Boundary Value Problems for Linear and Nonlinear Discrete Systems: Solvability Conditions and Input-to-State Stability

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Abstract: This paper investigates boundary value problems (BVPs) for both linear and nonlinear discrete systems with a focus on input-to-state stability (ISS). We begin by analyzing homogeneous and nonhomogeneous linear systems, deriving explicit solution formulas using mathematical induction. Solvability conditions for these systems are established, providing necessary and sufficient criteria for the existence of solutions. The study is then extended to nonlinear discrete systems, where perturbation terms are introduced, and equivalent transformed BVPs are derived. By applying solvability conditions, we analyze the transition from nonlinear to linear cases as perturbations approach zero. The results contribute to the understanding of ISS properties in discrete-time dynamical systems and serve as a foundation for stability analysis and numerical methods in solving BVPs in discrete settings.

Key-Words: Boundary value problems, discrete systems, solvability conditions, input-to-state stability (ISS), nonlinear dynamics, difference equations, perturbation analysis, Moore-Penrose pseudoinverse operators

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1 Introduction

The notion of exponential dichotomy and its connection with nonlinear dynamics goes back to [1] on stable manifolds. This fact was known and used by [2], [3] in the study of transversal intersections for stable and unstable manifolds [4]. One of the main results in this direction was obtained by [5], [6], and then was generalised in the works [7] (see also [8], [9], [4], [10] and recent preprints [11] (here the corresponding assertions were obtained for the interconnected system in Banach space [12], [8]). Later, in the paper of [13](thesis of conference) and the known paper of [14] the notion of Moore-Penrose pseudoinverse matrix was introduced. This concept has since been thoroughly studied and extended in modern operator theory and matrix analysis [15], [16], [17], [18]. This definition gave the possibility to investigate the so-called incorrect problems [19] (such tasks can have a non-unique unstable solution under perturbations solution not for any right parts of the considered equations, for more about ill-posed problems, see [20]). Such correctness conditions were formulated by [21], [22] (existence of solutions of the corresponding problem, uniqueness and continuous dependence from the right-hand parts (stability) see also [23], [9], [20]), see also works on regularization theory [23]. We are going to investigate systems which can admit dichotomy (or have bounded solutions). Last year, among of such systems interconnected systems become more popular (see [24], [25]). Interconnected systems represent the class of systems such that components or elements of subsystems interact with each other (see also recent results in discrete systems [12]). Such systems, sometimes called aggregated systems, were studied as early as the 1980s of the 20th century. It turns out that many structured systems can be modeled as special cases of more general interconnected systems. The stability of an interconnected system consists of two or more switched systems.

The theory of generalized invertible and Moore-Penrose pseudoinverse operators [7] gives the possibility to investigate the case of resonance boundary-value problems (incorrect by Hadamard). Furthermore, recent advances in the study of discrete dynamical systems [26], [27], [28], [29] contribute to better understanding of the complexity and stability phenomena addressed in our problem settings.

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This framework, originally laid out by Tikhonov and Arsenin in their foundational monograph [30], established rigorous methodologies for analyzing and solving ill-posed problems. These include approaches to ensuring stability, regularization, and well-posedness, all of which are central to the study of discrete boundary-value problems in systems exhibiting instability under perturbations.

2 Statement of the problem. Neural networks

In the past few years, neural networks such as Hopfield neural networks [31], [32] (single layer neural network with general non-linear and additional internal linear connections among neurons), cellular neural networks and associative memory neural networks have attracted the attention of many mathematicians, physicts and computer scientists due to the wide range of applications in pattern recognition associative memory and combinatorial optimisation. Among them the Cohen-Grossberg neural networks are an important one, which can be described by the system of ODE (signal processing and the design of networks) [33]. The general form of such equations has the following form

$$x_i'(t) = -a_i(x_i(t)) \left[b_i(x_i) - \sum_{j=1}^{n} c_{ij} g_j(x_j) \right], i \geq 2, \tag{2.1}$$

