Optimal Control Problem for a Nonhomogeneous Equation of Vibrations of a Three-Layer Plate

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Abstract: - The paper considers a quadratic criterion optimal control problem for an inhomogeneous equation of vibrations of a three-layer plate. The existence and uniqueness of the solution of a boundary value problem were studied for each fixed admissible control. Then the existence theorem of optimal control was proved, and a necessary condition for optimality was derived in the form of an integral inequality.

Key-Words: - There-layer plate, generalized solution, optimal control, adjoint problem, optimality conditions, variational inequality.

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1 Introduction

It is known that many practical problems are described by means of fourth-order partial differential equations. For example, vibrations of a tuning fork, elastic plate, thin plate, and three-layer plate can be an example, [1], [2], [3].

Note that in the works, [4], [5], [6], [7], [8], [9], [10], some related problems were considered. Namely, in [4], the propagation of elastic waves in a three-layer plate with a high contrast of mechanical and geometric properties of the layers was analyzed, and in [5], the problem of precise controllability of a three-dimensional linear-elastic thin plate with control at the boundary and inside the plate was considered. In [9], necessary and sufficient optimality conditions were obtained for a fairly wide class of structural design problems, in [10], nonlinear vibrations of the Karman plate were studied using the optimal dynamic inversion technique.

It should be noted that for processes described by such equations, there is a need to study boundary value problems.

Therefore, studying optimal control problems for such equations, i.e., proving the existence of optimal control, deriving necessary and sufficient conditions for optimality, studying approximate construction of optimal control and appropriate optimal state, is of great importance. Optimal control problems with a control function in the coefficient of higher-order derivatives are of special interest. In the paper, an optimal control problem for an inhomogeneous equation of a three-layer plate when a co ntrol function is contained in the coefficient of higher-order derivatives, is considered

2 Problem Statement

It is known that the equation of vibrations of a three-layer plate is in the form, [3]

$$\Delta(D\Delta u) + (1-\nu)\left(2\frac{\partial^2 D}{\partial x\partial y}\cdot\frac{\partial^2 u}{\partial x\partial y} - \frac{\partial^2 D}{\partial x^2}\cdot\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 D}{\partial y^2}\cdot\frac{\partial^2 u}{\partial x^2}\right) = q. \quad (1)$$

Here $D = \frac{h(h_0+h)^2 E_f}{2(1-\nu^2)}$ - is cylindrical stiffness, h_0 - is the thickness of the main layer, h is the thickness of the top and bottom layers, E_f - is a Young module, $\nu\left(0<\nu<\frac{1}{2}\right)$ - is a Poisson ratio, $u(x,y,t)$ - is plate displacement at the point (x,y) a time t, Δ - is a Laplace operator with respect to the variables x, y and q is a loading intensity.

Assuming $h \ll h_0$, $D = \frac{h_0^2 E_f}{2(1-\nu^2)} h$.

Taking the force q as the inertia force $-\big(h_0\rho_c +$ $h\rho f\partial 2u\partial t2$, the equation (1) takes the following form:

$$\Delta(h\Delta u) + (1-\nu)\left(2\frac{\partial^2 h}{\partial x\partial y}\cdot\frac{\partial^2 u}{\partial x\partial y} - \frac{\partial^2 h}{\partial x^2}\cdot\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 h}{\partial y^2}\cdot\frac{\partial^2 u}{\partial x^2}\right)$$

$$+2\big(h_0\rho_c + h\rho_f\big)\frac{(1-\nu^2)}{h_0^2 E_f}\cdot\frac{\partial^2 u}{\partial t^2} = 0. \qquad (2)$$

Here, $\rho_c(x,y) > 0$ is the density of the layer, $\rho_f(x,y) > 0$ is the density of the material of the top and bottom layers.

In equation (2), denoting the constant number $\frac{2(1-\nu^2)}{h_0^2 E_f}$ by α we obtain the equation,

$$\alpha\big(h_0\rho_c + h\rho_f\big)\cdot\frac{\partial^2 u}{\partial t^2} + \Delta(h\Delta u) + (1-\nu)\left(2\frac{\partial^2 h}{\partial x\partial y}\cdot\frac{\partial^2 u}{\partial x\partial y} - \frac{\partial^2 h}{\partial x^2}\cdot\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 h}{\partial y^2}\cdot\frac{\partial^2 u}{\partial x^2}\right) = 0. \quad (3)$$

