Numerical Integral Equation Approaches for the ARL and ATS for the Cumulative Sum Control Chart with Seasonal Time-Series Models

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Abstract: - In this study, the performance of a proposed CUSUM control chart was evaluated using the average run length (ARL) and average time to signal (ATS) First, the ARL was approximated using four numerical integral equation (NIE) methods based on the Gauss–Legendre quadrature, Midpoint, Trapezoidal, and Simpson's rules to establish a unified computational framework for assessing CUSUM control chart behavior under diverse process conditions. Primary emphasis was placed on the ARL performance associated with rapid detections of process mean shifts, while ATS was used to characterize the temporal efficiency of signaling the out-of-control ARL state. Both ARL and ATS evaluations were conducted for the CUSUM control chart with a FISMAX(D, Q, r)$_s$ model, a long-memory process with exponential white noise pertinent to real-world processes. The numerical results reveal notable differences among the NIE-based methods. Simpson's rule provided the smallest ARL and expected ARL values, followed by the Midpoint, Trapezoidal, and Gauss–Legendre methods. In contrast, when computational efficiency was considered, the Midpoint method consistently produced the lowest ATS and expected ATS values, with Gauss–Legendre, Trapezoidal, and Simpson's rule requiring progressively more computational time. These findings highlight trade-offs between approximation accuracy and computational speed. Considering both detection speed and computational efficiency, the Midpoint method emerged as the most appropriate and practically effective option for the scenarios investigated in this study.

Key-Words: - Approximated ARL, numerical integral equation (NIE), average run length (ARL), average time to signal (ATS), long-memory FISMAX(D, Q, r)$_s$ process, exponential white noise.

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1 Introduction

Control charts are often used for monitoring processes in various industries. They are designed to promptly trigger an out-of-control signal when the process becomes unstable based on a change in a process parameter, using the mean. Several control charts for monitoring process parameters have been proposed, including the Shewhart [1], cumulative sum (CUSUM), and exponentially weighted moving-average (EWMA), [2]. They are classified as either memoryless or memory-type depending on whether historical data are involved. Although the Shewhart control chart is memoryless as it solely depends on current data and is adept at identifying significant shifts in a process parameter, it is relatively insensitive to minor or moderate variations. In contrast, the CUSUM and EWMA control charts are classified as memory-type, [3], [4], employing both current and historic data to rapidly detect small and moderate shifts in process parameters, which has led to their widespread adoption in the processing and services industries. Specifically, the CUSUM control chart has been used for manufacturing (e.g., precision machining, electronics), healthcare (e.g., monitoring surgical outcomes), pharmaceuticals (e.g., ensuring drug quality), and finance (e.g., identifying shifts in time-series data), [5]. Consequently, we evaluated the efficacy of the CUSUM control chart in terms of detecting process parameter shifts and performance monitoring.

The average run length (ARL), a key metric used to assess the performance of a control chart, is the average number of observations taken before an out-of-control signal is detected. It comprises two primary components. The in-control ARL (ARL_0), representing stable process operation, is set sufficiently high to avoid false alarms. In the CUSUM control chart context, this is configured by modifying the threshold parameter. Conversely, the out-of-control ARL (ARL_1) represents when the process parameter deviates from the standard value.

In conclusion, a highly sensitive chart is configured using a sufficiently large ARL_0 to reduce false alarms (e.g., 370) and the smallest ARL_1 possible to detect changes in the process parameter signaling that the process is out of control, [6].

Various methods to approximate the ARL or the run-length distribution on CUSUM charts for detecting a shift in the mean of a process where the observations follow a normal distribution have been reported. An approximation of the ARL for the CUSUM chart with no head start was obtained by solving two integral equations (IEs), [7]. This approach was extended for the same scenario with a head start using both Markov chain and IE methods, [8]. Subsequently, [9] geometrically approximated the run-length distribution for the CUSUM control chart. More recently, an algorithm based on the IE method was used to evaluate the run-length distribution for the CUSUM control chart that was both fast and accurate, [10].

There have also been similar investigations where the observations follow a non-normal distribution. Exact expressions for the ARL of the one-sided CUSUM chart where the observations follow an exponential distribution were provided by [11]. Meanwhile, the ARL for the CUSUM control chart where the observations follow a chi-square distribution was defined using the Gaussian–Legendre quadrature and its performance assessed in terms of computational efficiency by [12]. Furthermore, the Markov chain approach has been applied to compute the ARLs for EWMA and CUSUM control charts, [13], [14]. However, non-normal run length distributions for the CUSUM control chart have not been extensively studied, with notable exceptions including exponential and Erlang run length distributions for the one-sided CUSUM control chart by [15] and [16], respectively.

To determine the ARL for a process on the CUSUM control chart, we usually assume that the observations are independent and identically distributed (i.i.d.) and follow a normal distribution. However, in practice, process data can be autocorrelated or conform to a time-series model. In this context, stochastic analysis of autoregressive (AR) and moving-average (MA) time series models [17], [18] was extended by [19] to stationary ARMA models, which became essential for applied time-series research. Moreover, the advent of long-memory processes marked the beginning of a promising research trajectory for time-series analysts and econometricians, [20], [21]. A further extension of ARMA, AR fractionally integrate MA (ARFIMA), in which a fractional amount of differencing is employed, has been utilized in various areas, including energy pricing and demand, telecommunications, environmental science, and financial volatility, [22,], [23], [24].

Although the ARL for CUSUM control chart performance evaluation is relevant in these cases, the formulas used to calculate it are more complex. Nevertheless, researchers have developed ARL formulas for various time-series models, such as AR(1) [25], ARIMA(p, d, q) [26], and ARFIMA(p, d, q) [27], among others. In the present study, we examined seasonal ARFIMA (SARFIMA) models, in which the parameter values are not necessarily stationary over time. It has been applied to analyze various real-world phenomena with long-range dependence [28], such as river flows [29], money supply [30], income trends over time, inflation rates [31], quarterly gross national product, and shipping data [32].

ARFIMA models with incorporated exogenous variables (i.e., ARFIMAX) offer enhanced performance. [33] was the first to exploit an ARFIMAX model to estimate volatility within an investment portfolio. Meanwhile, [34] focused on forecasting daily price changes using both ARFIMAX and ARFIMAX-threshold autoregressive conditional heteroskedasticity (TARCH) models. White noise must be considered when modeling seasonal time series. Although it is typically Gaussian distributed, it can also be exponentially distributed, as has been utilized in numerous research studies, [35], [36]. As discussed, by disregarding the seasonal autoregressive component, this study focuses on the long-memory FISMAX(D, Q, r)$_s$ model, which is characterized by exponential white noise and long-memory behavior, for monitoring process mean shifts using the CUSUM control chart.

Many numerical IE (NIE) methods based on the midpoint, Gaussian, trapezoidal, and Simpson's quadrature rules have been applied to compute or approximate the ARL for the CUSUM control chart. These methods offer practical solutions for approximating the ARL, particularly when analytical expressions are intractable, and have been widely used in studies involving both short- and long-memory processes, as well as non-Gaussian white noise. Explicit formulas for the ARL for a lower-sided negative CUSUM control chart while assuming that the observations follow an exponential distribution were provided by [37]. The authors compared the accuracy and efficiency of these expressions against ARL approximations obtained through the midpoint rule-based NIE approach. Building on this, [38] approximated the ARL for an SARX(P,1)$_S$ process with exponential white noise on the CUSUM control chart. They provided four NIEs based on various quadrature rules and evaluated each in terms of computational efficiency and accuracy. Most recently, [39] applied the Gauss-Legendre quadrature rule-based NIE approach to approximate the ARL for an ARFIX model. This work marked a significant extension of the NIE approach to more complex time series structures. Moreover, by integrating both the sampling and computational requirements, the average time to signal (ATS) offers a well-rounded metric for assessing the effectiveness of control charts, particularly within multistage or time-dependent processes, [40]. In this study, ATS is employed to examine the monitoring capability of the CUSUM chart. Owing to its strong responsiveness to small and moderate shifts in the process mean, the CUSUM approach—used in conjunction with ATS—provides a rigorous basis for detecting such variations. Furthermore, ATS has long been recognized as a widely used performance metric across industrial applications and academic investigations alike, [41], [42].