The number of neurons, a_i - are the amplification functions, b_i are appropriately behaved functions, and g_j are activation functions. Firstly proposed and studied by [33]. It includes a lot of models from evolutionary theory, population biology, and neurobiology. In a special case (when $a_i(x_i) = 1, b_i(x_i) = \frac{c_i}{R_i} x_i(t) + I_i$ we obtain a Hopfield network (see recent paper [32])). It should be noted a priori that we must investigate the considered problem in the case when provided that the boundedness of the solutions is guaranteed (it is a natural assumption for such tasks). It should be noted that the Hopfield network is an interconnected system with the linear part that admits an exponential dichotomy, and the proposed method in [11] can be applied to such a system and to Cohen-Grossberg in special cases.

In the presented proposal we try to investigate the following boundary-value problem for the following systems of differential and difference equations, which model the simple version of the neural networks

$$x_i^{'}(t,\varepsilon) = d_i I_i x_i(t,\varepsilon) - \varepsilon c_i d_i x_i^2(t,\varepsilon) +$$
$$x_i(n+1,\varepsilon) = d_i I_i x_i(n,\varepsilon) - \varepsilon c_i d_i x_i^2(n,\varepsilon) +$$
$$+\varepsilon d_i x_i(t,\varepsilon) \sum_{j=1}^{n} a_{ij} f_j(x_j(t,\varepsilon)), \tag{2.2}$$

with additional boundary conditions

$$x_i(0,\varepsilon) - x_i(T,\varepsilon) = \alpha_i, \tag{2.3}$$

where a_{ij} can be considered as a control parameters. In this model $x_i(t,\varepsilon)$ denotes the state of the i-th neuron at time t, $d_i x_i$ and $c_i x_i$ represent the amplification and behavior functions at time t, respectively, ε is a small parameter. Moreover, in further considerations, we suppose that $d_i I_i T \neq 0$. Condition (2.2) represents the two-point boundary condition, α_i are the set of parameters. We can consider such boundary-value problem as in the finite-dimensional case ($i = \overline{1,n}$), as in the countable case ($i = \overline{1,\infty}$) and in the case when we consider the given problem in the case of Hilbert or Banach spaces (in this case, quadratic nonlinearity can be interpreted in another way). A discrete analogue of such a problem can be represented in the following form

$$x_i(n+1,\varepsilon) = d_i I_i x_i(n,\varepsilon) - \varepsilon c_i d_i x_i^2(n,\varepsilon) +$$
$$+\varepsilon d_i x_i(n,\varepsilon) \sum_{j=1}^{n} a_{ij} f_j(x_j(n,\varepsilon)), \tag{2.4}$$
$$x_i(0,\varepsilon) - x_i(n,\varepsilon) = \alpha_i. \tag{2.5}$$

2.1. Linear Case

Let us now consider the following generated linear system of equations:

$$(x_i^o)'(t,\varepsilon) = d_i I_i x_i^0(t) \tag{2.6}$$

subject to the boundary condition

$$x_i^o(0) - x_i^o(T) = \alpha_i. \tag{2.7}$$

With using of standard technique of differential equations we obtain the following assertion.

Theorem 1 *(Existence and Uniqueness of Solution - Linear Case). The boundary-value problem (2.6), (2.7) is always solvable. The general solution is given by:*

$$x_i^o(t,\alpha_i) = \frac{e^{d_i I_i t}\cdot \alpha_i}{1-e^{d_i I_i T}}. \tag{2.8}$$

In the discrete case:

$$x_i^o(n+1) = d_i I_i x_i^o(n) \tag{2.9}$$

with the boundary condition:

$$x_i^o(0) - x_i^o(m) = \alpha_i \tag{2.10}$$

Suppose that

$$x_i^o(0) = \frac{\alpha_i}{1-(d_i I_i)^m} \tag{2.11}$$

Theorem 2 *(Solvability Conditions)*
A. 1) Boundary Value Problem (BVP) (2.9), (2.10) is solvable if and only if the condition (2.11) is satisfaied.
2) The set of solutions of (2.9), (2.10) has the form

$$x_i^o(n) = (d_i I_i)^n x_i^o(0) + \sum_{j=0}^{n}(d_i I_i)^{n-j} h_i(n) \tag{2.12}$$

B. 1) BVP (2.9), (2.10) is solvable if and only if the next condition is satisfied

$$x_i^o(0) = \frac{\alpha_i - \sum_{j=0}^{n}(d_i I_i)^{n-j} h_i(n)}{1-(d_i I_i)^m} \tag{2.13}$$

2) The set of solutions of (2.9), (2.10) has the form (2.12).