In the present paper, we consider the solution of the inhomogeneous equation, [3]

$$\alpha\big(h_0\rho_c + h\rho_f\big)\cdot\frac{\partial^2 u}{\partial t^2} + \Delta(h\Delta u) + (1-\nu) + \left(2\frac{\partial^2 h}{\partial x\partial y}\cdot\frac{\partial^2 u}{\partial x\partial y} - \frac{\partial^2 h}{\partial x^2}\cdot\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 h}{\partial y^2}\cdot\frac{\partial^2 u}{\partial x^2}\right) = f(x,y,t), (x,y,t) \in Q \qquad (4)$$

of a three-layer plate under the initial conditions and boundary conditions

$$u(x,y,0) = \varphi_0(x,y), \frac{\partial u(x,y,0)}{\partial t} = \varphi_1(x,y), \quad (x,y) \in \Omega, \qquad (5)$$

$$u(0,y,t) = 0, \frac{\partial u(0,y,t)}{\partial x} = 0,$$
$$u(a,y,t) = 0, \frac{\partial u(a,y,t)}{\partial x} = 0,$$
$$0 \le y \le b, 0 \le t \le T,$$
$$u(x,0,t) = 0, \frac{\partial u(x,0,t)}{\partial y} = 0,$$
$$u(x,b,t) = 0, \frac{\partial u(x,b,t)}{\partial y} = 0,$$
$$0 \le x \le b, 0 \le t \le T, \qquad (6)$$

here $(x,y) \in \Omega = \{(x,y): 0 < x < a, 0 < y < b\}$, $t \in (0,T)$, $Q = \Omega \times (0,T)$, a, b, T are the described positive numbers, $f(x,y,t) \in L_2(Q)$ is a given function, $\varphi_0(x,y) \in \overset{o}{W}{}_2^2(\Omega)$, $\varphi_1(x,y) \in L_2(\Omega)$ are given initial functions, $h(x,y)$ is a control function.

We take

$$U_{ad} = \{h(x,y): h(x,y) \in W_2^3(\Omega), 0 < \alpha, \le h(x,y) \le \beta, \left|\frac{\partial h}{\partial x}\right| \le M, \left|\frac{\partial h}{\partial y}\right| \le M,$$
$$\left|\frac{\partial^2 h}{\partial x^2}\right| \le M, \left|\frac{\partial^2 h}{\partial x\partial y}\right| \le M, \left|\frac{\partial^2 h}{\partial y^2}\right| \le M,$$
$$a.e.\ in\ \Omega, \|h(x,y)\|_{W_2^3(\Omega)} \le M\Big\}.$$

Here, as a c lass of admissible controls, M is given a positive number.

We take the following optimal control problem: find a control from the class of admissible controls U_{ad} that of the minimum to the functional

$$J(h) = \int_Q [u(x,y,t;h) - z(x,y,t)]^2 dxdydt + \beta\int_\Omega [h(x,y) - \omega(x,y)]^2 dxdy \qquad (7)$$

together with the solution of the problem (4)-(6), here $z(x,y,t) \in L_2(Q)$, $\omega(x,y) \in L_2(\Omega)$ are the given functions, $\beta > 0$ is a given number.

In the paper, we consider a generalized solution of problem (4)-(6).

Definition. The generalized solution of the problem (4)-(6) corresponding to the admissible control $h(x,y)$ is called such a function $u(x,y,t) \in W_2^{2,1}(Q)$ that for arbitrary function $\eta(x,y,t) \in$ *C2,1(Q)*, *η(x,y,T)=0*

$$\eta(0,y,t) = 0, \frac{\partial\eta(0,y,t)}{\partial x} = 0,$$
$$\eta(a,y,t) = 0, \frac{\partial\eta(a,y,t)}{\partial x} = 0,$$
$$0 \le y \le b, 0 \le t \le T,$$
$$\eta(x,0,t) = 0, \frac{\partial\eta(x,0,t)}{\partial y} = 0,$$
$$\eta(x,b,t) = 0, \frac{\partial\eta(x,b,t)}{\partial y} = 0,$$
$$0 \le x \le b, 0 \le t \le T$$

satisfies the integral identity

$$\int_Q \left[-\alpha(h_0\rho_c + h\rho_f)\cdot\frac{\partial u}{\partial t}\cdot\frac{\partial\eta}{\partial t} + h\Delta u\Delta\eta + (1-\nu)\left(2\frac{\partial^2 h}{\partial x\partial y}\cdot\frac{\partial^2 u}{\partial x\partial y} - \frac{\partial^2 h}{\partial x^2}\cdot\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 h}{\partial y^2}\cdot\frac{\partial^2 u}{\partial x^2}\right)\eta\right] dxdydt -$$

$$-\int_\Omega \alpha\left(h_0\rho_c + h\rho_f\right)\varphi_1(x,y)\eta(x,y,0)dxdy$$

$$= \int_Q f(x,y,t)\eta(x,y,t)dxdyt$$

and the initial condition $u(x,y,0) = \varphi_0(x,y)$.

3 Solvability of Initial-Boundary Value Problem

Theorem 1. Suppose that the conditions above on dates is satisfies. Then for each fixed admissible control $h(x,y)$ the problem (4)-(6) has a unique generalized solution in the space $W_2^{2,1}(Q)$.

