1.279               | 1.363              |
| <b>EARL</b> | 94.427              | 99.381              | 115.678             | 122.094            |
| <b>AEQL</b> | 0.081               | 0.088               | 0.081               | 0.088              |
| <b>PCI</b>  | 1                   | 1.089               | 1                   | 1.089              |

Source: created by the authors

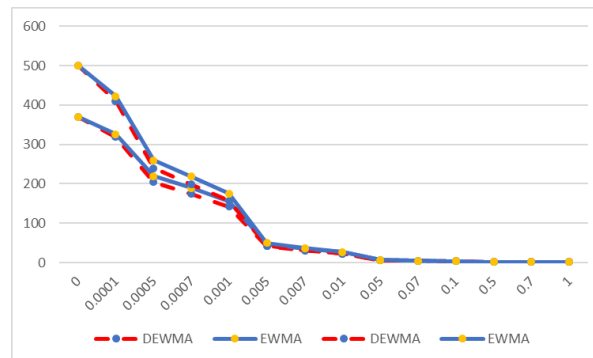


Fig. 3: ARL of DEWMA and EWMA control charts for ARX(1,1) model with  $\lambda_1 = 0.05$ ,  $\lambda_2 = 0.1$ ,  $\phi_1 = 0.847$  and  $\alpha_1 = 0.062$  for Natural Gas data

Source: created by the authors

Table 7. The ARX (1,1) model parameters and model fit for NaturalGAs and Exponential white noise Testing

| Model    | Estimation    | Coefficient | t                    | Sig.  | Model fit |       |
|----------|---------------|-------------|----------------------|-------|-----------|-------|
|          |               |             |                      |       | MAPE      | RMSE  |
| ARX(1,1) | $\phi_1$      | 0.847       | 12.133               | 0.000 | 8.181     | 0.601 |
|          | Crude Oil WTI |             |                      |       |           |       |
|          | $(\alpha_1)$  | 0.062       | 8.371                | 0.000 |           |       |
|          |               | Mean        | Kolmogorov-smirnov Z |       | Sig.      |       |
| residual |               | .41059      |                      | .688  | .732      |       |

Source: created by the authors