New Structure of Intuitionistic Fuzzy Subgroups under Complex Realm

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Abstract: - Complex intuitionistic fuzzy sets (CIFS) can represent human information that contains uncertainty and periodicity semantics concurrently. In this research, the generalization concepts of CIF-subgroup (CIFSG) and anti-CIFSG of a given group are introduced, and their significance lies in assigning the grade of membership and non-membership functions in the complex plane. Also, CIFSG and anti-CIFSG are an inspiring expansion of the intuitionistic fuzzy subgroup (IFSG) and anti-IFSG. Our modification is highlighted in the phase term for membership and non-membership functions. Subsequently, a numerical example illustrates the inherent conditions of CIFSG is given. Also, we prove the relation between the complement of CIFSG and anti-CIFSG. After applying the operations $\diamondsuit$ and $\square$ to a CIFSG, the results are CIFSG. Also, the intersection of two CIFSGs is as well a CIFSG. Moreover, the complex normal intuitionistic fuzzy subgroup (CNIFSG) and its algebraic properties are presented and investigated. Some equivalent conditions to show that a CIFSG is CNIFSG are obtained.

Key-Words: - Fuzzy sets, Intuitionistic fuzzy set, Complex Intuitionistic fuzzy set, complex fuzzy subgroup, complex intuitionistic fuzzy subgroup, anti-complex intuitionistic fuzzy subgroup, and complex normal intuitionistic fuzzy subgroup.

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1 Introduction

The first result in the field of uncertainty sets is structured by the notion of fuzzy sets (FS) and its generalization intuitionistic fuzzy sets (IFS). FS and IFS are exceptionally good for modeling and simplifying a complex system that is inherited from different real-life situations. This modeling and

simplification are designed with minimum loss of information. Reference [1] suggested FS and reference [2] generalized FS's works and introduced IFS. An IFS has a membership function and non-membership function to represent a degree of belongingness and a degree of non-belongingness to a certain set for every element at the universe of discourse by using values lies in [0,1]. IFS has been applied in numerous fields including multi-criteria and attributes decision-making, railway train selection, evaluation of the utility values of the alternatives, classification, etc., [3], [4], [5], [6], [7]. The FS and IFS purely deal with one aspect of vagueness and uncertainties but are incapable to deal with two dimensions. Thus, A new kind of uncertainty set, named a complex fuzzy set (CFS) [8] was suggested, to decrease the amount of minimum loss of information in fuzzy set by stretching the image values of membership function from [0, 1] to $\{a/a \in \mathbb{C}: |a| \leq 1\}$. Several researchers generalized CFS approach in many fields. In [9], [10] some researchers studied and introduced each of hedges, linguistic variables, and many distances on CFS and CIFS. Also, relation with its properties were investigated. In [11], they incorporated CFS and a multi-fuzzy soft set. Researchers in [12] applied the CFS in the realm of prediction. Some other researchers established and studied the concept of complex fuzzy subgroups, [13], [14], [15].

CIFS was proposed as a generalization of CFS's work. A CIFS has membership and non-membership functions with a codomain that lies within the unit disk in $\mathbb{C}$. The extra range is helpful to represent the same information that has different meanings in different phases or affected factors with no wasting of full meaning of information. FSs, CFSs, and their generalizations were established to merge human knowledge and mathematical modeling in the engineering sciences and other fields. Both [16] and [17] introduced and discussed the notion of homomorphism in intuitionistic fuzzy multi-group situations. The notion of IFSG and its properties were first introduced in [18]. (α, β)-cut of IFG was exhibited, [19]. In 2020, [20], [21] some algebraic aspects of n-IFSG were studied. Also, the motivation of introducing the notion of homomorphism in an intuitionistic fuzzy multi-groups framework and examining its properties was presented in 2020. The idea of quasi-coincidence of a fuzzy point with an FS was successfully used in both, [22], [23].

The first definition of the concept of fuzzy subgroup was given in [24]. After that, many researchers generalized, discussed, and studied fuzzy subgroup's approach. In [25], the construction of fuzzy space was created. Therefore, several notions like fuzzy normal subgroups, fuzzy cosets concepts, and fuzzy subgroup Abelian group were presented in [26], [27].

In [28], [29] they introduced the conception of a fuzzy normal subgroup and a fuzzy coset. They proved that for a group G, a fuzzy subgroup is constant on the conjugate classes of G if and only if it is a fuzzy normal subgroup of G. Group theoretic notions were employed to the notion of IFS to create a new theory of IFSGs. In 2009 and 2010, [30], [31], they generalized the theory of intuitionistic fuzzy groups (IFG) and intuitionistic fuzzy normal subgroup (IFNSG) by following fuzzy group's Approach, which is different from IFSG. They introduce IFG and IFNSG depending on the perception of intuitionistic fuzzy space which is replaced by the universal set at conventional algebra. Some scholars have extensively revised IFSG and its properties, [32], [33], [34]. In [35], [36] The notion of Abelian IFSG and cyclic group under t-IFSG were intended, with their properties and relations examined. The notion of the intuitionistic fuzzy bi-ideal of a near ring was classified and reviewed. Also, their properties were researched in [37]. Also, some operations were generalized to the form of CAIFS, [38].

In 2017, Complex fuzzy subgroups were generalized and established and studied some notions and their characteristics. In 2020, as a more general approach, Authors denoted the amplitude and phase terms by FSs with images of values contained in [0,1] in the complex realm. They started their research by specifying the theory of complex fuzzy subgroups, the algebraic structure of complex fuzzy normal subgroups, and the complex fuzzy left (right) ideal. Then, they ended their research by introducing and studying the complex fuzzy subgroups theory. Also, in 2020, [39] introduced and studied the notion of anti-subgroups and normal anti-subgroups under CFS and s-norms. Recently, in reference [40], they initiated the concept of complex intuitionistic fuzzy subgroups. They introduced CIFSG by giving different conditions and studied the properties and relation of level subsets, the homomorphic image, and pre-image of a complex intuitionistic fuzzy set. Furthermore, the value of the phase term in [41] is a real valued, in contrast to the present work, we consider the phase term as a fuzzy value lies in $[0, 1]$. The expansion of the phase term from real value to fuzzy value helps us to solve the problem of directly translating an uncertainty of complex-valued physical quantities into words from natural

language [41], which can be solved by proposing several membership functions instead one membership function. In this research, CIFSG uses four membership functions (two of amplitude terms and two of phase terms) to describe different real-valued characteristics of this kind of quantity. Note that, both amplitude and phase terms carry intuitionistic fuzzy information. In 2022, [42] studied group regard to how complex fuzzy normal subgroup forms a group and establishes a regular homomorphism from group to its factor group regard to complex fuzzy normal subgroup. Also, [43] explored the notion of interval-valued spherical fuzzy (IVSF) sets (IVSFS), a modified version of intuitionistic and picture fuzzy sets. In 2022, [44] presented some operators as the CPF Hamacher prioritized averaging operator, CPF Hamacher prioritized ordered geometric operator and others.