Proof. We apply the Faedo-Galerkin method, [11]. We search for an approximate solution to the problem (3)-(5) by the fixed control $h(x,y)$ in the following form

$$u^N(x,y,t) = \sum_{i=1}^N c_i^N(t)\omega_i(x,y),$$

where $\{\omega_i(x,y)\}_{i=1}^\infty$ is a b asis in $\overset{o}{W}{}_2^2(\Omega)$ and $c_1^N(t),\dots,c_N^N(t)$ are defined from the system of ordinary differential equations

$$\int_\Omega 2(h_0\rho_c + h\rho_f)\frac{(1-\nu^2)}{h_0^2E_f}\frac{\partial^2 u^N}{\partial t^2}\omega_j(x,y)dxdy$$
$$+\int_\Omega h\Delta u^N \Delta\omega_j(x,y)dxdy$$
$$+(1-\nu)\int_\Omega\left(2\frac{\partial^2 h}{\partial x\partial y}\frac{\partial^2 u^N}{\partial x\partial y} - \frac{\partial^2 h}{\partial x^2}\frac{\partial^2 u^N}{\partial y^2} - \frac{\partial^2 h}{\partial y^2}\frac{\partial^2 u^N}{\partial x^2}\right)\omega_j(x,y)dxdy$$
$$= \int_\Omega f(x,y,t)\omega_j(x,y)dxdy,\ 1\le j\le N, \quad (8)$$
$$c_i^N(0) = \alpha_i^N,\ \frac{d}{dt}c_i^N(t)\Big|_{t=0} = \beta_i^N,$$

where α_i^N, β_i^N are the coefficients of the sums

$$\varphi_0^N(x,y) = \sum_{i=1}^N \alpha_i^N\,\omega_i(x,y),$$
$$\varphi_1^N(x,y) = \sum_{i=1}^N \beta_i^N\,\omega_i(x,y)$$

approximating the functions $\varphi_0(x,y)$ and $\varphi_1(x,y)$ in the norm $W_2^2(\Omega)$, and $L_2(\Omega)$ correspondingly as $N\to\infty$.

Multiplying both the equalities (8) by the corresponding $\frac{d}{dt}c_i^N(t)$ and taking a su m over j from 1 to N, we obtain the equality

$$\int_\Omega 2(h_0\rho_c + h\rho_f)\frac{(1-\nu^2)}{h_0^2E_f}\frac{\partial^2 u^N}{\partial t^2}\frac{\partial u^N}{\partial t}dxdy$$
$$+\int_\Omega h\Delta u^N\frac{\partial(\Delta u^N)}{\partial t}dxdy + (1-\nu)$$
$$\int_\Omega\left(2\frac{\partial^2 h}{\partial x\partial y}\frac{\partial^2 u^N}{\partial x\partial y} - \frac{\partial^2 h}{\partial x^2}\frac{\partial^2 u^N}{\partial y^2} - \frac{\partial^2 h}{\partial y^2}\frac{\partial^2 u^N}{\partial x^2}\right)$$
$$\cdot\frac{\partial u^N}{\partial t}dxdy = \int_\Omega f(x,y,t)\frac{\partial u^N}{\partial t}dxdy.$$

Hence

$$\frac{1}{2}\frac{d}{dt}\int_Q\left[2(h_0\rho_c + h\rho_f)\frac{(1-\nu^2)}{h_0^2E_f}\left(\frac{\partial u^N}{\partial t}\right)^2\right.$$
$$\left.+h(\Delta u^N)^2dxdy\right.$$
$$= -(1-\nu)\int_\Omega\left(2\frac{\partial^2 h}{\partial x\partial y}\frac{\partial^2 u^N}{\partial x\partial y} - \frac{\partial^2 h}{\partial x^2}\frac{\partial^2 u^N}{\partial y^2}\right.$$
$$\left.-\frac{\partial^2 h}{\partial y^2}\frac{\partial^2 u^N}{\partial x^2}\right)\frac{\partial u^N}{\partial t}dxdy$$
$$+\int_\Omega f(x,y,t)\frac{\partial u^N}{\partial t}dxdy = 0.$$

Integrating this relation over t from 0 t o t and taking $h(x,y)\in U_{ad}$, $\rho_c(x,y)>0$, $\rho_f(x,y)>0$

$$\int_\Omega\left[\left(\frac{\partial u^N}{\partial t}\right)^2\right]dxdy \le C$$
$$+C\int_0^t\int_\Omega\left[(u^N)^2 + \left(\frac{\partial u^N}{\partial x}\right)^2 + \left(\frac{\partial u^N}{\partial y}\right)^2 +\right.$$
$$+\left(\frac{\partial u^N}{\partial t}\right)^2 + \left(\frac{\partial^2 u^N}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u^N}{\partial x\partial y}\right)^2 +$$
$$\left.+\left(\frac{\partial^2 u^N}{\partial y^2}\right)^2\right]dxdyds,\forall t\in[0,T],$$

here and further on we denote by C different constants.

Hence, by virtue of the equivalence of norms in $\overset{o}{W}{}_2^2(\Omega)$:

$$\int_\Omega\left[(u^N)^2 + \left(\frac{\partial u^N}{\partial x}\right)^2 + \left(\frac{\partial u^N}{\partial y}\right)^2 + \left(\frac{\partial u^N}{\partial t}\right)^2\right]dxdy$$
$$\le C + C\int_0^t\int_\Omega\left[(u^N)^2 + \left(\frac{\partial u^N}{\partial x}\right)^2 + \left(\frac{\partial u^N}{\partial y}\right)^2 +\right.$$

∂uN∂t2+∂2uN∂x22+∂2uN∂x∂y2+∂2uN∂y22d xdyds.