Herein, we propose numerical integral equation NIE approaches based on the Gauss–Legendre quadrature, Midpoint, Trapezoidal, and Simpson's rules to approximate the average run time (ARL) of a long-memory FISMAX(D, Q, r)$_s$ model with exponential white noise on the CUSUM control chart. The approximated ARL methods were compared numerically, with a particular focus on their detection speed using the average time to signal (ATS) metric and computational efficiency.

2 The CUSUM Control Chart with FISMAX Processes

2.1 Long-Memory FISMAX Models

The FISMAX model is particularly well-suited for processes that exhibit both long-term dependence and seasonal patterns, which are frequently observed in industrial and economic scenarios. The inclusion of exogenous variables provides a flexible framework for modeling the influence of external factors, while the assumption of exponential white noise enables the model to capture non-Gaussian stochastic behavior that is often present in real-world data. By integrating fractional differencing, seasonal patterns, and exogenous variables, the proposed CUSUM control chart becomes an effective tool for detecting shifts in the mean of FISMAX processes. Let Y_t be a FISMAX process of order (D, Q, r), (i.e., FISMAX(D, Q, r)$_s$) defined as:

$$(1-B^s)^D Y_t - \sum_{i=1}^{r}\beta_i x_{it} = \Theta_s(B)\varepsilon_t, \qquad (1)$$

where ε_t is the exponential white noise ($\varepsilon_t \sim Exp(\alpha)$), s is the seasonal period (e.g. $s=12$ for monthly data with annual seasonality); B is a backshift operator; $BY_t = Y_{t-1}$.; seasonal fractional differencing parameter D exhibits long-memory behavior when $0<D<0.5$, (which is of particular interest in the present study; $x_{it}, i=1,2,..., r$ represents the exogenous variables at time t,; $\beta_i, i=1,2,..., r$ denotes corresponding coefficient variables; and $\Theta_s(B)$ are finite moving average (MA) polynomials of order Q represented by

$$\Theta_s(B) = 1-\Theta_1 B^s - ... - \Theta_Q B^{sQ} = (1-\sum_{j=1}^{Q}\Theta_j B^{sj}) \qquad (2)$$

Seasonal fractional differencing operator $(1-B^s)^D$ can be expanded using the binomial series as follows, [21]:

$$(1-B^s)^D = \sum_{k=0}^{\infty}\binom{D}{k}(-1)^k B^{sk},$$

where $\binom{D}{k} = \dfrac{D!}{k!(D-k)!}$, which can be represented in operator form as

$$(1-B^s)^D = 1 - DB^s + \frac{D(D-1)}{2!}B^{2s} - ..., \qquad (3)$$

A generalized long-memory FISMAX(D, Q, r)$_s$ model can be derived from simplified forms of Equations (1) and (3), as follows:

$$Y_t = \varepsilon_t - \sum_{j=1}^{Q}\Theta_j B^{sj} + \sum_{i=1}^{r}\beta_i x_{it} + DY_{t-s} - \frac{D(D-1)}{2!}Y_{t-2s} + ..., \qquad (4)$$

2.2 The one-sided CUSUM Control Chart

The memory-type CUSUM control chart is widely employed for detecting small-to-moderate shifts in the mean of a monitored process, [3]. In this study, we propose a version of the CUSUM control chart tailored specifically for long-memory FISMAX processes. The one-sided upper CUSUM statistic at time t can be expressed as

$$Z_t = \max\{0, Z_{t-1} + Y_t - \kappa\}, \qquad (5)$$

where κ is the reference parameter and Y_t represents the observed value of a long-memory FISMAX(D, Q, r)$_s$ process at time t. The initial value of the CUSUM control chart statistic is set as

$Z_0 = \psi$, where $0 \le \psi \le b,$ and b is the upper control limit (UCL) of the control chart. Decision interval b is computed to provide a predetermined in-control ARL (ARL_0) value.

In the framework for sequential detection, the upper-sided CUSUM stopping time with a prescribed threshold b is defined as:

$$\tau_b = \inf\left\{ t > 0;\, Z_t > b \right\}, \tag{6}$$

where τ_b represents the earliest time at which the CUSUM statistic exceeds b or the upper control limit (UCL). The CUSUM statistics ($Z_t,$) are plotted against time t along with b.

The control chart signals an upward shift in the process mean when $Z_t > b.$ This approach provides a robust mechanism for the early detection of upward shifts in the process mean, thus making it particularly effective for monitoring long-memory FISMAX processes.

The performance of the proposed CUSUM control chart was evaluated in terms of the ARL, defined as the expected number of observations before the control scheme issues its first signal. To assess detection capability, a simulation-based study was conducted. The CUSUM control chart was designed to detect only changes in the mean of a long-memory FISMAX processes. Out-of-control scenarios were simulated by introducing shifts of varying magnitudes in the long-memory FISMAX process parameters. Specifically, the explored out-of-control values of the exponential white noise parameter $(\alpha = \alpha_1)$, where $\alpha_1 = (1+\delta)\alpha_0$ consisted of upward shifts from its in-control value $(\alpha = \alpha_0)$. This evaluation framework ensures a reliable assessment of the proposed CUSUM control chart's sensitivity and effectiveness under various process conditions.

3 Methodology for Performance Evaluation

The performance of the proposed CUSUM control chart was evaluated using several key metrics. The ARL was first considered, defined as the number of observations before the control chart signals for the first time. To account for situations in which the shift magnitude is unknown, the expected ARL (EARL) was also employed to measure detection performance over a specified interval of possible mean shifts.

The ATS (the expected time required for the control chart to signal following a process mean shift of known magnitude) was utilized to assess computational efficiency, [41].

To ensure an accurate and comprehensive comparison of the proposed monitoring procedures under both known and unknown shift conditions, a simulation-based study was carried out following the protocol outlined below.

3.1 Approximation of ARL using NIE–based Approaches

The ARL for FISMAX(D, Q, r)$_s$ time-series models on a CUSUM control chart can be approximated using four NIE–based methods: Gauss–Legendre quadrature, Midpoint, Trapezoidal, and Simpson's rules.

Let $\mathrm{L}(\psi) = E_\infty(\tau_b) < \infty$ denote the ARL of the upper-sided CUSUM control chart associated with an initial charting value (ψ) for a long-memory FISMAX(D, Q, r)$_s$ model with exponential white noise. The control chart operates with the lower control limit fixed at 0 and the upper control limit defined as $b.$ For the in-control process, the CUSUM statistic satisfies $0 \le C_t \le b,$ where a C_t value exceeding b signals the out-of-control condition. Function $\mathrm{L}(\psi)$ defined over interval $0 \le \psi \le b,$ is obtained by solving a Fredholm integral equation of the second kind [43], which can be expressed as

$$\begin{aligned}\mathrm{L}(\psi) = 1 + \int_0^b \mathrm{L}(g)\, f\Bigg(& g + \kappa - \psi - (-\sum_{j=1}^{Q} \Theta_j B^{sj} + \sum_{i=1}^{r} \beta_i x_{it} \\ & + DY_{t-s} - \frac{D(D-1)}{2!} Y_{t-2s} + \ldots) \Bigg) dg + \mathrm{L}(0) F\Big(\kappa - \psi \\ & -(-\sum_{j=1}^{Q} \Theta_j B^{sj} + \sum_{i=1}^{r} \beta_i x_{it} + DY_{t-s} - \frac{D(D-1)}{2!} Y_{t-2s} + \ldots) \Big)\end{aligned} \tag{7}$$

where $f(\psi) = \alpha e^{-\alpha\psi},\ F(\psi) = 1 - e^{-\alpha\psi}.$

By applying the NIE-based approximation, the ARL derived using the NIE method with initial value $Z_0 = \psi$ can be rewritten as

$$\begin{aligned}\mathrm{L}(\varphi) = 1 + \alpha e^{\alpha\left(\varphi - K + (-\sum_{j=1}^{Q}\Theta_j B^{sj} + \sum_{i=1}^{r}\beta_i x_{it} + DY_{t-s} - \frac{D(D-1)}{2!}Y_{t-2s} + \ldots)\right)} . \int_0^b \mathrm{L}(g) e^{-\alpha g} dg \\ + \mathrm{L}(0)\, (1 - e^{-\alpha\left(K - \varphi - (-\sum_{j=1}^{Q}\Theta_j B^{sj} + \sum_{i=1}^{r}\beta_i x_{it} + DY_{t-s} - \frac{D(D-1)}{2!}Y_{t-2s} + \ldots)\right)}).\end{aligned} \tag{8}$$