Here, the concept of complex intuitionistic fuzzy subgroup (CIFSG) adds a unique viewpoint to the fields of CFS theory and algebraic structures, notably group theory. Traditional fuzzy and complex fuzzy subgroups are well-documented, but CIFSGs add an additional dimension of complexity by allowing for uncertainty and hesitation in subgroup membership. This exceptional idea stretches the applicability of fuzzy and complex fuzzy subgroups by providing a more advanced representation of imprecise and complex information inside group-theoretic settings. There is a wealth of research on fuzzy subgroups, complex fuzzy subgroups, Bipolar complex fuzzy subalgebras and ideals of BCK/BCI-algebras, [45]. The rise of CIFSG highlights the area's ongoing evolution by incorporating elements of intuitionistic logic, which improves its ability to model and solve real-world problems involving ambiguity and complex information. As a result, Complex intuitionistic fuzzy subgroups offer a viable route for investigating more complicated and subtle applications within the larger terrain of CFS theory and algebraic structures.

The motivation of this research can be highlighted by emerging both notions of CIFS and the theory of groups to create a modern group theory in the complex realm. The advantage that can be touched from this combination, i.e. combination complex fuzzy subgroup and intuitionistic fuzzy subgroup, is that both the amplitude and phase terms in each of the membership and non-membership functions of CIFS have values lies in [0, 1]. Group theory is playing an important rule to build the modern algorithm in the field of cryptography. Also, the wide presence of CIF information in multi applications requires the need to introduce the algebraic structure of the notion CIFSGs. Subsequently, the ability to represent CIF information encourages cryptography algorithms to have the proper ability to deal with this type of information. So, the future application should be a new algorithm using CIFSGs to crypt and encrypt this type of information. However, the concepts of the CIFSG and the anti-CIFSG of a given group are presented. Also, the relation that appears between the complement of CIFSG and anti-CIFSG is demonstrated. The intersection of the two CIFSG and the operations $\diamondsuit$ and $\square$ are applied to the notion of CIFSG, then the results also give a CIFSG. Furthermore, the formal definition of CNIFSG and its properties are expressed and studied. Besides, some equivalent conditions show that CIFSG is CNIFSG are achieved.

2 Preliminaries

This section recalls some main results associated with the presented works. This section is considered as a first step to combine and establish the notion of CIFSGs.

Definition 2.1. [1] Let U be a nonempty set. A fuzzy set is just a function mapping any element in U into [0, 1]

Definition 2.2. [9]. A complex intuitionistic fuzzy set A, defined by $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \epsilon X\}$, where

$$\mu_A(x): X \to \{a | a \in \mathbb{C},\ |a| \leq 1\},$$
$$\gamma_A(x): X \to \{\acute{a} | \acute{a} \in \mathbb{C},\ |\acute{a}| \leq 1\},$$

and

$$|\mu_A(x) + \gamma_A(x)| \leq 1,$$

and $i = \sqrt{-1}$, each of $r_A(x)$, $k_A(x)$ belong to the interval [0, 1] such that $0 \leq r_A(x) + k_A(x) \leq 1$, also $w^r{}_A(x)$ and $w^k{}_A(x)$ are real-valued.

Definition 2.3. Let $A = \{\langle x, \mu_A(o), \gamma_A(o)\rangle : o \epsilon O\}$ be a complex intuitionistic fuzzy set. Define the complement of A, $c(A)$, as

$$c(A) = \ddot{A} = \{\langle x, \gamma_A(o), \mu_A(o)\rangle : x \epsilon X\}$$
$$\left\{\langle x, k_A(o).e^{i(w^k{}_A(o))}, r_A(o)e^{i(w^r{}_A(o))}\rangle : o \epsilon O\right\}$$

, where $w^k{}_A(o) = w_A(o), 2\pi - w_A(o)$ or $w_A(o) + \pi$.

Definition 2.4. [9] Let $\mu_A(o) = r_A(o)e^{i(w^r{}_A(o))}$, $\gamma_A(o) = k_A(o).e^{i(w^k{}_A(o))}$, and $\mu_B(o) = r_B(o)e^{i(w^r{}_B(o))}$, $\gamma_B(o) = k_B(o).e^{i(w^k{}_B(o))}$ be two membership and non-membership functions of complex intuitionistic fuzzy sets A and B respectively, on O.

B is subset of A, "$A \supseteq B$ or $B \subseteq A$", if for any $o \in O$, $r_A(o) \leq r_B(o), k_A(o) \geq k_B(o)$,

$$\omega^r{}_A(o) \leq \omega^r{}_B(o), and \omega^k{}_A(o) \geq \omega^k{}_B(o).$$

A union B, $A \cup B$, as

$$A \cup B = \{x, max(r_A(o), r_B(o))e^{i\,max(\omega^r{}_A(o),\omega^r{}_B(o))}, min(k_A(o), k_B(o))e^{i\,min(\omega^k{}_A(o),\omega^k{}_B(o))}\}.$$

A intersection B, denoted by $A \cap B$, as

$$A \cap B = \{x, min(r_A(o), r_B(o))e^{i\,min(\omega^r{}_A(o),\omega^r{}_B(o))}, max(k_A(o), k_B(o))e^{i\,max(\omega^k{}_A(o),\omega^k{}_B(o))}\}.$$