It is clear, [11]

$$\int_\Omega\left[\left(\frac{\partial^2 u^N}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u^N}{\partial x\partial y}\right)^2 +\right.$$

∂2uN∂y22dxdy≤Ω∂2uN∂x2+∂2uN∂y22dxdy.

Then

$$\int_\Omega\left[(u^N)^2 + \left(\frac{\partial u^N}{\partial x}\right)^2 + \left(\frac{\partial u^N}{\partial y}\right)^2 + \left(\frac{\partial u^N}{\partial t}\right)^2\right.$$
$$\left.+\left(\frac{\partial^2 u^N}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u^N}{\partial x\partial y}\right)^2 + \left(\frac{\partial^2 u^N}{\partial y^2}\right)^2\right]dxdy$$
$$\le C + C\int_0^t\int_\Omega\left[(u^N)^2 + \left(\frac{\partial u^N}{\partial x}\right)^2 + \left(\frac{\partial u^N}{\partial y}\right)^2 +\right.$$

∂uN∂t2+∂2uN∂x22+∂2uN∂x∂y2+∂2uN∂y22d xdyds.

Hence, applying Gronwall`s lemma, one may get

$\int_{\Omega}\left[(u^N)^2+\left(\frac{\partial u^N}{\partial x}\right)^2+\left(\frac{\partial u^N}{\partial y}\right)^2+\left(\frac{\partial u^N}{\partial t}\right)^2+\right.$
$\partial 2uN\partial x22+\partial 2uN\partial x\partial y2+\partial 2uN\partial y22dxdy\leq cons$
$t, \forall t\in[0,T]$.

From the last, it follows, that the sequence $\{u^N\}$ is bounded in $W_2^{2,1}(Q)$. Therefore, it is possible to choose such a sequence from $\{u^N\}$ (we`ll denote it by $\{u^N\}$ also) for which

$$u^N\to u, \frac{\partial u^N}{\partial t}\to\frac{\partial u}{\partial t}, \frac{\partial u^N}{\partial x}\to\frac{\partial u}{\partial x}, \frac{\partial u^N}{\partial y}\to\frac{\partial u}{\partial y},$$
$$\frac{\partial^2 u^N}{\partial x^2}\to\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u^N}{\partial x\partial y}\to\frac{\partial^2 u}{\partial x\partial y}, \frac{\partial^2 u^N}{\partial y^2}\to\frac{\partial^2 u}{\partial y^2}$$

weakly in $L_2(Q)$ by $N\to\infty$

Then, by virtue of the embedding theorem

$$u^N\to u,\ \frac{\partial u^N}{\partial x}\to\frac{\partial u}{\partial x}, \frac{\partial u^N}{\partial y}\to\frac{\partial u}{\partial y}$$

strongly in $L_2(Q)$.

Let us take $u=u^N$ in the definition of the generalized solution of problem (4)-(6):

$$\int_Q\left[-\alpha(h_0\rho_c+h\rho_f)\cdot\frac{\partial u^N}{\partial t}\cdot\frac{\partial\eta}{\partial t}+h\Delta u^N\Delta\eta\right.$$
$$\left.+(1-\nu)\left(2\frac{\partial^2 h}{\partial x\partial y}\cdot\frac{\partial^2 u^N}{\partial x\partial y}-\frac{\partial^2 h}{\partial x^2}\cdot\frac{\partial^2 u^N}{\partial y^2}-\frac{\partial^2 h}{\partial y^2}\cdot\frac{\partial^2 u^N}{\partial x^2}\right)\eta\right]dxdydt-$$
$$-\int_\Omega\alpha\big(h_0\rho_c+h\rho_f\big)\varphi_1(x,y)\eta(x,y,0)dxdy$$
$$=\int_Q f(x,y,t)\eta(x,y,t)dxdyt.$$

Passing to limit who $N\to\infty$ from here we have

$$\int_Q\left[-\alpha(h_0\rho_c+h\rho_f)\cdot\frac{\partial u}{\partial t}\cdot\frac{\partial\eta}{\partial t}+h\Delta u\Delta\eta\right.$$
$$\left.+(1-\nu)\left(2\frac{\partial^2 h}{\partial x\partial y}\cdot\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial^2 h}{\partial x^2}\cdot\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 h}{\partial y^2}\cdot\frac{\partial^2 u}{\partial x^2}\right)\eta\right]dxdydt-$$
$$-\int_\Omega\alpha\big(h_0\rho_c+h\rho_f\big)\varphi_1(x,y)\eta(x,y,0)dxdy$$
$$=\int_Q f(x,y,t)\eta(x,y,t)dxdyt.$$

So, we`ve shown, that the function $u(x,y,t)$ is the generalized solution of problem (4)-(6) from $W_2^{2,1}(Q)$.

The uniqueness of the solution to the problem (4)-(6) may be proved by a standard way. Theorem is proved.

4 Existence of the Optimal Control

Theorem 2. Within the above conditions imposed on the data of the problem (4)-(7), this problem has an optimal control.