The Quadrature rule can be used to approximate the integral via finite summation. To this end, the integral over interval $[0,b]$ is partitioned into m

subintervals; i.e., $0 \le \ell_1 \le \ell_2 \le .... \le \ell_m \le b.$ Thus, the integral equation can be approximated using the NIE-based formulation as follows:

$$\int_0^b W(u)\, f(u)du \approx \sum_{j=1}^{m} w_j\, f(\ell_j),\ j = 1,2,...,m, \quad (9)$$

where ℓ_j and w_j represent the selected quadrature nodes and their corresponding weights, respectively, and $W(u)$ denotes the weight function. When $W(u) = 1,$ the set of nodes $\{\ell_j\},$ and the corresponding constant weights $\{w_j\}$ are determined according to specific optimality criteria. The weight function is chosen such that a suitable class of polynomials can effectively approximate integrand $f(\ell)$ over the integration domain. The points and weights are selected to ensure that the Quadrature rule is exact for polynomials of the highest achievable degree for a chosen set of points. For $\hat{\mathrm{L}}(\psi)$ denoting the NIE-based approximation of the ARL (function $\mathrm{L}(\psi).$ in Equation (8)), we can transform the original integral equation into a system of linear algebraic equations as follows:

$$\begin{aligned}
\hat{\mathrm{L}}(\ell_i) = 1 + \sum_{j=1}^{m} w_j \hat{\mathrm{L}}(\ell_j)\, f\Bigg(& \ell_j + \kappa - \ell_i - (-\sum_{j=1}^{Q}\Theta_j B^{sj} + \sum_{i=1}^{r}\beta_i x_{it} \\
&+ DY_{t-s} - \frac{D(D-1)}{2!}Y_{t-2s} + ...)\Bigg) + \hat{\mathrm{L}}(\ell_1) F\Bigg(\kappa - \ell_i - (-\sum_{j=1}^{Q}\Theta_j B^{sj} \\
&+ \sum_{i=1}^{r}\beta_i x_{it} + DY_{t-s} - \frac{D(D-1)}{2!}Y_{t-2s} + ...)\Bigg) \quad (10)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\hat{\mathrm{L}}(\ell_1) = 1 + \sum_{j=2}^{m} w_j \hat{\mathrm{L}}(\ell_j)\, f\Bigg(& \ell_j + \kappa - \ell_1 - (-\sum_{j=1}^{Q}\Theta_j B^{sj} \\
&+ \sum_{i=1}^{r}\beta_i x_{it} + DY_{t-s} - \frac{D(D-1)}{2!}Y_{t-2s} + ...)\Bigg) \\
+ \hat{\mathrm{L}}(\ell_1)\Bigg[& F\Big(\kappa - \ell_1 - (-\sum_{j=1}^{Q}\Theta_j B^{sj} + \sum_{i=1}^{r}\beta_i x_{it} + DY_{t-s} \\
&- \frac{D(D-1)}{2!}Y_{t-2s} + ...) + w_1 f\Bigg(\kappa - (-\sum_{j=1}^{Q}\Theta_j B^{sj} \\
&+ \sum_{i=1}^{r}\beta_i x_{it} + DY_{t-s} - \frac{D(D-1)}{2!}Y_{t-2s} + ...)\Bigg)\Bigg]
\end{aligned}$$

$$\begin{aligned}
\hat{\mathrm{L}}(\ell_2) = 1 + \sum_{j=2}^{m} w_j \hat{\mathrm{L}}(\ell_j)\, f\Bigg(& \ell_j + \kappa - \ell_2 - (-\sum_{j=1}^{Q}\Theta_j B^{sj} \\
&+ \sum_{i=1}^{r}\beta_i x_{it} + DY_{t-s} - \frac{D(D-1)}{2!}Y_{t-2s} + ...)\Bigg) \\
+ \hat{\mathrm{L}}(\ell_1)\Bigg[& F\Big(\kappa - \ell_2 - (-\sum_{j=1}^{Q}\Theta_j B^{sj} + \sum_{i=1}^{r}\beta_i x_{it} + DY_{t-s} \\
&- \frac{D(D-1)}{2!}Y_{t-2s} + ...) + w_1 f\Bigg(\ell_1 + \kappa - \ell_2 - (-\sum_{j=1}^{Q}\Theta_j B^{sj} \\
&+ \sum_{i=1}^{r}\beta_i x_{it} + DY_{t-s} - \frac{D(D-1)}{2!}Y_{t-2s} + ...)\Bigg)\Bigg]
\end{aligned}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\begin{aligned}
\hat{\mathrm{L}}(\ell_m) = 1 + \sum_{j=2}^{m} w_j \hat{\mathrm{L}}(\ell_j)\, f\Bigg(& \ell_j + \kappa - \ell_m - (-\sum_{j=1}^{Q}\Theta_j B^{sj} \\
&+ \sum_{i=1}^{r}\beta_i x_{it} + DY_{t-s} - \frac{D(D-1)}{2!}Y_{t-2s} + ...)\Bigg) \\
+ \hat{\mathrm{L}}(\ell_1)\Bigg[& F\Big(\kappa - \ell_m - (-\sum_{j=1}^{Q}\Theta_j B^{sj} + \sum_{i=1}^{r}\beta_i x_{it} + DY_{t-s} \\
&- \frac{D(D-1)}{2!}Y_{t-2s} + ...) + w_1 f\Bigg(\ell_1 + \kappa - \ell_m - (-\sum_{j=1}^{Q}\Theta_j B^{sj} \\
&+ \sum_{i=1}^{r}\beta_i x_{it} + DY_{t-s} - \frac{D(D-1)}{2!}Y_{t-2s} + ...)\Bigg)\Bigg].
\end{aligned}$$

The equation can be rewritten in matrix form as:

$$\mathbf{L}_{m\times1} = \mathbf{1}_{m\times1} + \mathbf{K}_{m\times m}\mathbf{L}_{m\times1}, \quad (11)$$

where $\mathbf{1}_{m\times1} = [1,1,...,1]'$ is a vector of ones,

$\mathbf{L}_{m\times1} = \left[\hat{\mathrm{L}}(\ell_1), \hat{\mathrm{L}}(\ell_2), ..., \hat{\mathrm{L}}(\ell_m)\right]' \in \mathbb{R}^m$ is a vector of the approximated ARL values at the quadrature nodes, $\mathbf{K}_{m\times m}$ is the matrix containing the kernel evaluations weighted using the corresponding quadrature weights; i.e.,

$$\mathbf{K}_{m\times m} = \begin{bmatrix} w_1 f(\kappa - Y_t) & \cdots + w_m f(\psi_m + \kappa - \psi_1 - Y_t) + F(\kappa - \psi_1 - Y_t) \\ w_1 f(\psi_1 + \kappa - \psi_2 - Y_t) & \ldots + w_m f(\ell_m + \kappa - \psi_2 - Y_t) + F(\kappa - \psi_2 - Y_t) \\ \vdots & \vdots \qquad\qquad \vdots \\ w_1 f(\psi_1 + \kappa - \psi_m - Y_t) & \ldots + w_m f(\ell_m + \kappa - \psi_m - Y_t) + F(\kappa - \psi_m - Y_t) \end{bmatrix},$$

where

$$Y_t = -\sum_{j=1}^{Q}\Theta_j B^{sj} + \sum_{i=1}^{r}\beta_i x_{it} + DY_{t-s} - \frac{D(D-1)}{2!}Y_{t-2s} + ...$$

Solving for $\mathbf{L}_{m\times1}$ yields $\mathbf{L}_{m\times1} = (\mathbf{I}_m - \mathbf{R}_{m\times m})^{-1}\mathbf{1}_{m\times1},$ which provides the NIE-based approximation of the ARL.

By substituting $\ell_i = \psi$ into Equation (11), we obtain

$$\hat{\mathrm{L}}(\varphi) = 1 + \sum_{j=1}^{m} w_j \hat{\mathrm{L}}(\upsilon_j)\, f(\upsilon_j + K - \varphi - Y_t) + \hat{\mathrm{L}}(\upsilon_1) F(K - \varphi - Y_t). \quad (12)$$

Using Equation (12), the ARL can then be approximated by specifying the corresponding nodes and weights determined using a specific NIE method (namely, Gauss–Legendre quadrature, Midpoint, Trapezoidal, or Simpson's rule). Each provides distinct sets of nodes and weights, thereby allowing the integral equation to be evaluated numerically with varying degrees of accuracy and computational complexity.