Proposition 2.1 [9]. If A and B are two CIFS of E, then

$i.\ (\ddot{\ddot{A}}) = A$

$ii.\ (A \ddot{\cup} B) = (\ddot{A} \cap \ddot{B})$

$iii.\ (A \ddot{\cap} B) = (\ddot{A} \cup \ddot{B})$

Definition 2.5. [38]. For every CAIFS A, the following two operations transform every CAIFS into CFS form. The operations $\diamondsuit$ and $\square$ to a CIFSG

$$\square A = \{\langle x, r_A(x)e^{i(w^r{}_A(x))}\rangle : x \epsilon X\} = \{\langle x, r_A(x)e^{i(w^r{}_A(x))}, 1 - r_A(x)e^{i2\pi(1-w^r{}_A(x))}\rangle : x \epsilon X\}$$

$$\square A = \left\{\langle x, k_A(x).e^{i\left(w^k{}_A(x)\right)}\rangle : x \epsilon X\right\} = \left\{\left\langle x, 1 - k_A(x).e^{i\left(w^k{}_A(x)\right)}e^{i2\pi\left(1-w^k{}_A(x)\right)}, k_A(x).e^{i\left(w^k{}_A(x)\right)}\right\rangle : x \epsilon X\right\}$$

Definition. 2.6. [18] Let $A = (A, \mu_A, \vartheta_A)$ be IFS, G be a group. Then A is called an IFSG of G if

$$\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}$$
$$\mu_A(x^{-1}) = \mu_A(x)$$
$$\vartheta_A(xy) \leq \max\{\vartheta_A(x), \vartheta_A(y)\}$$
$$\vartheta_A(x^{-1}) = \vartheta_A(x), \text{ are satisfied } \forall\, x, y \in G.$$

Definition 2.7. [32] Let G be a group, An IFSG $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \epsilon G\}$ of G is called intuitionistic fuzzy normal subgroup (IFNSG) of G if

(i) $\mu_A(xy) = \mu_A(yx)$ (ii) $\gamma_A(xy) = \gamma_A(yx)$ for all $x, y \epsilon G$.

Definition 2.8. [40] Let A be an IFSG over group G, If

$$\mu_A(op) \leq min\{\mu_A(o), \mu_A(p)\}, and\ \nu_A(op) \geq max\{\nu_A(o), \nu_A(p)\}, \forall\, o, p \in G, \quad (1)$$

$$\mu_A(o^{-1}) \leq \mu_A(o), \nu_A(o^{-1}) \geq \nu_A(o), \text{satisfied } \forall\, o \in G \quad (2)$$

Then A is said to be an intuitionistic anti-fuzzy subgroup on G and IAFG $[G]$ is the set that contains all intuitionistic anti-fuzzy subgroups on G.

Definition 2.9. [13] Let O be a group and $A = \{(o, \mu_A(o)) : o \in O\}$ be a homogeneous CFS. Then A is a complex fuzzy subgroup of G if:

$\mu_A(op) \geq \min\{\mu_A(o), \mu_A(p)\}$, for all o, p in O.

$\mu_A(o^{-1}) \geq \mu_A(o)$, hold for all o in G.

Definition 2.10. [39] Let G be a group and μ be a CFS on G. Next, under s-norm S, $\mu = r_A(x)e^{i(w^r{}_A(x))}$ is an anti complex fuzzy subgroup of G if,

$r_A(xy) \leq S\{r_A(x), r_A(y)\}$,

$$r_A(x^{-1}) \leq r_A(x)$$

$w_A(xy) \leq max\{w_A(x), w_A(y)\}$,

$w_A(x^{-1}) \leq w_A(x)$, hold $\forall$ x, y $\in$ G.

Note. ACFS (G) is the set containing all anti-complex fuzzy subgroups of G under s-norm S.

Definition 2.11. [5] A complex fuzzy subgroup $A = \{(x, \mu_A(x)) : x \in G\}$ can be formed a normal complex fuzzy subgroup of G if $\mu_A(xy) = \mu_A(xy)$.

3 Complex Intuitionistic Fuzzy Subgroups

The formal notion of complex intuitionistic fuzzy subgroup (CIFSG) and anti-complex intuitionistic fuzzy subgroup of a group G is defined. The structure of Definition 3.1, Definition 3.2, and Definition 4.1 contain the parameter α, where α ∈ (0, 2π]. We consider α equal to 2π in this research. The parameter α helps to keep the recognizing of the phase's values lie in the unit disk and the interval (0, 2π]. The existing of the parameter α supports that the phase terms may indicate the IF information, and in this illustration case the phase terms ∈ [0, 1] and fulfilled IFS restriction. Furthermore, an explicit example is illustrated.

Definition 3.1. Let G be a group, and $A = \left(x, s_A(x)e^{i\alpha w^s_A(x)}, l_A(x)e^{i\alpha w^l_A(x)} : x \in G\right)$ be a CIFSG of G. Then A is called a CIFSG of G if the following are satisfied $\forall x, y \epsilon G$.

$$s_A(xy)e^{i\alpha w^s_A(xy)} \tilde{\geq} \min\{s_A(x), s_A(y)\}e^{i\alpha \min\{w^s_A(x), w^s_A(y)\}},$$

where the operation "$\tilde{\geq}$" is defined by $s_A(xy) \geq \min\{s_A(x), s_A(y)\}$,and

$$w^s_A(xy) \geq \min\{w^s_A(x), w^s_A(y)\}.$$

$s_A(x^{-1})e^{i\alpha w^s_A(x^{-1})} = s_A(x)e^{i\alpha w^s_A(x)}$,

where $s_A(x^{-1}) = s_A(x)$, and $w_A^S(x^{-1}) = w_A^S(x)$.

$$l_A(xy)e^{i\alpha w_A^l(xy)} \tilde{\leq} \max\{l_A(x), l_A(y)\}e^{i\alpha\max\{w_A^l(x),w_A^l(y))\}}$$

, where the operation "$\tilde{\leq}$" is defined by $l_A(xy) \leq \max\{l_A(x), l_A(y)\}$, and $w_A^l(xy) \leq \max\{w_A^l(x), w_A^l(y)\}$.
$l_A(x^{-1})e^{i\alpha w_A^l(x^{-1})} = l_A(x)e^{i\alpha w_A^l(x)}$,
where, $l_A(x^{-1}) = l_A(x)$, and $w_A^l(x^{-1}) = w_A^l(x)$.