Proof. Let $\{h_n\}\in U_{ad}$ be a minimizing sequence, i.e.

$$\lim_{n\to\infty} J(h_n)=\inf_{h\in U_{ad}} J(h).$$

It is clear that

$$\|h_n\|_{W_2^3(\Omega)}\leq const. \tag{9}$$

Taking it into account, for solutions of problem (4)-(6) corresponding to h_n, we obtain the estimation (see. part II)

$$\|u_n\|_{W_2^{2,1}(Q)}\leq const. \tag{10}$$

By virtue of (9) and (10), the property of weak compactness in the Hilbert spaces and the embedding theorem, it is possible to consider, that as $n\to\infty$

$$h_n\to h_0, \frac{\partial h_n}{\partial x}\to\frac{\partial h_0}{\partial x}, \frac{\partial h_n}{\partial y}\to\frac{\partial h_0}{\partial y}$$

uniformly in $C(\bar\Omega)$,

$$\frac{\partial^2 h_n}{\partial x^2}\to\frac{\partial^2 h_0}{\partial x^2}, \frac{\partial^2 h_n}{\partial x\partial y}\to\frac{\partial^2 h_0}{\partial x\partial y}, \frac{\partial^2 h_n}{\partial y^2}\to\frac{\partial^2 h_0}{\partial y^2}$$

strongly in $L_2(\Omega)$ and

$$u_n\to u_0, \frac{\partial u_n}{\partial x}\to\frac{\partial u_0}{\partial x}, \frac{\partial u_n}{\partial y}\to\frac{\partial u_0}{\partial y}$$

strongly in $L_2(Q)$,

$$\frac{\partial^2 u_n}{\partial x^2}\to\frac{\partial^2 u_0}{\partial x^2}, \frac{\partial^2 u_n}{\partial x\partial y}\to\frac{\partial^2 u_0}{\partial x\partial y}, \frac{\partial^2 u_n}{\partial y^2}\to\frac{\partial^2 u_0}{\partial y^2}$$

weakly in$L_2(Q)$.

Considering these relations, in the definition of the generalized solution for the problem (4)-(6), by $h=h_n$, $u=u_n$, passing to the limit as $n\to\infty$ we have

$$\int_Q\left[-\alpha(h_0\rho_c+h\rho_f)\cdot\frac{\partial u}{\partial t}\cdot\frac{\partial\eta}{\partial t}+h\Delta u\Delta\eta\right.$$
$$\left.+(1-\nu)\left(2\frac{\partial^2 h}{\partial x\partial y}\cdot\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial^2 h}{\partial x^2}\cdot\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 h}{\partial y^2}\cdot\frac{\partial^2 u}{\partial x^2}\right)\eta\right]dxdydt$$
$$-\int_\Omega\alpha\big(h_0\rho_c+h\rho_f\big)\varphi_1(x,y)\eta(x,y,0)dxdy$$

$$= \int_Q f(x,y,t)\eta(x,y,t)dxdydt.$$

Hence, it follows, that $u_0 = u(h_0)$, i.e. $u_0(x,y,t)$ is a generalized solution for the problem (4)-(6), corresponding to the control $h_0(x,y)$.

The functional $J(h)$ is continuous in U_{ad} by h. Therefore,

$$\lim_{n\to\infty} J(h_n) = \inf_{h\in U_{ad}} J(h) = J(h_0).$$

It shows, that $h_0(x,y)$ it provides the minimum to functional (7), i.e., is an optimal control.

The theorem is proved.

5 Necessary Conditions of Optimality

Then we introduce the following adjoint problem corresponding to the admissible $h(x,y)$ given for obtaining the necessary condition for optimality:

$$\alpha\left(h_0\rho_c + h\rho_f\right)\cdot\frac{\partial^2\psi}{\partial t^2} + \Delta(h\Delta\psi)$$
$$+(1-\nu)\left[2\frac{\partial^2}{\partial x\partial y}\cdot\left(\psi\frac{\partial^2 h}{\partial x\partial y}\right) - \frac{\partial^2}{\partial x^2}\cdot\left(\psi\frac{\partial^2 h}{\partial y^2}\right) - \frac{\partial^2}{\partial y^2}\cdot\left(\psi\frac{\partial^2 h}{\partial x^2}\right)\right]$$
$$= -2(u-z), (x,y,t)\in Q, \quad (11)$$

$$\psi(x,y,T) = 0, \frac{\partial\psi(x,y,T)}{\partial t} = 0, (x,y)\in\Omega, \quad (12)$$

$$\psi(0,y,t) = 0, \frac{\partial\psi(0,y,t)}{\partial x} = 0,$$
$$\psi(a,y,t) = 0, \frac{\partial\psi(a,y,t)}{\partial x} = 0,$$
$$0\le y\le b, 0\le t\le T,$$
$$\psi(x,0,t) = 0, \frac{\partial\psi(x,0,t)}{\partial y} = 0,$$
$$\psi(x,b,t) = 0, \frac{\partial\psi(x,b,t)}{\partial y} = 0,$$
$$0\le x\le b, 0\le t\le T, \quad (13)$$

here the function $u(x,y,t)$ is the solution of the problem (4)-(6) corresponding to the control $h(x,y)$.