3.1.1 The Gauss-Legendre Quadrature Rule

$\hat{L}_G(\psi)$ denoting the ARL approximation obtained using the NIE approach with the Gauss–Legendre quadrature rule is computed as

$$\begin{aligned}\hat{L}_G(\psi) &= 1+\sum_{j=1}^{m} w_j \hat{L}(\ell_j)\, f\Bigg(\ell_j+\kappa-\psi-(-\sum_{j=1}^{Q}\Theta_j B^{sj} \\ &\quad +\sum_{i=1}^{r}\beta_i x_{it}+DY_{t-s}-\frac{D(D-1)}{2!}Y_{t-2s}+...)\Bigg) \\ &\quad +\hat{L}(\ell_1)F\Bigg(\kappa-\psi-(-\sum_{j=1}^{Q}\Theta_j B^{sj}+\sum_{i=1}^{r}\beta_i x_{it}+DY_{t-s} \\ &\quad -\frac{D(D-1)}{2!}Y_{t-2s}+...)\Bigg), \qquad (13)\end{aligned}$$

where $\ell_j=\frac{b}{m}\left(j-\frac{1}{2}\right)$, $w_j=\frac{b}{m}$; $j=1,2,...,m.$

$u_m=b$

3.1.2 The Midpoint Rule

Let $f(u)$ be a function defined on interval $[0,b]$, which is divided into m, $\{[u_{j-1},u_j], j=1,2,...,m\}$ subintervals of equal width ($H=b/m.$). The subintervals are spaced equally $\{u_j=u_0+jH,\ j=0,1,2,...,m\}$, such that $z_0=0$ and $u_m=b.$ The Midpoint rule approximates the integral as

$$M(f,H)=H\sum_{j=1}^{m} f\left(\kappa+\left(j-\frac{1}{2}\right)H\right);\ j=0,1,2,...,m$$

The integral of $f(u)$ over interval $[0,b]$ can be approximated in the form:

$$\int_0^b f(u)du \approx M(f,H).$$

$\hat{L}_M(\psi)$ denoting the ARL approximation obtained using the NIE approach with the Midpoint rule is given by

$$\begin{aligned}\hat{L}_M(\psi) &= 1+\sum_{j=1}^{m} w_j \hat{L}(\ell_j)\, f\Bigg(\ell_j+\kappa-\psi-(-\sum_{j=1}^{Q}\Theta_j B^{sj} \\ &\quad +\sum_{i=1}^{r}\beta_i x_{it}+DY_{t-s}-\frac{D(D-1)}{2!}Y_{t-2s}+...)\Bigg) \\ &\quad +\hat{L}(\ell_1)F\Bigg(\kappa-\psi-(-\sum_{j=1}^{Q}\Theta_j B^{sj}+\sum_{i=1}^{r}\beta_i x_{it}+DY_{t-s} \\ &\quad -\frac{D(D-1)}{2!}Y_{t-2s}+...)\Bigg), \qquad (14)\end{aligned}$$

where $\ell_j=w_j\left(j-\frac{1}{2}\right)$, and $w_j=\frac{b}{m}$; $j=1,2,...,m.$

3.1.3 The Trapezoidal Rule

The Trapezoidal rule is defined as function $f(u)$ on interval $[0,b]$. It is divided into m subintervals, $\{[u_{j-1},u_j], j=1,2,...,m\}$, with width $H=b/m$, and spaced equally: $\{u_j=u_0+jH,\ j=0,1,2,...,m\}$. Therefore, given that $u_0=0$ and $u_m=b$, the composite Trapezoidal rule can be written as

$$T(f,H)=\frac{H}{2}\left(f(0)-f(b)\right)+H\sum_{j=1}^{m} f(u_j).$$

The integral of $f(u)$ from 0 to b can be approximated in the form $\int_0^b f(u)du \approx T(f,H).$

$\hat{L}_T(\psi)$ denoting the ARL approximation obtained via the NIE approach using the Trapezoidal rule is computed as

$$\begin{aligned}\hat{L}_T(\psi) &= 1+\sum_{j=1}^{m} w_j \hat{L}(\ell_j)\, f\Bigg(\ell_j+\kappa-\psi-(-\sum_{j=1}^{Q}\Theta_j B^{sj} \\ &\quad +\sum_{i=1}^{r}\beta_i x_{it}+DY_{t-s}-\frac{D(D-1)}{2!}Y_{t-2s}+...)\Bigg) \\ &\quad +\hat{L}(\ell_1)F\Bigg(\kappa-\psi-(-\sum_{j=1}^{Q}\Theta_j B^{sj}+\sum_{i=1}^{r}\beta_i x_{it}+DY_{t-s} \\ &\quad -\frac{D(D-1)}{2!}Y_{t-2s}+...)\Bigg), \qquad (15)\end{aligned}$$

where $\ell_j=jw_j$ and $w_j=\frac{b}{m}$; $j=1,2,...,m-1,$

or $w_j=b/2m.$

3.1.4 Simpson's Rule

Let $[0,b]$ be partitioned into $2m$ equal subintervals $\{[u_{j-1},u_j], j=1,2,...,2m\}$ with width $H=b/2m$ and spaced equally $\{u_j=u_0+jH,\ j=0,1,2,...,2m\}$. Therefore, given $u_0=0$ and $u_{2m}=b$, the composite Simpson's functional can be written as

$$\begin{aligned}S(f,H) &= \frac{H}{3}\left(f(0)-f(b)\right)+\frac{2H}{3}\sum_{j=1}^{m} f(u_{2j}) \\ &\quad +\frac{4H}{3}\sum_{j=1}^{m} f(u_{2j-1}),\end{aligned}$$

The integral of $f(u)$ from 0 to b is approximated in the form $\int_0^b f(u)du \approx S(f,H)$.

$\hat{\mathrm{L}}_S(\psi)$ denoting the ARL approximation using the NIE approach with Simpson's rule can be expressed as

$$\hat{\mathrm{L}}_S(\psi) = 1 + \sum_{j=1}^{m} w_j \hat{\mathrm{L}}(\ell_j)\, f\Bigg(\ell_j + \kappa - \psi - (-\sum_{j=1}^{Q}\Theta_j B^{sj} + \sum_{i=1}^{r}\beta_i x_{it} + DY_{t-s} - \frac{D(D-1)}{2!}Y_{t-2s} + ...)\Bigg) + \hat{\mathrm{L}}(\ell_1) F\Bigg(\kappa - \psi - (-\sum_{j=1}^{Q}\Theta_j B^{sj} + \sum_{i=1}^{r}\beta_i x_{it} + DY_{t-s} - \frac{D(D-1)}{2!}Y_{t-2s} + ...)\Bigg), \quad (16)$$

where

$\ell_j = jw_j$ and $w_j = \frac{4}{3}\left(\frac{b}{2m}\right);\; j = 1,3,...,2m-1,$

$w_j = \frac{2}{3}\left(\frac{b}{2m}\right);\; j = 2,4,...,2m-2,$

or $w_j = \frac{1}{3}\left(\frac{b}{2m}\right).$

3.2 Expected ARL (EARL)

This criterion can be used to evaluate the performances of the proposed methods when the magnitude of the process shift is unknown. Specifically, it can be employed to assess control chart performance over a continuous range of plausible shift magnitudes $\delta \in (\delta_{\min},\ \delta_{\max})$, where $\delta_{\min}$ and $\delta_{\max}$ denote the lower and upper bounds of the mean shift, respectively. Accordingly, the EARL for the CUSUM control chart over the interval $(\delta_{\min},\ \delta_{\max})$ is defined as

$$\mathrm{EARL}(\delta_{\min}, \delta_{\max}) = \int_{\delta_{\min}}^{\delta_{\max}} f_\delta(\delta).\mathrm{ARL}(\delta)d\delta, \quad (17)$$

where $f_\delta(\delta)$ as in Equation (17) denotes the probability density function (pdf) of shift magnitude δ and $\mathrm{ARL}(\delta)$ comprises the ARL approximations based on the four NIE methods.