Example 3.1. Let $G = \{id, a, b, c: a^2 = b^2 = c^2 = id, ab = c, ac = b, bc = a\}$ be a group, where "id" is the identity element, with complex membership values for a set CIFS A is given as follow:

$$A = \left\{\left(id, 0.9e^{i\alpha 0.8,0.1e^{i\alpha 0.2}}\right), \left(a, 0.2e^{i\alpha 0.5,0.3e^{i\alpha 0.4}}\right) \left(b, 0.5e^{i\alpha 0.7,0.3e^{i\alpha 0.2}}\right)\left(c, 0.2e^{i\alpha 0.4,0.2e^{i\alpha 0.4}}\right)\right\}$$

Then the CIFS A forms a CIFSG of G.

Theorem 3.1.
If $A = \left(x, s_A(x)e^{i\alpha w_A^S(x)}, l_A(x)e^{i\alpha w_A^l(x)}: x \in G\right)$, is a CIFG of group G, then $\forall x \epsilon G$, and the identity element denoted by "id", we have

$s_A(id)e^{i\alpha w_A^S(id)} \tilde{\geq} s_A(x)e^{i\alpha w_A^S(x)}(x), \forall x \epsilon G.$
$l_A(id)e^{i\alpha w_A^l(id)} \tilde{\leq} l_A(x)e^{i\alpha w_A^l(x)}(x), \forall x \epsilon G,$

Proof. For (i), Let us start by L.H.S. $= s_A(id)e^{i\alpha w_A^S(id)} = s_A(xx^{-1})e^{i\alpha w_A^S(xx^{-1})}$

$\tilde{\geq} \min\{s_A(x), s_A(x^{-1})\}e^{i\alpha\min\{w_A^S(x),w_A^S(x^{-1})\}}$

$= \min\{s_A(x), s_A(x)\}e^{i\alpha\min\{w_A^S(x),w_A^S(x)\}}$

$= s_A(x)e^{i\alpha w_A^S(x)}$ = R.H.S.
For (ii), we have L.H.S. $= l_A(id)e^{i\alpha w_A^l(id)}$
$= l_A(xx^{-1})e^{i\alpha w_A^l(xx^{-1})}$

$\tilde{\leq} \max\{l_A(x), l_A(x^{-1})\}e^{i\alpha\max\{w_A^l(x),w_A^l(x^{-1})\}}$

$= \max\{l_A(x), l_A(x)\}e^{i\alpha\min\{w_A^l(x),w_A^l(x)\}}$

$= l_A(x)e^{i\alpha w_A^l(x)}$ = R.H.S.

Theorem 3.2. Let G be a group, If A is a CIFSG of G, then ∀ x, y ∈ G.
If $s_A(xy)e^{i\alpha w_A^S(xy)} = s_A(id)e^{i\alpha w_A^S(id)}$, and $l_A(xy)e^{i\alpha w_A^l(xy)} = l_A(id)e^{i\alpha w_A^l(id)}$,then
$s_A(x)e^{i\alpha w_A^S(x)} = s_A(y)e^{i\alpha w_A^S(y)}$and
$l_A(x)e^{i\alpha w_A^l(x)} = l_A(y)e^{i\alpha w_A^l(y)}$.

Proof. For the membership function, we have

$s_A(x)e^{i\alpha w_A^S(x)}$
$= s_A(xyy^{-1})e^{i\alpha w_A^S(xyy^{-1})} \tilde{\geq} \min(s_A(xy), s_A(y^{-1}))e^{i\alpha\min(w_A^S(xy),w_A^S(y^{-1}))}$

$= \min(s_A(id), s_A(y))e^{i\alpha\min(w_A^S(id),w_A^S(y))}$

$= s_A(y)e^{i\alpha w_A^S(y)}$.

Then, $s_A(x)e^{i\alpha w_A^S(x)} \tilde{\geq} s_A(y)e^{i\alpha w_A^S(y)}$.
On the other hand $s_A(y)e^{i\alpha w_A^S(y)} = s_A(x^{-1}xy)e^{i\alpha w_A^S(x^{-1}xy)} \tilde{\geq} \min(s_A(x^{-1}), s_A(xy))e^{i\alpha\min(w_A^S(x^{-1}),w_A^S(xy))}$

$= \min(s_A(x), s_A(id))e^{i\alpha\min(w_A^S(x),w_A^S(id))}$

$= s_A(x)e^{i\alpha w_A^S(x)}$.

Then, $s_A(y)e^{i\alpha w_A^S(y)} \tilde{\geq} s_A(x)e^{i\alpha w_A^S(x)}$
Now, for the non-membership function, we have

$l_A(x)e^{i\alpha w_A^l(x)} = l_A(xyy^{-1})e^{i\alpha w_A^l(xyy^{-1})} \tilde{\leq} \max(l_A(xy), l_A(y^{-1}))e^{i\alpha\max(w_A^l(xy),w_A^l(y^{-1}))}$

$= \max(l_A(id), l_A(y))e^{i\alpha\max(w_A^l(id),w_A^l(y))}$

$= l_A(y)e^{i\alpha w_A^l(y)}$

Then, $l_A(x)e^{i\alpha w_A^l(x)} \tilde{\leq} l_A(y)e^{i\alpha w_A^l(y)}$.
On the other hand $l_A(y)e^{i\alpha w_A^l(y)}$

$= l_A(x^{-1}xy)e^{i\alpha w_A^l(x^{-1}xy)} \tilde{\leq} \max(l_A(x^{-1}), l_A(xy))e^{i\alpha\max(w_A^l(x^{-1}),w_A^l(xy))}$

$= \max(l_A(x), l_A(id))e^{i\alpha\max(w_A^l(x),w_A^l(id))}$

$= l_A(x)e^{i\alpha w_A^l(x)}$.

Then, $l_A(y)e^{i\alpha w_A^l(y)} \tilde{\leq} l_A(x)e^{i\alpha w_A^l(x)}$
Thus By (1), (2), (3), and (4), we get $s_A(x)e^{i\alpha w_A^S(x)} = s_A(y)e^{i\alpha w_A^S(y)}$
and $l_A(x)e^{i\alpha w_A^l(x)} = l_A(y)e^{i\alpha w_A^l(y)}$.