Moreover, the structural conditions for global asymptotic stability of discrete-time monotone systems, as established in [34], provide an essential theoretical foundation that can be directly applied to the solvability analysis of equations (2.9)–(2.12). The criteria presented therein offer rigorous justification for convergence in such systems and further support the stability mechanisms discussed throughout this section.

2.2. Nonlinear Case

We aim to derive a necessary condition for the existence of a solution $x_i(t,\varepsilon)$ for the following boundary value problem (BVP) in the nonlinear case:

$$x_i^{'}(t,\varepsilon) = d_i I_i x_i(t,\varepsilon) - \varepsilon c_i d_i x_i^2(t,\varepsilon) +$$

$$+\varepsilon d_i x_i(t,\varepsilon)\sum_{j=0}^{n} a_{ij} f_j(x_j(t,\varepsilon)) \tag{2.13}$$

subject to the boundary condition

$$x_i(0,\varepsilon) - x_i(T,\varepsilon) = \alpha_i \tag{2.14}$$

From the Cauchy formula, the general solution to equation (2.13) is

$$x_i(t,\epsilon) = e^{d_i I_i t}\cdot \bar{x_i}^o + \epsilon\int_0^t e^{d_i I_i (t-\tau)} h_i(\tau,\epsilon)d\tau \tag{2.15}$$

Theorem 3 *(Nesessary condition) Suppose that the nonlinear BVP (2.13), (2.14) has a solution $x_i(t,\varepsilon)$, which for $\varepsilon = 0$ reduces to one of the solutions in (2.8) and (2.15), denoted as $x_i(t,\alpha_i)$.*

Then element α_i must satisfy the system for generating parameters α_i

$$F_i(\alpha_i) = 0, i = \overline{1,n},$$

where

$$F_i(\alpha_i) = \frac{d_i\alpha_i e^{d_i I_i T}}{1-e^{d_i I_i T}}\left(\sum_{j=0}^{n} a_{ij}\int_0^T f_j\left(\frac{e^{d_i I_i \tau}\cdot\alpha_i}{1-e^{d_i I_i T}}\right)d\tau + \frac{c_i\alpha_i}{d_i I_i}\right)$$

3 Examples

Example 1

Let us consider the following system of differential equations:

$$x_1^{'}(t,\varepsilon) = d_1 I_1 x_1(t,\varepsilon) - \varepsilon c_1 d_1 x_1^2(t,\varepsilon) +$$

$$+\varepsilon d_1 x_1(t,\varepsilon) a_{12} x_2(t,\varepsilon)$$

with the boundary condition:

$$x_1(0,\varepsilon) - x_1(T,\varepsilon) = \alpha_1$$

Similarly, we consider the second equation:

$$x_2^{'}(t,\varepsilon) = d_1 I_2 x_2(t,\varepsilon) - \varepsilon c_2 d_2 x_2^2(t,\varepsilon) +$$

$$+\varepsilon d_2 x_2(t,\varepsilon) a_{21} x_1(t,\varepsilon)$$

with the corresponding boundary condition:

$$x_2(0,\varepsilon) - x_2(T,\varepsilon) = \alpha_2$$

That is, we define the functions:

$$f_1(x_1) = x_1, f_2(x_2) = x_2$$

For $\varepsilon = 0$, the exact solutions are given by:

$$x_1(t,\alpha_1) = \frac{e^{d_1 I_1 t\cdot\alpha_1}}{1-e^{d_1 I_1 T}}$$

$$x_2(t,\alpha_2) = \frac{e^{d_2 I_2 t\cdot\alpha_2}}{1-e^{d_2 I_2 T}}$$

To analyze the case when $\varepsilon \neq 0$, we introduce the following substitution:

$$x_1(t,\varepsilon) = e^{d_1 I_1 t} \cdot \bar{x_1}^o + \varepsilon \int_0^t e^{d_1 I_1 (t-\tau)} h_1(\tau) d\tau$$

$$x_2(t,\varepsilon) = e^{d_2 I_2 t} \cdot \bar{x_2}^o + \varepsilon \int_0^t e^{d_2 I_2 (t-\tau)} h_1(\tau) d\tau$$

Applying the next solvability condition from Theorem 2, we derive:

$$\frac{d_i \alpha_i e^{d_i I_i T}}{1 - e^{d_i I_i T}} \cdot \left(\sum_{j=0}^{n} a_{ij} \int_0^T f_j \left(\frac{e^{d_i I_i \tau} \cdot \alpha_i}{1 - e^{d_i I_i T}} \right) d\tau + \frac{c_i \alpha_i}{d_i I_i} \right) = 0$$

$$\sum_{j=0}^{2} a_{1j} \int_0^T f_j \left(\frac{e^{d_1 I_1 \tau} \cdot \alpha_1}{1 - e^{d_1 I_1 T}} \right) d\tau + \frac{c_1 \alpha_1}{d_1 I_1} = 0$$

$$\int_0^T f_1 \left(\frac{e^{d_1 I_1 \tau} \cdot \alpha_1}{1 - e^{d_1 I_1 T}} \right) d\tau + a_{12} \int_0^T f_2 \left(\frac{e^{d_1 I_1 \tau} \cdot \alpha_1}{1 - e^{d_1 I_1 T}} \right) d\tau + \frac{c_1 \alpha_1}{d_1 I_1} = 0$$

Under notebook

$$F_1(\alpha_1) = \int_0^T e^{d_1 I_1 \tau} \left(\left(\frac{e^{d_1 I_1 \tau} \cdot \alpha_1}{1 - e^{d_1 I_1 T}} \right) + \frac{d_1 a_{12} e^{d_1 I_1 \tau} \alpha_1}{1 - e^{d_1 I_1 T}} \cdot \frac{e^{d_2 I_2 \tau} \alpha_2}{1 - e^{d_2 I_2 T}} \right) d\tau - \int_0^T \frac{e^{d_1 I_1 \tau}}{(1 - e^{d_1 I_1 T})^2} \cdot \alpha_1^2 \cdot e^{2 d_1 I_1 \tau}$$

View each -

1) $\int_0^T \frac{e^{d_1 I_1 \tau} \cdot \alpha_1^2 \cdot e^{2 d_1 I_1 \tau}}{(1 - e^{d_1 I_1 T})^2} d\tau =$

$$= \frac{\alpha_1^2 (e^{3 d_1 I_1 T} - 1)}{3 d_1 I_1 (1 - e^{d_1 I_1 T})^2}$$

2) $\int_0^T \left(\frac{d_1 a_{12} e^{d_1 I_1 \tau} \alpha_1}{1 - e^{d_1 I_1 T}} \cdot \frac{e^{d_2 I_2 \tau} \alpha_2}{1 - e^{d_2 I_2 T}} \right) d\tau =$

$$= \frac{d_1 a_{12} \alpha_1 \alpha_2 (e^{d_2 I_2 T} - 1)}{d_2 I_2 (1 - e^{d_1 I_1 T})(1 - e^{d_2 I_2 T})}$$