As in problem (4)-(6), we can show that the adjoint problem (4)-(6) also has a unique generalized solution in the space $W_2^{2,1}(Q)$.

For deriving a necessary condition for optimality, we have two admissible controls $h(x,y)$ and $h(x,y)+\delta h(x,y)$. Denoting the solutions of the problem (4)-(6) corresponding to these controls by $u(x,y,t)$ and $u(x,y,t)+\delta u(x,y,t)$ we obtain that the function $\delta u(x,y,t)$ is the solution of the boundary value problem

$$\alpha\left(h_0\rho_c + h\rho_f\right)\cdot\frac{\partial^2(\delta u)}{\partial t^2} + \Delta(h\Delta(\delta u))$$
$$+(1-\nu)\left(2\frac{\partial^2 h}{\partial x\partial y}\cdot\frac{\partial^2(\delta u)}{\partial x\partial y} - \frac{\partial^2 h}{\partial x^2}\cdot\frac{\partial^2(\delta u)}{\partial y^2} - \frac{\partial^2 h}{\partial y^2}\cdot\frac{\partial^2(\delta u)}{\partial x^2}\right)$$
$$= -\alpha(\delta h)\rho_f\cdot\frac{\partial^2 u}{\partial t^2} + \Delta(\delta h\Delta u)$$
$$+(1-\nu)\left(2\frac{\partial^2(\delta h)}{\partial x\partial y}\cdot\frac{\partial^2 u}{\partial x\partial y} - \frac{\partial^2(\delta h)}{\partial x^2}\cdot\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2(\delta h)}{\partial y^2}\cdot\frac{\partial^2 u}{\partial x^2}\right)$$
$$-\alpha(\delta h)\rho_f\cdot\frac{\partial^2(\delta u)}{\partial t^2} + \Delta\left(\delta h\Delta(\delta u)\right)$$
$$+(1-\nu)\left(2\frac{\partial^2(\delta h)}{\partial x\partial y}\cdot\frac{\partial^2(\delta u)}{\partial x\partial y} - \frac{\partial^2(\delta h)}{\partial x^2}\cdot\frac{\partial^2(\delta u)}{\partial y^2} - \frac{\partial^2(\delta h)}{\partial y^2}\cdot\frac{\partial^2(\delta u)}{\partial x^2}\right),$$
$$(x,y,t)\in Q, \quad (14)$$

$$\delta u(x,y,0) = 0, \frac{\partial\delta u(x,y,0)}{\partial t} = 0,$$
$$(x,y)\in\Omega, \quad (15)$$

$$\delta u(0,y,t) = 0, \frac{\partial\left(\delta u(0,y,t)\right)}{\partial x} = 0,$$
$$\delta u(a,y,t) = 0, \frac{\partial\left(\delta u(a,y,t)\right)}{\partial x} = 0,$$
$$0\le y\le b, 0\le t\le T,$$
$$\delta u(x,0,t) = 0, \frac{\partial\left(\delta u(x,0,t)\right)}{\partial y} = 0$$
$$\delta u(x,b,t) = 0, \frac{\partial\left(\delta u(x,b,t)\right)}{\partial y} = 0,$$
$$0\le x\le b, 0\le t\le T \quad (16)$$

For the solution of the problem (14)-(16), it is clear derived the estimation:

$$\|\delta u\|_{W_2^{2,1}(Q)} \le C\|\delta h\|_{W_2^3(\Omega)}.$$

Then, we calculate the increment of the functional $J(h)$:

$$\delta J(h) = 2\int_Q (u-z)\delta u dxdydt$$

$$2\beta\int_\Omega (h-\omega)\delta h dxdy + \int_Q (\delta u)^2 dxdydt$$

$$+\int_\Omega (\delta h)^2 dxdy$$

$$= 2\int_Q (u-z)\delta u dxdydt + 2\beta\int_\Omega (h-\omega)\delta h dxdy$$

$$+R_1,$$

here

$$R_1 = \int_Q (\delta u)^2 dxdydt + \beta \int_\Omega (\delta h)^2 dxdy$$

denotes a residual.

Since the function δu is the generalized solution of the boundary value problem (14)-(16) for an arbitrary function $\eta(x,y,t) \in W_2^{2,1}(Q)$, $\eta(x,y,T) = 0$,

$$\eta(0,y,t) = 0, \frac{\partial \eta(0,y,t)}{\partial x} = 0$$
$$\eta(x,0,t) = 0, \frac{\partial \eta(x,0,t)}{\partial y} = 0,$$
$$0 \le y \le b, 0 \le t \le T,$$
$$\eta(a,y,t) = 0, \frac{\partial \eta(a,y,t)}{\partial x} = 0$$
$$\eta(x,b,t) = 0, \frac{\partial \eta(x,b,t)}{\partial y} = 0,$$
$$0 \le x \le b, 0 \le t \le T$$