In this study, shifts in the process mean are considered to occur with equal probability; i.e.,

$$f_\delta(\delta) = \frac{1}{\delta_{\max} - \delta_{\min}},\ \delta \in (\delta_{\min},\ \delta_{\max}).$$

Consequently, the monitoring scheme that yields the smallest EARL over this interval is regarded as providing the most desirable overall performance. Under this assumption, Equation (17) can be equivalently expressed as

$$\mathrm{EARL}(\delta_{\min}, \delta_{\max}) = \frac{1}{\delta_{\max} - \delta_{\min}} \sum_{\delta=\delta_{\min}}^{\delta_{\max}} \mathrm{ARL}(\delta), \quad (18)$$

In the present study, both the ARL and EARL criteria were employed to evaluate the performance of the CUSUM control chart when the process mean shift is known. For computational purposes, note that the lower limit of δ was excluded from the summation when $\delta_{\min} = 0$.

3.3 Average Time to Signal (ATS)

This is a key performance metric for control charts, representing the expected time for the control chart to signal the out-of-control condition following a process shift. It is unlike the ARL, which measures the number of samples until a signal occurs. Mathematically, ATS can be expressed as:

$$\mathrm{ATS} = \mathrm{ARL} \times h \quad (19)$$

where ARL in Equations (13)–(16) represents the approximated ARL computed using the four NIE-based methods, and represents the time interval between successive observations. A smaller ATS value indicates faster detection of process mean shifts, reflecting greater responsiveness of the control chart.

3.4 Expected ATS (EATS)

EATS, used to assess control chart performance over shift interval $(\delta_{\min},\ \delta_{\max})$, where $\delta_{\min}$ and $\delta_{\max}$ denote the lower and upper bounds of the mean shift magnitude, respectively, is defined as follows:

$$\mathrm{EATS}(\delta_{\min}, \delta_{\max}) = \int_{\delta_{\min}}^{\delta_{\max}} f_\delta(\delta).\mathrm{ATS}(\delta)d\delta, \quad (20)$$

where $f_\delta(\delta)$ in Equation (17) denotes the pdf of the shift magnitude δ and $\mathrm{ATS}(\delta)$ in Equation (19). It is assumed that all shifts in the process mean occur with equal probability; i.e., $f_\delta(\delta) = \frac{1}{\delta_{\max} - \delta_{\min}}$.

Under this assumption, Equation (17) can be equivalently expressed as

$$\mathrm{ETS}(\delta_{\min}, \delta_{\max}) = \frac{1}{\delta_{\max} - \delta_{\min}} \sum_{\delta=\delta_{\min}}^{\delta_{\max}} \mathrm{ATS}(\delta), \quad (21)$$

From Equation (19), ATS(0), and ATS(δ), represent the ATS when the process is in-control and out-of-control, respectively. Similarly, the in-control and out-of-control EATS are represented by EATS(0) and EATS($\delta_{\min}$, $\delta_{\max}$), respectively.

3.5 Simulation Design, Parameter Settings, and Implementation Procedures

The performance of the proposed CUSUM control chart was evaluated through the ARL, which is the primary metric for quantifying detection capability. For each configuration of the FISMAX(D, Q, r)$_s$ model, the ARL was approximated using the numerical procedures described in the preceding subsections.

To assess the control chart's ability to detect process deterioration, out-of-control scenarios were generated by introducing mean-shift magnitudes ranging from 0 to 1.6, with increments of 0.2. These settings mirror those adopted by [44], thereby ensuring methodological consistency and facilitating meaningful comparison with existing findings.

3.6 The Algorithm for Determining the In-Control and Out-Of-Control ARL Values for Long-Memory FISMAX Models with Exponential White Noise on the CUSUM Control Chart

Stage 1: Search for the optimal value of *b*

Step 1. Specify target ARL$_0$ = 370, and the number of division points $m = 500$ used in the numerical integration method.

Step 2. Solve the generalized long-memory FISMAX(D, Q, r)$_s$ model with exponential white noise on the CUSUM control chart.

- Set the initial values for the model: $Y_{t-s}, Y_{t-2s}, ..., Y_{t-sQ} = 1;\ x_{1t}, x_{2t}, ..., x_{rt} = 1.$
- Specify the coefficient values for the model:
  - $\text{FISMAX}(D,1,1)_s : \Theta_1 = 0.1,\ \beta_1 = 0.1$ or 0.4.
  - $\text{FISMAX}(D,2,1)_s : \Theta_1 = 0.1,\ \Theta_2 = 0.2,$ $\beta_1 = 0.1$ or 0.4. when $D = 1/3$, $1/5$ or $1/7$; $s = 12$.
- Specify the mean for the exponential parameter $(\alpha = \alpha_0)$, where $\alpha_0 = 1$, for an in-control mean shift size of $(\delta) = 0$.

Step 3. Compute the CUSUM statistic (Z_t) using specified reference parameter κ, set to 1.50, 1.75, or 2.00. This step produces the in-control ARL corresponding to each combination of κ and b.

Step 4. Design the structure of the upper-sided CUSUM control chart for long-memory FISMAX(D, Q, r)$_s$ processes with exponential white noise.

- Set the initial value of $(\psi) = 1$.

Step 5. Search for design parameter b, in conjunction with κ, computed using the four NIE approaches based on Equations (13)–(16) that most closely approximates the specified target ARL$_0$. Once this value has been identified, designate b as the optimal design parameter.

Step 6. Repeat Steps (2)–(5) for all scenarios under consideration until the corresponding optimal b values have been obtained.

The resulting optimal parameters were then used to compute the out-of-control ARL values for various mean-shift magnitudes

Stage 2: Compute the characteristics of the out-of-control ARL and ATS.

Step 7. Compute ARL$_1$ for each mean shift magnitude using Equations (13)–(16). Adjust the mean of the exponential parameter from its in-control value α_0 to $\alpha_1 = (1+\delta)\alpha_0$, where δ can be 0.2 , 0.4 , 0.6, 0.8, 1, 1.2, 1.4, or 1.6.

Step 8. Iteratively apply Steps (6)–(7) to all designated mean-shift magnitudes to calculate the corresponding ARL values. Record the computational time required to evaluate the ATS, as defined in Equation (19).

The Mathematica program was used to compute charting-parameter combinations (κ, b) for the CUSUM control chart under the four NIE-based methods. These combinations were selected to minimize the ARL, EARL, ATS, and EATS values.

4 Results and Discussion

We evaluated the performance of the proposed CUSUM control chart based on the ARL approximated using the four NIE-based methods introduced in this study. These methods provide a comprehensive assessment framework for quantifying the control chart performance under various process conditions. Primary emphasis was placed on the ARL associated with the fastest detection of a shift in the process mean, as rapid detection is a critical criterion for effective process monitoring. Complementing this analysis, the ATS values were also recorded to capture the efficiency of the control chart in signaling the out-of-control state

Both ARL and ATS were evaluated for long-memory FISMAX(D, Q, r)$_s$ models with exponential white noise on the proposed CUSUM control chart. The details of the simulation settings and the considered long-memory FISMAX(D, Q, r)$_s$ with exponential white noise scenarios are addressed below.

Table 1 (Appendix) provides design parameter b or UCL values for the upper-sided CUSUM control chart for monitoring the process mean. The values of b are determined for different values of κ (1.5, 1.75, or 2) under the in-control ARL for long-memory FISMAX(D, Q, r)$_s$ processes with exponential white noise. The results indicate a decrease in design parameter b values for the FISMAX(D, Q, r)$_s$ processes in several scenarios. Specifically, as the reference value k was increased from 1.5 to 2, the corresponding b values decreased progressively across all scenarios.

The out-of-control ARL (ARL_1) values computed using the four NIE-based methods to evaluate the performance of the proposed CUSUM control chart when process mean shifts occurred due to changes in the mean parameter are reported for the long-memory FISMAX(D, 1, 1)$_{12}$ model in Table 2 (Appendix) and the long-memory FISMAX(D, 2, 1)$_{12}$ model in Table 3 (Appendix) with varying D values $(1/3, 1/5, 1/7)$. Specifically, variations in b were considered in conjunction with different reference κ values under the out-of-control state, with κ ranging from 1.5 to 2.0 in increments of 0.25. For small mean-shift magnitudes $(0 < \delta \le 0.2)$, the ARL_1 values decreased rapidly, whereas for larger shifts $(0.2 < \delta \le 1.6)$, the ARL_1 values changed only slightly, and for very large shift values $(1.2 < \delta \le 1.6)$, the ARL_1 values remained nearly constant.