Equating the obtained solutions:

$$\frac{\alpha_1^2 (e^{3 d_1 I_1 T} - 1)}{3 d_1 I_1 (1 - e^{d_1 I_1 T})^2} = \frac{d_1 a_{12} \alpha_1 \alpha_2 (e^{d_2 I_2 T} - 1)}{d_2 I_2 (1 - e^{d_1 I_1 T})(1 - e^{d_2 I_2 T})}$$

or

$$\frac{\alpha_1 (e^{3 d_1 I_1 T} - 1)}{3 d_1 I_1 (1 - e^{d_1 I_1 T})} = \frac{d_1 a_{12} \alpha_2 (e^{d_2 I_2 T} - 1)}{d_2 I_2 (1 - e^{d_2 I_2 T})}$$

or

$$\frac{\alpha_1 (1 - e^{3 d_1 I_1 T})}{3 d_1 I_1 (1 - e^{d_1 I_1 T})} = \frac{d_1 a_{12} \alpha_2}{d_2 I_2}$$

Thus, we obtain –

$$\frac{c_1 \alpha_1 (1 - e^{3 d_1 I_1 T})}{3 d_1 I_1 (1 - e^{d_1 I_1 T})} = \frac{d_1 a_{12} \alpha_2}{d_2 I_2},$$

$$\frac{c_2 \alpha_2 (1 - e^{3 d_2 I_2 T})}{3 d_2 I_2 (1 - e^{d_2 I_2 T})} = \frac{a_{21} \alpha_1}{d_1 I_1}.$$

Let us consider the roots of the system, preferably continuous ones.

Make the assumption that the solution is not unique.

After further simplifications, the system can be rewritten in matrix form:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \mathbf{0}$$

where the coefficients are given by:

$$A = \frac{c_1 (1 - e^{3 d_1 I_1 T})}{3 d_1 I_1 (1 - e^{d_1 I_1 T})},$$

$$B = \frac{a_{12}}{d_2 I_2},$$

$$C = \frac{c_2 (1 - e^{3 d_2 I_2 T})}{3 d_2 (1 - e^{d_2 I_2 T})},$$

$$D = \frac{a_{21}}{d_1 I_1}.$$

To obtain a nontrivial solution, we require the determinant of the matrix to be zero:

$$AD - BC = 0.$$

This condition ensures that the system possesses an infinite number of solutions, implying the presence of a continuous spectrum of solutions. The general form of the solution vector can thus be expressed as:

$$\alpha_1 \alpha_2 = \gamma \frac{B}{A} 1, \quad \forall \gamma \in R$$

Thus, the system exhibits non-uniqueness in its solutions under the given conditions.

Example 2

Consider the system of difference equations:

$$x_i(n+1,\varepsilon) = d_i I_i x_i(n,\varepsilon) - \varepsilon c_i d_i x_i^2(n,\varepsilon) +$$

$$+\varepsilon d_i x_i(n,\varepsilon)\sum_{j=1}^{n} a_{ij}x_j(n,\varepsilon)$$

with the boundary condition:

$$x_i(0,\varepsilon) - x_i(m,\varepsilon) = \alpha_i$$

Our objective is to derive all possible solutions for this system. Setting $\varepsilon = 0$, the system simplifies to:

$$x_i^o(n+1) = d_i I_i x_i^o(n) =$$
$$= (d_i I_i)^2 x_i^o(n-1)$$

By iterating this equation, we obtain:

$$x_i^o(n) = (d_i I_i)^{n+1} x_i^o(0)$$

To verify this hypothesis for $n = 1$, we substitute into the boundary condition:

$$x_i^o(0) - x_i^o(m) = (1 - (d_i I_i)^m)\, x_i^o(0) = \alpha_i$$

Solving for $x_i^o(0)$, we get:

$$x_i^o(0) = \frac{\alpha_i}{1-(d_i I_i)^m}$$

Thus, the general solution to the homogeneous equation with non-homogeneous boundary conditions is:

$$x_i^o(n) = \frac{\alpha_i (d_i I_i)^n}{1-(d_i I_i)^m}$$

Now, we seek the general solution for the non-homogeneous equation. Consider:

$$f_j(x_j) = x_j$$

A general solution in this case takes the form:

$$x_i^o(n) = (d_i I_i)^n \frac{\alpha_i + \sum_{j=0}^{m}(d_i I_i)^j h(m-j)}{1-(d_i I_i)^m} +$$

$$+\sum_{j=0}^{n}(d_i I_i)^j h(m-j)$$

To derive this result rigorously, consider:

$$x_i(n+1,\varepsilon) = d_i I_i \left(d_i I_i x_i(n-1,\varepsilon) + \varepsilon h_i(n-1,\varepsilon)\right)$$
$$+\varepsilon h(n,\varepsilon)$$