the integral identity

$$\int_Q \Big[-\alpha(h_0\rho_c + h\rho_f) \cdot \frac{\partial(\delta u)}{\partial t} \cdot \frac{\partial \eta}{\partial t} + h\Delta(\delta u)\Delta\eta$$
$$+(1-\nu)\Big(2\frac{\partial^2 h}{\partial x \partial y} \cdot \frac{\partial^2(\delta u)}{\partial x \partial y} - \frac{\partial^2 h}{\partial x^2} \cdot \frac{\partial^2(\delta u)}{\partial y^2}$$
$$-\frac{\partial^2 h}{\partial y^2} \cdot \frac{\partial^2(\delta u)}{\partial x^2}\Big)\eta\Big] dxdydt$$
$$= \int_Q \Big[-\alpha(\delta h)\rho_f \cdot \frac{\partial u}{\partial t} \cdot \frac{\partial \eta}{\partial t} + \delta h \Delta u \Delta \eta$$
$$+(1-\nu)\Big(2\frac{\partial^2(\delta h)}{\partial x \partial y} \cdot \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2(\delta h)}{\partial x^2} \cdot \frac{\partial^2 u}{\partial y^2}$$
$$-\frac{\partial^2(\delta h)}{\partial y^2} \cdot \frac{\partial^2 u}{\partial x^2}\Big)\eta\Big] dxdydt$$
$$+ \int_Q \Big[-\alpha(\delta h)\rho_f \cdot \frac{\partial(\delta u)}{\partial t} \cdot \frac{\partial \eta}{\partial t} + \delta h \Delta(\delta u)\Delta\eta$$
$$+(1-\nu)\Big(2\frac{\partial^2(\delta h)}{\partial x \partial y} \cdot \frac{\partial^2(\delta u)}{\partial x \partial y} - \frac{\partial^2(\delta h)}{\partial x^2} \cdot \frac{\partial^2(\delta u)}{\partial y^2}$$
$$-\frac{\partial^2(\delta h)}{\partial y^2} \cdot \frac{\partial^2(\delta u)}{\partial x^2}\Big)\eta\Big] dxdydt \qquad (17)$$

is valid.

As the function $\psi(x,y,t)$ is a g eneralized solution of the adjoint problem (11)-(13) for an arbitrary function $\chi(x,y,t) \in W_2^{2,1}(Q)$, $\chi(x,y,0) = 0$,

$$\chi(0,y,t) = 0, \frac{\partial \chi(0,y,t)}{\partial x} = 0$$
$$\chi(x,0,t) = 0, \frac{\partial \chi(x,0,t)}{\partial y} = 0,$$
$$0 \le y \le b, 0 \le t \le T,$$
$$\chi(a,y,t) = 0, \frac{\partial \chi(a,y,t)}{\partial x} = 0$$
$$\chi(x,b,t) = 0, \frac{\partial \chi(x,b,t)}{\partial y} = 0,$$
$$0 \le x \le b, 0 \le t \le T$$

the integral identity

$$\int_Q \Big[-\alpha(h_0\rho_c + h\rho_f) \cdot \frac{\partial \psi}{\partial t} \cdot \frac{\partial \chi}{\partial t} + h\Delta\psi\Delta\chi$$
$$+(1-\nu)2\psi\frac{\partial^2 h}{\partial x \partial y} \cdot \left(\frac{\partial^2 \chi}{\partial x \partial y}\right) - \psi\frac{\partial^2 h}{\partial x^2} \cdot \left(\frac{\partial^2 \chi}{\partial y^2}\right)$$
$$-\psi\frac{\partial^2 h}{\partial y^2} \cdot \left(\frac{\partial^2 \chi}{\partial x^2}\right)\Big] dxdydt$$
$$= -2\int_Q (u-z)\chi dxdydt \qquad (18)$$

is valid.

Having written η instead of ψ in identity (17), χ instead of δu in identity (18) and subtracting them

$$\int_Q \Big[-\alpha(\delta h)\rho_f \cdot \frac{\partial u}{\partial t} \cdot \frac{\partial \psi}{\partial t} + \delta h \Delta u \Delta \psi$$
$$+(1-\nu)\Big(2\frac{\partial^2(\delta h)}{\partial x \partial y} \cdot \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2(\delta h)}{\partial x^2} \cdot \frac{\partial^2 u}{\partial y^2}$$
$$-\frac{\partial^2(\delta h)}{\partial y^2} \cdot \frac{\partial^2 u}{\partial x^2}\Big)\psi\Big] dxdydt$$
$$+ \int_Q \Big[-\alpha(\delta h)\rho_f \cdot \frac{\partial(\delta u)}{\partial t} \cdot \frac{\partial \psi}{\partial t} + \delta h \Delta(\delta u)\Delta\psi$$
$$+(1-\nu)\Big(2\frac{\partial^2(\delta h)}{\partial x \partial y} \cdot \frac{\partial^2(\delta u)}{\partial x \partial y} - \frac{\partial^2(\delta h)}{\partial x^2} \cdot \frac{\partial^2(\delta u)}{\partial y^2}$$
$$-\frac{\partial^2(\delta h)}{\partial y^2} \cdot \frac{\partial^2(\delta u)}{\partial x^2}\Big)\psi\Big] dxdydt$$
$$+2\int_Q (u-z)\delta u dxdydt = 0$$