Definition 3.2. Let G be a group. A complex intuitionistic fuzzy subset A of G is called an anti-complex intuitionistic fuzzy subgroup of G if $\forall x, y \epsilon G$.

$$s(xy)e^{i\alpha w_A^S(xy)} \tilde{\leq} \max\{s_A(x), s_A(y)\}e^{i\alpha\max\{w_A^S(x),w_A^S(y)\}}$$

, where the operation "$\tilde{\leq}$" is defined by $s_A(xy) \leq \max\{s_A(x), s_A(y)\}$, and
$w_A^S(xy) \leq \max\{w_A^S(x), w_A^S(y)\}$.
$s_A(x^{-1})e^{i\alpha w_A^S(x^{-1})} = s(x)e^{i\alpha w_A^S(x)}$,where
$s_A(x^{-1}) = s(x)$ and $w_A^S(x^{-1}) = w_A^S(x)$.

$$l_A(xy)e^{i\alpha w_A^l(xy)} \tilde{\geq} \min\{l_A(x),\ l_A(y)\}e^{i\alpha \min\{w_A^l(x), w_A^l(y)\}}$$

, where the operation "$\tilde{\geq}$" is defined by $l_A(xy) \geq \min\{l_A(x), l_A(y)\}$,
and $w_A^l(xy) \geq \min\{w_A^l(x), w_A^l(y)\}$.
$l_A(x^{-1})e^{i\alpha w_A^l(x^{-1})} = l_A(x)e^{i\alpha w_A^l(x)}$,
where $l_A(x^{-1}) = l(x)$ and $w_A^l(x^{-1}) = w_A^l(x)$.

Proposition 3.1.
If $A = \left(x, s_A(x)e^{i\alpha w_A^S(x)}, l_A(x)e^{i\alpha w_A^l(x)} : x \in G\right)$ is a CIFG of group G, then $s_A(x)e^{i\alpha w_A^l(x)}$ is a CFSG of G, and $l_A(x)e^{i\alpha w_A^l(x)}$ is an anti-CFSG of G.

Proof. Trivial.

Proposition 3.2. If A is a CIFG of a group G, then $(\ddot{A})$ is an anti CIFSG of G.

Proof. Let
$A = \left\{\left(x, s_A(x)e^{i\alpha w_A^S(x)}, l_A(x)e^{i\alpha w_A^l(x)}\right) \middle| x \in G\right\}$ is a CIFSG of group G, then
$\ddot{A} = \left\{\left(x, l_A(x)e^{i\alpha w_A^l(x)}, s_A(x)e^{i\alpha w_A^S(x)}\right) \middle| x \in G\right\}$
For condition (i) in Definition 3.1. We need to show

$l(xy)e^{i\alpha w_A^l(xy)} \tilde{\geq} \min\{l_A(x), l_A(y)\}e^{i\alpha \min\{w_A^l(x), w_A^l(y)\}}$,
Which is satisfied condition (iii) in Definition 3.2.
For condition (iii) in Definition 3.1. We need to show

$$s_A(xy)e^{i\alpha w_A^S(xy)} \tilde{\leq} \max_{e^{i\alpha \max\{w_A^S(x), w_A^S(y)\}}} \{s_A(x), s_A(y)\}$$

Which is satisfied condition (i) in Definition 3.2. For condition (ii) and (iv) in Definition 3.1. It's clear that similar condition (iv) and (ii), respectively in Definition 3.2. This completes the proof.

Theorem 3.3. An alternative condition to show that a CIFS A of a group G to be a CIFSG of G is

$$s_A(xy^{-1})e^{i\alpha w_A^S(xy^{-1})} \tilde{\geq} \min\{s_A(x), s_A(y)\}e^{i\alpha \min\{w_A^S(x), w_A^S(y)\}}$$

,and

$$l_A(xy^{-1})e^{i\alpha w_A^l(xy)} \tilde{\leq} \max\{l_A(x), l_A(y)\}e^{i\alpha max\{w_A^l(x), w_A^l(y)\}}$$

, $\forall x, y \in G$.

Proof.
For membership function, we have:
L. H. S. $= s_A(xy^{-1})e^{i\alpha w_A^S(xy^{-1})}$
$\tilde{\geq} \min\{s_A(x), s_A(y^{-1})\}e^{i\alpha \min\{w_A^S(x), w_A^S(y^{-1})\}}$,
$=$
$\min\{s_A(x), s_A(y)\}e^{i\alpha \min\{w_A^S(x), w_A^S(y)\}}$ = R. H. S.

For non-membership function, we have:
L.H.S. $= l_A(xy^{-1})e^{i\alpha w_A^l(xy^{-1})} \tilde{\leq}$
$\max\{l_A(x), l_A(y^{-1})\}e^{i\alpha \max\{w_A^l(x), w_A^l(y^{-1})\}}$,

$= \max\{l_A(x), l_A(y)\}e^{i\alpha \max\{w_A^l(x), w_A^l(y)\}}$
= R. H. S.

Theorem 3.4. Let G be a group, IF A and B are two CIFSGs of G, so A ∩ B is also a CIFSG of G.

Proof. Clearly, $s_A(x)e^{i\alpha w_A^S(x)}$ and $s_B(x)e^{i\alpha w_B^S(x)}$ are complex fuzzy subgroups of G.
Then $s_A(x)e^{i\alpha w_A^S(x)} \cap s_B(x)e^{i\alpha w_B^S(x)}$ is also so.

Again $l_A(x)e^{i\alpha w_A^l(x)}$ and $l_B(x)e^{i\alpha w_B^l(x)}$, are anti complex fuzzy subgroups of G.
Then $l_A(x)e^{i\alpha w_A^l(x)} \cap l_B(x)e^{i\alpha w_B^l(x)}$ is also so.
Then
$(A \cap B, s_A(x)e^{i\alpha w_A^S(x)} \cap s_B(x)e^{i\alpha w_B^S(x)}, l_A(x)e^{i\alpha w_A^l(x)} \cap l_B(x)e^{i\alpha w_B^l(x)})$ is a CIFG of G

Theorem 3.5. Let G be a group, If A is a CIFSG of G, so

i. □ A is a CIFG of G.
ii. ◇ A is a CIFG of G.