When comparing the four NIE-based approximation methods, Simpson's rule consistently yielded the smallest ARL_1 values across all levels of process mean shifts. The Midpoint, Trapezoidal, and Gauss–Legendre methods followed in order of increasing ARL_1 when $\kappa = 1.5$. As κ was increased to 1.75 and 2.0, the same ranking was maintained, with only slight improvements in detection. These patterns were observed consistently across all scenarios examined, highlighting the relative effectiveness of Simpson's rule for rapid detection while also demonstrating only minor differences in performance among the alternative NIE methods.

When assessing computational efficiency through the ATS_1 values reported in Table 4 and Table 5, in Appendix, for simulations using the long-memory FISMAX(D, Q, r)$_s$ models, the Midpoint method demonstrated clear superiority over the other NIE-based approaches by yielding the lowest ATS_1 values across all scenarios. The Trapezoidal, Gauss–Legendre, and Simpson's rule methods followed in ascending order of computational time. Closer examination of the trade-off between approximation accuracy and computational speed further reinforces the conclusion that the Midpoint method consistently produced the smallest ATS_1 values under every set of experimental conditions.

Reference parameter κ plays an important role in detection efficiency. When $\kappa = 1.5$, the ATS_1 values were the lowest for all levels of process mean shifts, with slightly higher values observed for $\kappa = 1.75$ and $\kappa = 2.00$. This pattern remained consistent across all scenarios considered (Table 4 and Table 5, Appendix). Furthermore, the reference parameter κ = 1.5 produced results that are consistent with the optimal reference parameter settings for ARL detection efficiency reported in [45].

When considering EARL and EATS together, consistent patterns emerge across all shift magnitudes. For EARL, the Simpson's rule, Midpoint, and Trapezoidal methods yielded similar values, indicating comparable accuracy in approximating the ARL. However, once computational time was taken into account through the EATS metric, clear differences arose: the Midpoint method consistently produced the smallest EATS values, followed by the Trapezoidal and Gauss–Legendre methods, which performed similarly. In contrast, Simpson's rule yielded the largest EATS across all levels of process mean shifts. These results confirm that although the methods provide similar EARL accuracy, the Midpoint method offers superior computational efficiency.

Based on the ARL, ATS, EARL, and EATS results for detecting mean shifts in the long-memory FISMAX(D, Q, r)$_s$ processes, the Midpoint method emerged as the most appropriate and effective approach in this study.

5 Summary and Conclusions

The performance of the proposed CUSUM control chart was rigorously assessed using the ARL, which was approximated through four NIE-based methods introduced in this study. Collectively, these methods provide a comprehensive and accurate framework for quantifying control chart performance under diverse process conditions for long-memory FISMAX(D, Q, r)$_s$ models with exponential white noise. Particular emphasis was placed on ARL measures associated with the rapid detection of process mean shifts, as timely signaling is a central requirement in statistical process control. To further complement the analysis, the ATS for all four methods was evaluated, offering additional insight into the operational efficiency of the CUSUM control chart in identifying the out-of-control state. The numerical findings reveal clear distinctions among the four NIE-based ARL approximation methods. Simpson's rule yielded the smallest ARL and EARL values, thereby reflecting superior approximation precision. However, when computational efficiency, characterized by ATS and EATS, was considered, the Midpoint method outperformed the others by delivering the lowest values for both metrics.

When both detection performance and computational efficiency were jointly evaluated, the Midpoint method emerged as the most effective and practically advantageous approach for implementing the proposed CUSUM control chart for monitoring long-memory FISMAX(D, Q, r)$_s$ models with exponential white noise.

Overall, the findings provide valuable methodological guidance for practitioners in selecting suitable numerical techniques for ARL and ATS approximation and demonstrate the robustness and applicability of the proposed CUSUM control chart design in realistic process-monitoring scenarios. Although the present study focused on detecting unilateral shifts in the process mean, the framework could be readily adapted to detect bilateral shifts with minimal conceptual or computational modifications. Furthermore, the numerical strategy introduced in this research could be integrated with other control charts.

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Conflict of Interest

The authors declare no conflict of interest.

APPENDIX

Table 1. The optimized design parameters for b utilized to compute ARL$_0$ = 370 using the NIE approaches

	Long-memory FISMAX(D,1,1)$_{12}$ process for $\Theta_1 = 0.1$, $\omega_1 = 0.1$			Long-memory FISMAX(D,2,1)$_{12}$ process for $\Theta_1 = 0.1, \Theta_2 = 0.2$, $\omega_1 = 0.4$,		
	D = 1/3	D = 1/5	D = 1/7	D = 1/3	D = 1/5	D = 1/7
1.50	4.590187	4.202904	4.046396	4.154010	3.859068	3.728957
1.75	4.063893	3.782408	3.656657	3.743754	3.499209	3.386144
2.00	3.670998	3.432924	3.322192	3.399139	3.180510	3.076990

Source: Calculations conducted by the author using Wolfram Mathematica 8.

Table 2. ARL values for the CUSUM control chart with a long-memory FISMAX(D,1,1)$_{12}$ process for $\Theta_1 = 0.1$, ω_1 = 0.1, and ARL$_0$ = 370

FISMAX(D,1,1)$_{12}$	K	Rules	δ							
			0.2	0.4	0.6	0.8	1	1.2	1.4	1.6
1/3	1.50	Gauss	160.902	57.961	29.736	18.565	13.097	10.016	8.097	6.810
		Trapezoidal	128.013	52.712	28.677	18.517	13.354	10.371	8.478	7.189
		Simpson's	**103.784**	**47.497**	**26.995**	**17.809**	**12.998**	**10.167**	**8.350**	**7.104**
		Midpoint	111.508	49.029	27.456	17.996	13.090	10.220	8.384	7.127
	1.75	Gauss	137.274	58.560	31.888	20.285	14.328	10.890	8.724	7.268
		Trapezoidal	127.536	56.854	31.783	20.596	14.752	11.334	9.159	7.682
		Simpson's	**114.075**	**53.551**	**30.638**	**20.094**	**14.494**	**11.185**	**9.064**	**7.618**
		Midpoint	120.576	54.993	31.098	20.284	14.587	11.237	9.097	7.640
	2.00	Gauss	132.837	59.974	33.471	21.492	15.205	11.530	9.200	7.628
		Trapezoidal	129.180	59.589	33.788	21.975	15.714	12.021	9.663	8.061
		Simpson's	**120.434**	**57.282**	**32.950**	**21.597**	**15.515**	**11.904**	**9.588**	**8.010**
		Midpoint	125.641	58.523	33.363	21.771	15.601	11.953	9.618	8.030
1/5	1.50	Gauss	140.645	58.206	31.318	19.841	14.008	10.659	8.556	7.143
		Trapezoidal	127.251	55.813	31.011	20.072	14.394	11.083	8.978	7.549
		Simpson's	**111.562**	**52.077**	**29.739**	**19.521**	**14.112**	**10.920**	**8.876**	**7.480**
		Midpoint	118.464	53.566	30.207	19.712	14.206	10.973	8.909	7.502
	1.75	Gauss	133.551	59.530	33.029	21.158	14.962	11.351	9.066	7.526
		Trapezoidal	128.627	58.846	33.246	21.600	15.450	11.831	9.522	7.954
		Simpson's	**118.742**	**56.288**	**32.329**	**21.190**	**15.235**	**11.705**	**9.442**	**7.899**
		Midpoint	124.331	57.595	32.759	21.369	15.324	11.755	9.473	7.920
	2.00	Gauss	132.201	60.976	34.383	22.174	15.704	11.900	9.480	7.844
		Trapezoidal	130.472	61.079	34.870	22.731	16.250	12.413	9.956	8.285
		Simpson's	**123.750**	**59.237**	**34.183**	**22.416**	**16.082**	**12.313**	**9.892**	**8.241**
		Midpoint	128.137	60.326	34.556	22.575	16.162	12.358	9.920	8.259
1/7	1.50	Gauss	136.937	58.612	31.959	20.341	14.368	10.918	8.746	7.284
		Trapezoidal	127.586	56.982	31.878	20.660	14.797	11.366	9.182	7.699
		Simpson's	**114.381**	**53.731**	**30.748**	**20.165**	**14.541**	**11.218**	**9.088**	**7.636**
		Midpoint	120.830	55.166	31.207	20.354	14.635	11.270	9.120	7.657
	1.75	Gauss	132.768	60.033	33.527	21.534	15.236	11.552	9.217	7.642
		Trapezoidal	129.254	59.682	33.856	22.023	15.747	12.046	9.681	8.074
		Simpson's	**120.645**	**57.406**	**33.028**	**21.649**	**15.550**	**11.930**	**9.607**	**8.024**
		Midpoint	125.803	58.639	33.439	21.822	15.636	11.978	9.637	8.044
	2.00	Gauss	132.192	61.450	34.788	22.475	15.926	12.065	9.606	7.942
		Trapezoidal	131.093	61.724	35.337	23.060	16.486	12.586	10.087	8.386
		Simpson's	**125.151**	**60.068**	**34.713**	**22.771**	**16.331**	**12.494**	**10.027**	**8.345**
		Midpoint	129.167	61.083	35.064	22.922	16.408	12.537	10.054	8.363

Source: Calculations conducted by the author using Wolfram Mathematica 8.