From here, we obtain:

$$2\int_Q (u-z)\delta u dxdydt$$
$$= \int_Q \Big[-\alpha(\delta h)\rho_f \cdot \frac{\partial u}{\partial t} \cdot \frac{\partial \psi}{\partial t} + \delta h \Delta u \Delta \psi$$
$$+(1-\nu)\Big(2\frac{\partial^2(\delta h)}{\partial x \partial y} \cdot \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2(\delta h)}{\partial x^2} \cdot \frac{\partial^2 u}{\partial y^2}$$

$$-\frac{\partial^2(\delta h)}{\partial y^2}\cdot\frac{\partial^2 u}{\partial x^2}\bigg)\psi\bigg]\,dxdydt$$

$$+\int_Q\bigg[-\alpha(\delta h)\rho_f\cdot\frac{\partial(\delta u)}{\partial t}\cdot\frac{\partial\psi}{\partial t}+\delta h\Delta(\delta u)\Delta\psi$$

$$+(1-\nu)\bigg(2\frac{\partial^2(\delta h)}{\partial x\partial y}\cdot\frac{\partial^2(\delta u)}{\partial x\partial y}-\frac{\partial^2(\delta h)}{\partial x^2}\cdot\frac{\partial^2(\delta u)}{\partial y^2}$$

$$-\frac{\partial^2(\delta h)}{\partial y^2}\cdot\frac{\partial^2(\delta u)}{\partial x^2}\bigg)\psi\bigg]dxdydt.$$

Then the increment of the function $J(h)$ will be:

$$\delta J(h)=\int_Q\bigg[\alpha\rho_f(\delta h)\cdot\frac{\partial u}{\partial t}\cdot\frac{\partial\psi}{\partial t}-(\delta h)\Delta u\Delta\psi$$

$$-(1-\nu)\bigg(2\frac{\partial^2(\delta h)}{\partial x\partial y}\cdot\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial^2(\delta h)}{\partial x^2}\cdot\frac{\partial^2 u}{\partial y^2}$$

$$-\frac{\partial^2(\delta h)}{\partial y^2}\cdot\frac{\partial^2 u}{\partial x^2}\bigg)\psi\bigg]\,dxdydt$$

$$+2\beta\int_\Omega(h-\omega)\delta h dxdy+R$$

Here

$$R=R_1+R_2,$$

$$R_2=\int_Q\bigg[\alpha(\delta h)\rho_f\cdot\frac{\partial(\delta u)}{\partial t}\cdot\frac{\partial\psi}{\partial t}-\delta h\Delta(\delta u)\Delta\psi$$

$$-(1-\nu)\bigg[\bigg(2\frac{\partial^2(\delta h)}{\partial x\partial y}\cdot\frac{\partial^2(\delta u)}{\partial x\partial y}-\frac{\partial^2(\delta h)}{\partial x^2}\cdot\frac{\partial^2(\delta u)}{\partial y^2}$$

$$-\frac{\partial^2(\delta h)}{\partial y^2}\cdot\frac{\partial^2(\delta u)}{\partial x^2}\bigg)\psi\bigg]\bigg]dxdydt$$

denotes the residual, remainder term.

From the estimation R_1, R_2 following

$$|R|\le C\|\delta h\|^2_{W_2^3(\Omega)}.$$

Then, a necessary condition for the control $h(x,y)$ to be an optimal control is to satisfy the inequality

$$\delta J(h)=\int_Q\bigg\{\bigg[\alpha\rho_f\cdot\frac{\partial u}{\partial t}\cdot\frac{\partial\psi}{\partial t}-\delta h\Delta u\Delta\psi\bigg](\tilde h-h)$$

$$-(1-\nu)\bigg(2\frac{\partial^2(\tilde h-h)}{\partial x\partial y}\cdot\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial^2(\tilde h-h)}{\partial x^2}\cdot\frac{\partial^2 u}{\partial y^2}$$

$$-\frac{\partial^2(\tilde h-h)}{\partial y^2}\cdot\frac{\partial^2 u}{\partial x^2}\bigg)\psi\bigg\}dxdydt$$

$$\ge 0\ \forall\tilde h\in U_{ad}. \qquad (19)$$

So, we proved the following theorem

Theorem 3. A necessary condition for the control $h(x,y)$ be an optimal solution within the conditions imposed above on the data of problem (4)-(6) is to satisfy the inequality (19).

6 Conclusion

The paper first considers the general equation of a three-layer plate. Further, assuming that the upper and lower thicknesses of the plate are sufficiently small compared to the main thickness and taking the inertial force as the external force, the equations of a three-layer plate are reduced to a more convenient form of study.

We study a optimal control problem for an inhomogeneous equation of vibrations of a three-layer plate. At first, by the Feado-Galerkin method, we prove the existence of the boundary value problem for each fixed control, and prove the uniqueness by the standard method. Then a theorem on the existence of an optimal control was proved. The increment of the functional was calculated, and a necessary condition for optimality was derived.

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