Proof.
Let $A = \left\{\left(x, s_A(x)e^{i\alpha w_A^S(x)}, l_A(x)e^{i\alpha w_A^l(x)}\right) \middle| x \in G\right\}$ is a CIFSG of group G, then
$\square A = \left\{\left(x, s_{\square A}(x)e^{i\alpha w_{\square A}^S(x)} = s_A(x)e^{i\alpha w_A^S(x)}, l_{\square A}(x)e^{i\alpha w_{\square A}^l(x)} = (1 - s_A(x))e^{i\alpha(1-w_A^S(x))}\right)\right\}$ and
$\diamond A = \left\{\left(x, s_{\diamond A}(x)e^{i\alpha w_{\diamond A}^S(x)} = \left(1 - l_A(x)\right)e^{i\alpha\left(1-w_A^l(x)\right)}, l_{\diamond A}(x)e^{i\alpha w_{\diamond A}^l(x)} = l_A(x)e^{i\alpha w_A^l(x)}\right)\right\}$.

For (i), $\boxdot A$ have same membership of a CIFSG A, so the first two conditions in definition 3.1 are fulfilled. So, we need to show the non-membership part only,

For condition (iii) in definition 3.1. We have:

$$l_{\boxdot A}(xy)e^{i\alpha w^{l}_{\boxdot A}(xy)} = \big(1 - s_A(xy)\big)e^{i\alpha(1-w^{s}_{A}(xy))}$$
$$\tilde{\leqq} (1 - \min\{s_A(x), s_A(y)\})e^{i\alpha(1-\min\{w^{s}_{A}(x),w^{s}_{A}(y)\})},$$
$$= \max\{1 - s_A(x), 1 - s_A(y)\}e^{i\alpha \max\{1-w^{s}_{A}(x),1-w^{s}_{A}(y)\}}.$$
$$= \max\{l_{\boxdot A}(x), l_{\boxdot A}(y)\}e^{i\alpha \max\{w^{l}_{\boxdot A}(x),w^{l}_{\boxdot A}(y)\}}.$$

Also, for condition (iv) in definition 3.1., we have,

$$l_{\boxdot A}(x^{-1})e^{i\alpha w^{l}_{\boxdot A}(x^{-1})} = \big(1 - s_A(x^{-1})\big)e^{i\alpha(1-w^{s}_{A}(x^{-1}))}$$
$$= \big(1 - s_A(x)\big)e^{i\alpha(1-w^{s}_{A}(x))},$$
$$= l_{\boxdot A}(x)e^{i\alpha w^{l}_{\boxdot A}(x)}.$$

For (ii), $\diamondsuit A$ have same non-membership of a CIFSG A, so the third and fourth conditions in definition 3.1 are satisfied. So, we need to show only the membership part. The proof is like (i). This completes the proof.

4 Normality of Subgroup under CIFS

The normal subgroup definition is generalized to be in the realm of CIFS. Also, some of equivalent forms required to show CNIFSG are obtained. Besides some theorems support the new structure of CNIFSG are proved in this section.

Definition 4.1.

If $A = \left(x, s_A(x)e^{i\alpha w^{s}_{A}(x)}, l_A(x)e^{i\alpha w^{l}_{A}(x)} : x \in G\right)$ be a CIFSG of G, then A is called complex normal intuitionistic fuzzy subgroup of G if the following are satisfied:

$s_A(x^{-1}yx)e^{i\alpha w^{s}_{A}(x^{-1}yx)} \tilde{\geqq} s_A(y)e^{i\alpha w^{s}_{A}(y)}$, and $l_A(x^{-1}yx)e^{i\alpha w^{l}_{A}(x^{-1}yx)} \tilde{\leqq} l_A(y)e^{i\alpha w^{l}_{A}(y)}$ for all x, y ∈ G.

Example 4.1. Consider the cyclic group $Z_4 = \{0, 1, 2, 3\}$, where the operation is addition modulo 4. We can define a complex intuitionistic fuzzy subset of Z_4 based on membership μ and non-membership ν functions as follows.

$$s_A(0)e^{i\alpha w^{s}_{A}(0)} = 1e^{\alpha(1)}, l_A(0)e^{i\alpha w^{l}_{A}(0)} = 0e^{\alpha(0)},$$
$$s_A(1)e^{i\alpha w^{s}_{A}(1)} = 0.7e^{\alpha(0.4)}, l_A(1)e^{i\alpha w^{l}_{A}(1)} = 0.2e^{\alpha(0.5)},$$
$$s_A(2)e^{i\alpha w^{s}_{A}(2)} = 0.4e^{\alpha(0.7)}, l_A(2)e^{i\alpha w^{l}_{A}(2)} = 0.5e^{\alpha(0.2)},$$
$$s_A(3)e^{i\alpha w^{s}_{A}(3)} = 0.7e^{\alpha(0.6)}, l_A(3)e^{i\alpha w^{l}_{A}(3)} = 0.2e^{\alpha(0.3)}.$$

Thus, the set defined by $s_A(x)e^{i\alpha w^{s}_{A}(x)}$ and $l_A(x)e^{i\alpha w^{l}_{A}(x)}$ satisfies the complex intuitionistic fuzzy subgroup property. In an abelian group like Z_4 , all subgroups are normal. Since Z_4 is commutative, Therefore, for all $y \in Z_4$ and $\in Z_4$, the complex membership and non-membership functions are automatically preserved because:

$s_A(x + y - x)e^{i\alpha w^{s}_{A}(x+y-x)} = s_A(y)e^{i\alpha w^{s}_{A}(y)}$, and $l_A(x + y - x)e^{i\alpha w^{l}_{A}(x+y-x)} = l_A(y)e^{i\alpha w^{l}_{A}(y)}$, for all $y, x \in Z_4$.

This property ensures that the complex intuitionistic fuzzy subgroup is normal.

Corollary 4.1.

If $A = \left(x, s_A(x)e^{i\alpha w^{s}_{A}(x)}, l_A(x)e^{i\alpha w^{l}_{A}(x)} : x \in G\right)$ be an CIFSG of G, then

i. $s_A(x^{-1}yx)e^{i\alpha w^{s}_{A}(x^{-1}yx)} \tilde{\geqq} s_A(y)e^{i\alpha w^{s}_{A}(y)}$, and $l_A(x^{-1}yx)e^{i\alpha w^{l}_{A}(x^{-1}yx)} \tilde{\leqq} l_A(y)e^{i\alpha w^{l}_{A}(y)}$ for all x, y ∈ G.

ii. $s_A(x^{-1}yx)e^{i\alpha w^{s}_{A}(x^{-1}yx)} = s_A(y)e^{i\alpha w^{s}_{A}(y)}$, and $l_A(x^{-1}yx)e^{i\alpha w^{l}_{A}(x^{-1}yx)} = l_A(y)e^{i\alpha w^{l}_{A}(y)}$ for all x, y ∈ G.

iii. $s_A(xy)e^{i\alpha w^{s}_{A}(xy)} = s_A(yx)e^{i\alpha w^{s}_{A}(yx)}$, and $l_A(xy)e^{i\alpha w^{l}_{A}(xy)} \tilde{\leqq} l_A(yx)e^{i\alpha w^{l}_{A}(yx)}$ for all x, y ∈ G. Are equivalent.