Table 3. ARL values for the CUSUM control chart with a long-memory FISMAX(D,1,1)$_{12}$ process for $\Theta_1 = 0.1$, $\Theta_2 = 0.2$, $\omega_1 = 0.4$, and ARL$_0$ = 370

FISMAX($\underline{D}$,1,2)$_{12}$	K	Rules	δ							
			0.2	0.4	0.6	0.8	1	1.2	1.4	1.6
1/3	1.50	Gauss	139.307	58.319	31.519	19.998	14.121	10.740	8.615	7.187
		Trapezoidal	127.326	56.183	31.286	20.258	14.521	11.172	9.042	7.596
		Simpson's	**112.462**	**52.605**	**30.060**	**19.725**	**14.248**	**11.014**	**8.942**	**7.528**
		Midpoint	119.228	54.079	30.526	19.916	14.341	11.067	8.975	7.550
	1.75	Gauss	133.265	59.682	33.183	21.275	15.047	11.413	9.113	7.562
		Trapezoidal	128.813	59.107	33.436	21.732	15.542	11.897	9.571	7.991
		Simpson's	**119.339**	**56.639**	**32.548**	**21.333**	**15.333**	**11.775**	**9.493**	**7.938**
		Midpoint	124.796	57.923	32.972	21.510	15.421	11.825	9.524	7.958
	2.00	Gauss	132.183	61.121	34.508	22.267	15.773	11.951	9.518	7.874
		Trapezoidal	130.661	61.279	35.015	22.833	16.323	12.466	9.996	8.316
		Simpson's	**124.187**	**59.496**	**34.348**	**22.526**	**16.159**	**12.369**	**9.933**	**8.273**
		Midpoint	128.460	60.562	34.714	22.683	16.238	12.413	9.961	8.291
1/5	1.50	Gauss	134.253	59.241	32.721	20.925	14.792	11.226	8.973	7.456
		Trapezoidal	128.278	58.319	32.861	21.334	15.264	11.698	9.424	7.880
		Simpson's	**117.527**	**55.575**	**31.886**	**20.900**	**15.038**	**11.566**	**9.340**	**7.823**
		Midpoint	123.374	56.923	32.326	21.083	15.128	11.617	9.371	7.844
	1.75	Gauss	132.281	60.693	34.134	21.988	15.568	11.799	9.403	7.785
		Trapezoidal	130.103	60.678	34.579	22.527	16.105	12.306	9.876	8.223
		Simpson's	**122.868**	**58.716**	**33.853**	**22.195**	**15.929**	**12.202**	**9.809**	**8.177**
		Midpoint	127.482	59.849	34.237	22.359	16.011	12.248	9.837	8.196
	2.00	Gauss	132.354	62.050	35.285	22.844	16.198	12.269	9.762	8.064
		Trapezoidal	131.881	62.501	35.901	23.459	16.774	12.799	10.249	8.512
		Simpson's	**126.814**	**61.060**	**35.349**	**23.201**	**16.635**	**12.716**	**10.195**	**8.475**
		Midpoint	130.371	61.980	35.672	23.342	16.706	12.756	10.220	8.491
1/7	1.50	Gauss	133.167	59.740	33.242	21.319	15.079	11.437	9.130	7.575
		Trapezoidal	128.886	59.206	33.509	21.782	15.577	11.923	9.590	8.005
		Simpson's	**119.565**	**56.771**	**32.630**	**21.387**	**15.370**	**11.801**	**9.512**	**7.953**
		Midpoint	124.971	58.047	33.053	21.564	15.458	11.851	9.543	7.973
	1.75	Gauss	132.180	61.177	34.556	22.302	15.799	11.970	9.533	7.886
		Trapezoidal	130.734	61.355	35.070	22.872	16.351	12.487	10.012	8.328
		Simpson's	**124.352**	**59.594**	**34.410**	**22.568**	**16.189**	**12.390**	**9.949**	**8.285**
		Midpoint	128.582	60.652	34.774	22.724	16.267	12.435	9.977	8.304
	2.00	Gauss	132.559	62.478	35.631	23.100	16.388	12.412	9.872	8.150
		Trapezoidal	132.443	63.035	36.288	23.735	16.974	12.948	10.362	8.601
		Simpson's	**127.938**	**61.735**	**35.785**	**23.497**	**16.845**	**12.871**	**10.312**	**8.566**
		Midpoint	131.175	62.585	36.087	23.630	16.913	12.909	10.336	8.582

Source: Calculations conducted by the author using Wolfram Mathematica 8.

Table 4. ATS values for the CUSUM control chart with a long-memory FISMAX(D,1,1)$_{12}$ process for $\Theta_1 = 0.1$, ω_1 = 0.1, and ARL_0 = 370

FISMAX($\underline{D}$,1,1)$_{12}$	K	Rules	δ							
			0.2	0.4	0.6	0.8	1	1.2	1.4	1.6
1/3	1.50	Gauss	21.706	7.681	3.961	2.485	1.747	1.343	1.078	0.916
		Trapezoidal	18.429	7.769	4.095	2.625	1.873	1.438	1.226	1.006
		Simpson's	112.800	51.200	28.913	19.132	14.041	10.852	8.964	7.609
		Midpoint	**13.347**	**5.831**	**3.279**	**2.141**	**1.584**	**1.221**	**1.000**	**0.852**
	1.75	Gauss	18.323	**6.953**	4.295	2.718	1.913	1.479	1.216	0.981
		Trapezoidal	17.721	7.846	4.413	2.850	2.050	1.561	1.265	1.064
		Simpson's	145.111	61.919	33.097	21.894	15.870	12.187	9.807	8.210
		Midpoint	**14.100**	**6.498**	**3.683**	**2.381**	**1.744**	**1.317**	**1.082**	**0.975**
	2.00	Gauss	17.597	7.913	4.429	2.848	2.007	1.521	1.210	1.004
		Trapezoidal	17.854	8.231	4.648	3.016	2.165	1.689	1.340	1.128
		Simpson's	127.316	61.265	35.126	22.715	16.753	12.688	10.267	8.502
		Midpoint	**15.062**	**6.926**	**3.923**	**2.551**	**1.864**	**1.395**	**1.125**	**0.990**
1/5	1.50	Gauss	18.577	7.665	4.118	2.608	1.869	1.460	1.132	0.941
		Trapezoidal	19.431	7.718	4.289	2.781	1.994	1.539	1.241	1.034
		Simpson's	119.836	55.876	32.087	20.891	15.258	11.719	9.548	8.525
		Midpoint	**14.168**	**6.294**	**3.544**	**2.534**	**2.373**	**1.785**	**1.408**	**1.192**
	1.75	Gauss	17.634	7.832	4.340	2.787	1.977	1.500	1.193	0.991
		Trapezoidal	17.911	8.183	4.603	2.994	2.141	1.659	1.327	1.098
		Simpson's	127.680	60.491	34.757	22.629	16.398	12.524	10.118	8.501
		Midpoint	**14.630**	**6.758**	**3.849**	**2.525**	**1.835**	**1.382**	**1.172**	**0.930**
	2.00	Gauss	17.264	8.008	4.530	2.924	2.073	1.566	1.251	1.032
		Trapezoidal	18.072	8.474	4.820	3.137	2.251	1.719	1.375	1.159
		Simpson's	132.286	63.380	36.631	24.019	17.556	13.240	10.578	8.814
		Midpoint	**15.076**	**7.121**	**4.062**	**2.650**	**1.945**	**1.468**	**1.169**	**0.974**
1/7	1.50	Gauss	17.965	7.756	4.212	2.678	1.894	1.463	1.150	0.961
		Trapezoidal	17.586	7.884	4.522	2.878	2.053	1.574	1.276	1.080
		Simpson's	123.728	58.011	33.202	21.767	15.978	12.172	9.790	8.259
		Midpoint	**14.312**	**6.510**	**3.696**	**2.387**	**1.753**	**1.437**	**1.072**	**0.903**
	1.75	Gauss	17.489	7.956	4.441	2.856	2.010	1.518	1.209	1.011
		Trapezoidal	17.915	8.244	4.716	3.040	2.188	1.673	1.347	1.120
		Simpson's	129.674	61.991	35.161	23.422	16.938	12.819	10.409	8.685
		Midpoint	**14.822**	**6.929**	**3.955**	**2.568**	**1.874**	**1.413**	**1.144**	**0.951**
	2.00	Gauss	17.355	8.113	4.565	2.955	2.101	1.596	1.260	1.050
		Trapezoidal	18.213	8.560	4.890	3.230	2.406	1.881	1.487	1.201
		Simpson's	134.729	64.924	37.296	24.433	17.790	13.380	10.721	9.095
		Midpoint	**15.299**	**7.252**	**4.159**	**2.713**	**1.965**	**1.480**	**1.199**	**1.013**