Expanding iteratively,

$$x_i(n+1,\varepsilon) = (d_i I_i)^2 x_i(n-1,\varepsilon) + \varepsilon d_i I_i h_i(n-1,\varepsilon) +$$

$$+\varepsilon h(n,\varepsilon)$$

By induction, we obtain:

$$x_i(n+1,\varepsilon) = (d_i I_i)^{n+1} x_i(0,\varepsilon) + \varepsilon \sum_{j=0}^{n}(d_i I_i)^{n-j} h_i(j,\varepsilon)$$

Substituting this into the boundary condition:

$$x_i(0,\varepsilon) - x_i(m,\varepsilon) = x_i(0,\varepsilon) - (d_i I_i)^m x_i(0,\varepsilon) -$$
$$-\varepsilon \sum_{j=0}^{m}(d_i I_i)^{m-j} h_i(j,\varepsilon) = \alpha_i$$

Rearranging,

$$\alpha_i = \left(1-(d_i I_i)^m\right) x_i(0,\varepsilon) - \varepsilon \sum_{j=0}^{m}(d_i I_i)^{m-j} h_i(j,\varepsilon)$$

Taking the limit $\varepsilon \to 0$, we obtain:

$$\sum_{j=0}^{m}(d_i I_i)^{m-j} h_i(j,\varepsilon) = 0$$

Rewriting this in terms of the generating function:

$$\sum_{k=0}^{m}(d_i I_i)^{m-k}(d_i x_i(m,\varepsilon)\sum_{j=1}^{n} a_{ij} f_j(x_j(m,\varepsilon)) -$$
$$-c_i d_i x_i^2(m,\varepsilon)) = 0.$$

Since the generating function satisfies:

$$x_i^o(n) = \frac{(d_i I_i)^n \alpha_i}{1-(d_i I_i)^m}$$

we derive the necessary condition:

$$\sum_{j=0}^{m}(d_i I_i)^{m-j} \frac{(d_i I_i)^m \alpha_i}{1-(d_i I_i)^m} \cdot \sum_{j=1}^{n} a_{ij} \frac{(d_j I_j)^m \alpha_i}{1-(d_i I_i)^m} =$$

$$= \sum_{j=0}^{m}(d_i I_i)^{m-j} \left(\sum_{k=1}^{m} a_{ik} \frac{(d_i I_i)^k \alpha_k}{1-(d_i I_i)^m} - C_i\right) \cdot$$

$$\cdot \frac{(d_i I_i)^j \alpha_j}{1-(d_i I_i)^m} = 0$$

Summing over terms separately:

$$\sum_{j=0}^{m}(d_i I_i)^{m-j} \sum_{k=1}^{j} a_{ik}(d_i I_i)^k$$

1) $\sum_{k=1}^{j}(d_i I_i)^k = \frac{d_i I_i((d_i I_i)^{j+1}-1)}{d_i I_i - 1}$
2) $\sum_{k=1}^{j} a_{ik}(d_i I_i)^k = \frac{a_{i1} d_i I_i((d_i I_i)^{j+1}-1)}{d_i I_i - 1} =$

$$= \frac{d_i I_i((d_i I_i)^{j+1} - 1)}{d_i I_i - 1}$$

3) $\sum_{j=0}^{m} \frac{(d_i I_i)^m((d_i I_i)^{j+2} - d_i I_i)}{d_i I_i - 1} =$

$$\frac{\sum_{j=0}^{m}(d_i I_i)^{m+j+2} - \sum_{j=0}^{m}(d_i I_i)^{m+1}}{d_i I_i - 1} =$$