Proof: Trivial.

Theorem 4.1.

Let $A = \left(x, s_A(x)e^{i\alpha w^{s}_{A}(x)}, l_A(x)e^{i\alpha w^{l}_{A}(x)} : x \in G\right)$ be a CIFSG of G. Then A is a complex normal intuitionistic fuzzy subgroup if and only if $s_A(xy\,(yx)^{-1})e^{i\alpha w^{s}_{A}(xy\,(yx)^{-1})} \tilde{\geqq} s_A(y)e^{i\alpha w^{s}_{A}(y)}$, and $l_A(xy\,(yx)^{-1})e^{i\alpha w^{l}_{A}(xy\,(yx)^{-1})} \tilde{\leqq} l_A(y)e^{i\alpha w^{l}_{A}(y)}$ for all x, y ∈ G.

Proof: (⇒) Let that A is a complex normal intuitionistic fuzzy subgroup, let x, y ∈ G then:-

For the membership function we have:
$s_A(xy\,(yx)^{-1})e^{i\alpha w_A^S(xy\,(yx)^{-1})} =$
$s_A(xy\,x^{-1}y^{-1})e^{i\alpha w_A^S(xy\,x^{-1}y^{-1})}$

$\tilde{\geq} \min(s_A(xy\,x^{-1}), s_A(y^{-1}))\; e^{i\alpha min(w_A^S(xy\,x^{-1}), w_A^S(y^{-1}))}$
$=$
$\min(s_A(y^{-1}), s_A(y^{-1}))\; e^{i\alpha min(w_A^S(y^{-1}), w_A^S(y^{-1}))}$

$= s_A(y^{-1})\; e^{i\alpha w_A^S(y^{-1})}$
$= s_A(y)\; e^{i\alpha w_A^S(y)}$, $\forall$ x, y $\in$ G.

For the non-membership function we have:
$l_A(xy\,(yx)^{-1})e^{i\alpha w_A^l(xy\,(yx)^{-1})} =$
$l_A(xy\,x^{-1}y^{-1})e^{i\alpha w_A^l(xy\,x^{-1}y^{-1})}$

$\tilde{\leq} \max(l_A(xy\,x^{-1}), s_A(y^{-1}))\; e^{i\alpha max(w_A^l(xy\,x^{-1}), w_A^S(y^{-1}))}$
$=$
$\max(l_A(y^{-1}), s_A(y^{-1}))\; e^{i\alpha max(w_A^l(y^{-1}), w_A^l(y^{-1}))}$
$= l_A(y^{-1})\; e^{i\alpha w_A^l(y^{-1})}$
$= l_A(y)\; e^{i\alpha w_A^l(y)}$, $\forall$ x, y $\in$ G.
($\Leftarrow$) for any x, y $\in$G, Let the inequality holds on A.

For the membership function we have:
$s_A(xy\,x^{-1})e^{i\alpha w_A^S(xy\,x^{-1})}$
$= s_A(xy\,x^{-1}y^{-1}y)e^{i\alpha w_A^S(xy\,x^{-1}y^{-1}y)}$

$\tilde{\geq} \min(s_A(xy\,x^{-1}y^{-1})$
, $s_A(y))\; e^{i\alpha min(w_A^S(xy\,x^{-1}y^{-1}), w_A^S(y))}$

$\tilde{\geq} \min(s_A(y), s_A(y))\; e^{i\alpha min(w_A^S(y), w_A^S(y))}$
$= s_A(y)\; e^{i\alpha w_A^S(y)}$, $\forall$ x, y $\in$ G.

For the non-membership function, we have:
$l_A(xy\,x^{-1})e^{i\alpha w_A^l(xy\,x^{-1})}$
$= l_A(xy\,x^{-1}y^{-1}y)e^{i\alpha w_A^l(xy\,x^{-1}y^{-1}y)}$

$\tilde{\leq} \max(l_A(xy\,x^{-1}y^{-1})$
$, l_A(y))\; e^{i\alpha max(w_A^l(xy\,x^{-1}y^{-1}), w_A^l(y))}$

$\tilde{\leq} \max(s_A(y), s_A(y))\; e^{i\alpha max(w_A^l(y), w_A^l(y))}$
$= l_A(y)\; e^{i\alpha w_A^l(y)}$, $\forall$ x, y $\in$ G.

Then A is a CNIFSG of G.

Theorem 4.2. Let A be a complex intuitionistic fuzzy subgroup of G, and $s_A(x^2)e^{i\alpha w_A^S(x^2)} = s_A(id)e^{i\alpha w_A^S(id)}$ and $l_A(x^2)e^{i\alpha w_A^l(x^2)} = l_A(id)e^{i\alpha w_A^l(id)}$ for all x $\in$ G. Then A is a CNIFSG.

Proof. Let x, y $\in$ G, and A be a CIFSG of G and $s_A(x^2)e^{i\alpha w_A^S(x^2)} = s_A(id)e^{i\alpha w_A^S(id)}$ then for the membership function, we have:
$s_A(id)e^{i\alpha w_A^S(id)} = s_A((xy)^2)e^{i\alpha w_A^S((xy)^2)} =$
$s_A(xyxy)e^{i\alpha w_A^S(xyxy)}$
$=$
$s_A((xyx^{-1})x^2y)e^{i\alpha w_A^S((xyx^{-1})x^2y)}$

Thus it follows, by Theorem 3.2 that is
$s_A(x^{-1}yx)e^{i\alpha w_A^S(x^{-1}yx)} = s_A(x^2y)e^{i\alpha w_A^S(x^2y)}$
$\tilde{\geq} \min(s_A(x^2), s_A(y))e^{i\alpha \min(w_A^S(x^2), w_A^S(y))}$