Source: Calculations conducted by the author using Wolfram Mathematica 8.

Table 5. ATS values for the CUSUM control chart with a long-memory FISMAX(D,1,1)$_{12}$ process for $\Theta_1 = 0.1$, $\Theta_2 = 0.2$, $\omega_1 = 0.4$, and ARL$_0$ = 370

FISMAX($\underline{D}$,1,2)$_{12}$	K	Rules	δ							
			0.2	0.4	0.6	0.8	1	1.2	1.4	1.6
1/3	1.50	Gauss	18.259	7.703	4.144	2.625	1.851	1.41	1.132	0.945
		Trapezoidal	18.251	7.989	4.421	2.906	2.056	1.587	1.283	1.076
		Simpson's	122.125	56.333	32.261	21.152	15.472	11.805	9.638	8.146
		Midpoint	**13.925**	**6.352**	**3.562**	**2.32**	**1.704**	**1.297**	**1.052**	**0.886**
	1.75	Gauss	17.572	7.894	4.363	2.788	1.974	1.508	1.229	1.073
		Trapezoidal	17.743	8.204	4.648	3.019	2.151	1.658	1.332	1.107
		Simpson's	129.282	60.771	34.746	22.954	16.694	12.699	10.281	8.529
		Midpoint	**14.726**	**6.796**	**3.856**	**2.514**	**1.837**	**1.391**	**1.113**	**0.938**
	2.00	Gauss	17.786	8.039	4.54	2.935	2.08	1.594	1.262	1.042
		Trapezoidal	18.147	8.494	4.866	3.183	2.272	1.728	1.391	1.163
		Simpson's	134.354	63.366	36.929	24.225	17.574	13.285	10.694	8.977
		Midpoint	**15.085**	**7.122**	**4.076**	**2.667**	**1.96**	**1.456**	**1.169**	**0.974**
1/5	1.50	Gauss	17.678	7.796	4.305	2.749	1.944	1.484	1.182	0.981
		Trapezoidal	17.673	8.041	4.54	2.994	2.155	1.622	1.318	1.088
		Simpson's	125.998	59.605	34.174	22.496	16.38	12.416	9.927	8.38
		Midpoint	**14.488**	**6.691**	**3.805**	**2.473**	**1.811**	**1.366**	**1.158**	**0.93**
	1.75	Gauss	17.399	7.928	4.478	2.965	2.047	1.552	1.235	1.024
		Trapezoidal	17.924	8.298	4.786	3.086	2.209	1.693	1.358	1.132
		Simpson's	131.787	62.78	36.047	23.523	17.263	13.071	10.474	8.755
		Midpoint	**14.954**	**7.046**	**4.048**	**2.859**	**1.915**	**1.442**	**1.157**	**0.964**
	2.00	Gauss	17.408	8.147	4.637	3.014	2.138	1.607	1.289	1.061
		Trapezoidal	18.249	8.673	4.973	3.248	2.319	1.781	1.428	1.188
		Simpson's	136.032	65.159	37.69	24.952	18.052	13.653	10.908	9.062
		Midpoint	**15.476**	**7.297**	**4.204**	**2.743**	**1.997**	**1.51**	**1.194**	**1.03**
1/7	1.50	Gauss	17.816	7.945	4.442	2.847	2.015	1.538	1.22	1.012
		Trapezoidal	17.806	8.189	4.644	3.001	2.152	1.648	1.313	1.101
		Simpson's	127.958	61.604	35.159	23.022	16.65	12.766	10.235	8.575
		Midpoint	**14.671**	**6.855**	**3.881**	**2.529**	**1.832**	**1.389**	**1.121**	**0.94**
	1.75	Gauss	17.584	8.196	4.606	2.979	2.114	1.593	1.299	1.052
		Trapezoidal	18.075	8.663	4.924	3.172	2.262	1.722	1.377	1.171
		Simpson's	133.511	63.39	36.763	24.125	17.528	13.296	10.678	8.878
		Midpoint	**15.173**	**7.145**	**4.091**	**2.719**	**1.95**	**1.466**	**1.198**	**0.982**
	2.00	Gauss	17.711	8.373	4.781	3.084	2.19	1.659	1.327	1.103
		Trapezoidal	18.395	8.773	5.019	3.300	2.355	1.79	1.432	1.190
		Simpson's	139.961	67.378	39.095	25.629	18.765	14.012	11.243	9.174
		Midpoint	**15.552**	**7.380**	**4.260**	**2.816**	**2.029**	**1.543**	**1.224**	**1.011**

Source: Calculations conducted by the author using Wolfram Mathematica 8.

Table 6. EARL and EATS values for the CUSUM control chart with long-memory FISMAX$(D,Q,X)_{12}$ processes

Measurement	K	Rules	Long-memory FISMAX processes					
			$(1/3,1,1)_{12}$	$(1/3,2,1)_{12}$	$(1/5,1,1)_{12}$	$(1/5, 2,1)_{12}$	$(1/7,1,1)_{12}$	$(1/7, 2,1)_{12}$
EARL	1.50	Gauss	1525.92	1449.03	1451.88	1447.94	1445.83	1453.45
		Trapezoidal	1336.56	1386.92	1380.76	1425.29	1400.75	1442.39
		Simpson's	1173.52	1282.92	1271.44	1348.28	1307.54	1374.95
		Midpoint	1224.05	1328.41	1317.70	1388.33	1351.20	1412.30
	1.75	Gauss	1446.09	1452.70	1450.87	1468.26	1457.55	1477.02
		Trapezoidal	1398.48	1440.45	1435.38	1471.99	1451.82	1486.05
		Simpson's	1303.60	1371.99	1364.15	1418.75	1389.20	1438.69
		Midpoint	1347.56	1409.65	1402.63	1451.10	1424.99	1468.58
	2.00	Gauss	1456.69	1475.98	1473.31	1494.13	1482.22	1502.95
		Trapezoidal	1449.96	1484.45	1480.28	1510.38	1493.80	1521.93
		Simpson's	1386.40	1436.46	1430.57	1472.23	1449.50	1487.75
		Midpoint	1422.50	1466.61	1461.47	1497.69	1477.99	1511.09
EATS	1.50	Gauss	204.59	190.35	191.85	190.60	190.40	194.18
		Trapezoidal	192.31	197.85	200.14	197.16	194.27	199.27
		Simpson's	1267.56	1384.66	1368.70	1446.88	1414.54	1479.85
		Midpoint	146.28	155.49	166.49	163.61	160.35	166.09
	1.75	Gauss	189.39	192.01	191.27	193.14	192.45	197.12
		Trapezoidal	193.85	199.31	199.58	202.43	201.22	206.83
		Simpson's	1540.48	1479.78	1465.49	1518.50	1495.50	1540.85
		Midpoint	158.90	165.86	165.41	171.93	168.28	173.62
	2.00	Gauss	192.65	196.39	193.24	196.51	194.98	201.14
		Trapezoidal	200.36	206.22	205.04	209.30	209.34	211.27
		Simpson's	1473.16	1547.02	1532.52	1577.54	1561.84	1626.29
		Midpoint	169.18	172.55	172.33	177.26	175.40	179.08

Source: Calculations conducted by the author using Wolfram Mathematica 8.