$$= \frac{(d_i I_i)^{m+2}((d_i I_i)^m - 1) - ((d_i I_i)^{m+1}(m+1)}{d_i I_i - 1} =$$

$$= \frac{(d_i I_i)^{m+1}((d_i I_i)^{m+1} - d_i I_i - (m+1))}{d_i I_i - 1}$$

Finally, substituting these results into the original condition leads to:

$$\sum_{j=0}^{m}(d_i I_i)^{m-j} \sum_{k=1}^{j} a_{ik}(d_i I_i)^k =$$

$$= \frac{(d_i I_i)^{m+1}((d_i I_i)^{m+1} - d_i I_i - (m+1))}{d_i I_i - 1}$$

In recent developments, significant attention has been given to rigorous theoretical treatments of stability and ISS conditions in interconnected and monotone systems. In particular, [34] provided concrete criteria for global asymptotic stability of discrete-time monotone systems, demonstrating how structural monotonicity can be exploited in stability verification procedures. This complements the classical ISS framework elaborated by [35], whose monograph remains one of the cornerstones of modern nonlinear control theory.

Furthermore, discrete-time formulations of ISS have been thoroughly developed by [36], offering robust characterizations and constructive tools that apply directly to the types of boundary value problems investigated in this paper. The role of monotonicity and switching dynamics also emerged in applications of small-gain theorems to large-scale systems.

On a broader theoretical level, the thermodynamic perspective introduced by [37], [38], and [39] opens pathways for reinterpreting the stability of dynamical systems under time-irreversible processes. Their work highlights the deep relationship between entropy, fluctuations, and structural transitions in dissipative systems—a paradigm that resonates with the non-uniqueness and bifurcation properties encountered in ill-posed boundary value problems.

Finally, the foundational lecture notes by [40] continue to serve as a rigorous entry point into the theory of partial differential equations and their well-posedness—a background indispensable for bridging continuous and discrete formulations, as seen in Sections 2 and 3 of this work.

4 Additional Applications and Theoretical Context

Beyond the examples discussed, the results of this paper have broader applicability to modern mathematical modeling frameworks.

Our findings relate directly to stability theory for nonlinear systems and differential equations, as detailed in foundational texts such as [28], [41], [42]. Moreover, the proposed methods are relevant for the study of well-posedness and control of boundary systems [43], [44], and enrich the growing literature on ISS (input-to-state stability) [35], [36], [34].

Additionally, insights into neural network dynamics derived here resonate with earlier theoretical and cognitive modeling works on emergent behavior and learning [26], [27]. These connections confirm the versatility of the analytical techniques employed in this paper.

Therefore, this research extends classical methods and enriches the set of tools available for studying complex discrete and continuous systems, including those arising in mathematical biology, neurodynamics, and engineering control applications.

5 Conclusion

This paper investigates boundary value problems for Cohen-Grossberg neural networks in both linear and nonlinear settings. In the linear case, we have established explicit solvability theorems, while in the nonlinear case, necessary conditions for solvability are derived via perturbation analysis and pseudoinverse operators. This approach is flexible and generalizable, making it suitable for the analysis of broader classes of dynamical systems.

The theoretical foundation developed herein builds upon the classical framework of differential equations [28], [41], [42] and extends it to discrete-time systems, which are increasingly relevant in modeling digital signal processing and computational architectures. These results also contribute to ongoing research in well-posedness and input-to-state stability for nonlinear control systems [43], [44], confirming the viability of the proposed techniques for application in boundary control problems and systems with structural feedback.

Furthermore, the applicability to modern artificial intelligence is reinforced by connections with emergent behavior in Hopfield-type and Grossberg neural networks [26], [27]. These

networks are foundational in modeling associative memory, adaptive pattern recognition, and parallel computation in both biological and engineered systems.

Therefore, this study not only reinforces existing results in operator theory and ISS stability but also contributes to the understanding of how abstract mathematical tools, such as Moore-Penrose pseudoinverse operators, can be effectively applied to nonlinear boundary-value problems arising in modern control and neural computation frameworks.

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