$\tilde{\geq} \min(s_A(id), s_A(y))e^{i\alpha \min(w_A^S(id), w_A^S(y))}$

$= \min(s_A(id), s_A(y))e^{i\alpha \min(w_A^S(id), w_A^S(y))}$
$= s_A(y)e^{i\alpha(w_A^S(y))}$

Therefore,
$s_A(x^{-1}yx)e^{i\alpha w_A^S(x^{-1}yx)} \tilde{\geq} s_A(y)e^{i\alpha(w_A^S(y))}$.
Now, Let $x, y \in G$, and A be a CIFSG of G and $l_A(x^2)e^{i\alpha w_A^l(x^2)} = l_A(id)e^{i\alpha w_A^l(id)}$ then for the non-membership function, we have
$l_A(id)e^{i\alpha w_A^l(id)} = l_A((xy)^2)e^{i\alpha w_A^l((xy)^2)}$
$= l_A(xyxy)e^{i\alpha w_A^l(xyxy)}$

$= l_A((xyx^{-1})x^2y)e^{i\alpha w_A^l((xyx^{-1})x^2y)}$

Thus it follows, by Theorem 3.2 that is
$l_A(x^{-1}yx)e^{i\alpha w_A^l(x^{-1}yx)} = l(x^2y)e^{i\alpha w_A^l(x^2y)} \tilde{\leq}$
$\max(l_A(x^2), l_A(y))e^{i\alpha \max(w_A^l(x^2), w_A^l(y))}$

$\tilde{\leq} \max(l_A(id), l_A(y))e^{i\alpha \max(w_A^l(id), w_A^l(y))}$

$= \max(l_A(id), l_A(y))e^{i\alpha \max(w_A^l(id), w_A^l(y))}$

$= l_A(y)e^{i\alpha(w_A^l(y))}$,
therefore
$l_A(x^{-1}yx)e^{i\alpha w_A^l(x^{-1}yx)} \tilde{\leq} l_A(y)e^{i\alpha(w_A^l(y))}$.

Therefore, by (3), and (4), hence A is a CNIFSG of G.

5 Comparative Analysis and Discussion

A comparative analysis of the bibliography on fuzzy subgroups, intuitionistic fuzzy subgroups, and

complex fuzzy subgroups typically involves evaluating their foundational theories, definitions, and applications in various mathematical and applied contexts. Table 1 (Appendix) summarizes the above notions based on key points and illustrates a comparative analysis between them.

Fuzzy subgroups form the basis, recent advances in intuitionistic and complex fuzzy subgroups have seen broader and more complex applications. Also, this research adds more complexity by combining complex fuzzy subgroups and intuitionistic fuzzy subgroups. Also, Intuitionistic fuzzy subgroups provide more nuanced uncertainty modeling, while complex fuzzy subgroups offer sophisticated methods for dealing with cyclical or oscillatory behavior. So, the mathematical complexity was introduced by building the structure of CIFSG and their properties in this research in detail. While the new direction of Application Domains for CIFSG has shifted from classical theoretical exploration to real-world applications such as AI, signal processing, and complex decision-support systems.

6 Conclusion

Through the combination of CIFS and fuzzy subgroup characteristics, a novel framework for CIFSG theory has been developed. Various ideas and links, including the notion of anti-CIFSG inside a specific group, have been carefully researched and later validated within this complex subject. The use of the operations $\diamondsuit$, $\square$, and the intersection procedures to a CIFSG provided numerous important results. Furthermore, the notion of CNIFSG and its related properties have been grouped and examined in a systematic manner. Some limitations appear in this study, CIFSG cannot deal with negative information and multi parameters as in bipolar valued fuzzy subgroup and intuitionistic fuzzy soft subgroups. There is a potential for additional research into the integration of bipolar fuzzy sets with CIFSG to alleviate the limitation of expressing negative sentiments inside CIFSG theory. Furthermore, the suggested notion of complex intuitionistic fuzzy soft sets and multi-soft sets may be introduced and thoroughly examined to solve the problem of multi parameters in some type of information. Also, an application in the cryptography will be useful to apply CIFSG to encrypt and disrupt the human being information using generalizing cryptosystem.
forward to seeing you at the Conference.

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- Doaa Al-Sharo, Ayat I. Al-Zoubi wrote - original draft.
- Amani Sheimat validate the results.
- Sharhabeel Alaidi has presented the published work, specifically writing the initial draft.
- Alkouri gave the conceptualization of the paper and Alkhasawneh was the project administration.

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APPENDIX

Table 1. A comparative analysis between fuzzy subgroups, intuitionistic fuzzy subgroups and complex fuzzy subgroups

	Fuzzy subgroup	Intuitionistic fuzzy subgroup	Complex fuzzy subgroup
Foundational theories	The concept of fuzzy subgroups stems from fuzzy sets in 1965, and they were formalized in 1971. However, recent studies have focused on extending and refining the algebraic properties of fuzzy subgroups.	In 1986, intuitionistic fuzzy sets have since been applied in numerous fields. Recent research extends intuitionistic fuzzy subgroups in contexts such as optimization and decision analysis.	Complex fuzzy sets were first introduced in 2002, but recent studies have significantly expanded their use, especially in fields that require complex data analysis, such as engineering and quantum computing.
Definitions	A fuzzy subgroup of a group G is a function from G to the interval [0, 1] that satisfies conditions analogous to classical group theory.	Intuitionistic fuzzy subgroups involve two functions: a membership function and a non-membership function, giving a richer framework for handling uncertainty.	A complex fuzzy subgroup is characterized by a membership function taking values in the complex plane, which allows for modelling phenomena that involve oscillations or periodicity.
Applications in various mathematical and applied contexts	Current research emphasizes fuzzy subgroups in decision-making, artificial intelligence, and automated control systems, adapting the concept to modern computing applications, [46], [47], [48].	Currently, intuitionistic fuzzy subgroups are widely used in supply chain risk management, and multi-criteria decision-making,. [49], [50], [51]	Complex fuzzy subgroups are now being applied in fields like BCK/BCI-Algebra, quantum systems, and Graphs in Hospital infrastructure design where phase information is critical, [52], [53], [54].
Bibliographic focus	Recent work focuses on practical applications, computational efficiency, and hybrid models involving fuzzy logic in modern systems	Recent studies investigate their applications in more complex decision-support systems, particularly in AI-driven systems.	Recent papers focus on complex fuzzy logic's application in data-driven models and its combination with neural networks and other machine learning techniques.