Pole Assignment With Static Output Feedback using Algebrogeometric Methods

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Abstract: In this article a solution to the pole assignment problem with output feedback is proposed. Necessary and sufficient conditions are derived which are related to the controllability or observability of the initial system. These arise from the solution of the state-feedback problem using the output or input matrix of the system. For the initial open loop system a new matrix is calculated such that under output feedback the new closed loop system has the desired poles. In the proposed approach, multilineal algebra, algebraic geometry and the theory of generalized inverse matrices are employed. An illustrative example of the proposed method is also given. The main advantage of our approach is that it can be used to derive an algorithm which generates the whole family of output feedback matrices with the required specifications, while avoiding the use of transfer functions.

Key-Words: Control Theory, Pole assignment, Output feedback, Controllability, Observability, Generalized inverse matrix.

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1 Introduction

The pole-assignment problem with output feedback is a fundamental problem in control theory, which can be employed to develop design methods of feedback systems which guarantee desired dynamic behavior. In these systems only partial information (measured outputs) is available for feedback, rather than the full state vector. The challenge is to design feedback laws that assign the poles of the system in a way that meets performance objectives, despite limited state information. This problem is pivotal in modern control applications, ranging from robotics to aerospace and involves both analytical and computational approaches for finding effective solutions. The paper provides an alternative approach to the solution of the pole assignment problem with output feedback which relies on necessary and sufficient controllability and observability conditions of the initial problem. Firstly, a general description of the problem is given by assuming that we have the open loop system of the form:

$$\begin{cases} x'(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{1}$$

for $t \geq t_0$, where $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^l$ and $y(t) \in \mathbb{R}^m$ are the state vector, the input vector and the output vector respectively. It is assumed that $l \leq n$ and $m \leq n$. Also, $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times l}$ and $C \in \mathbb{R}^{m\times n}$ denote the state matrix, the input matrix and the output matrix, respectively. First we seek a matrix $F \in \mathbb{C}^{l\times m}$ such that under the output feedback of the form:

$$u(t) = -Fy(t) + v(t) \tag{2}$$

the closed loop system has the desired poles. Note that here $v(t) \in \mathbb{R}^l$ denotes the new reference input vector. The closed loop dynamics can be written as:

$$x'(t) = (A - BFC)x(t) + Bv(t) \tag{3}$$

The studies, [1], [2], [3], have shown independently that a necessary condition for almost generalized pole placement of a completely controllable and observable system is the validity of the inequality:

$$m + l \geq n + 1$$

The meaning of the term ”generalized pole placement”, which we also use in our study, is that the desired poles are discrete, i.e. $\lambda_i \neq \lambda_j$ if and only if $i \neq j$, while by the term ”almost” we mean that a very small deviation from the desired pole set is acceptable. More precisely, this means that the closed-loop poles are allowed to lie within arbitrarily tiny discs having at their centers the desired poles. So, in our study we try to solve the problem of pole placement via output feedback, under necessary and sufficient controllability and observability conditions of the initial system, related to the solution of the problem via state feedback and the relation between the applied feedback and the output or input matrix, respectively. The relation of the pole placement problem to the controllability and the observability of the system is expected, according to past studies, that the initial system has to be controllable and observable in order for the new system to have the desired poles. In our study of the pole assignment problem under output feedback we use multilineal algebra and algebraic geometry as well as the theory of generalized inverse matrices; in this way the use of transfer functions is avoided.

2 Mathematical Background of Pole Assignment

Let assume that we have the system (1) for $t \geq t_0$. It is mentioned that the dynamical behaviour of the system is defined by the nature and the position of the poles of the system, meaning the eigenvalues of the matrix A, which are the poles of the characteristic polynomial

$$\varphi(s) := \det(sI_n - A) \tag{4}$$

We aim to change the dynamical behaviour of the closed-loop system through the pole placement, i.e. by transferring the poles to appropriate locations of the complex plane. Thus, the pole placement problem under output feedback can be formally defined as follows: For a given monic polynomial $a(s)$ of degree n having the desired poles as roots, find a complex matrix $l \times m$ such that for an input function of the form (2)

$$u(t) = -Fy(t) + v(t)$$

the corresponding closed loop system

$$x'(t) = (A - BFC)x(t) + Bv(t)$$

has the desired poles, i.e. its characteristic polynomial

$$\varphi_{CL}(s) := \det(sI_n - A + BFC) \tag{5}$$

is equal to $a(s)$. As we have already mentioned the controllability and the observability of the initial system are the necessary conditions in order the problem to be completely solvable [4].

Proposition 1

If for the system of the form (1) and for every monic polynomial $a(s)$ of degree n there is a complex $l \times m$ matrix F, such that the equation (5) is valid, then the system (A, B) is completely controllable and the (A, C) is completely observable.

Proof: Let us select an arbitrary monic polynomial $a(s)$ of degree n. Then, there is a matrix $F \in \mathbb{C}^{l \times n}$ such that equation (5)

$$\varphi_{CL}(s) := \det(sI_n - A + BFC) = a(s)$$

is satisfied.

Setting $\tilde{F} := FC$ shows the existence of a state feedback matrix $\tilde{F}$ such that

$$\varphi_{CL}(s) := det(sI_n - A + B\tilde{F}) = a(s)$$

and so the system (A, B) is completely controllable. Similarly, by setting $\tilde{F} := BF$ we conclude that there exists a matrix $\tilde{F}$ such that

$$\varphi_{CL}(s) := det(sI_n - A + \tilde{F}C) = a(s)$$

and so the system (A, C) is completely observable.

3 Solution of the Problem when the System is Controllable

Let assume that the system (1) is completely controllable, meaning that

$$\text{rank}[sI_n - A, B] = n, \forall s \in \mathbb{C}$$

or equivalently

$$\text{rank}[B, AB, A^2B, ..., A^{n-1}B] = n$$

Set:

$$\tilde{F} := FC \tag{6}$$

By using equation (6), equation (5) can be written as:

$$\varphi_{CL}(s) = det(sI_n - A + B\tilde{F}) = a(s) \tag{7}$$

and the initial problem leads to the calculation of $\tilde{F} \in \mathbb{C}^{l \times n}$ which satisfies equation (7) for a given monic polynomial $a(s)$ and then to the solution of the equation (6) to obtain F.

If

$$a(s) = a_0 + a_1 s + ... + a_{n-1}s^{n-1} + s^n \tag{8}$$

then by setting

$$e(s) = [1, s, ..., s^n]^t, \; a = [a_0, ..., a_{n-1}, 1]^t$$

we get

$$a(s) = e^t(s) \cdot a \tag{9}$$

Moreover, the characteristic polynomial of the closed loop system is written:

$$\begin{aligned}\varphi_{CL}(s) &:= \det(sI_n - A + B\tilde{F}) \\ &\equiv C_n(sI_n - A + B\tilde{F}) \\ &= C_n\left([sI_n - A, B] \cdot \begin{bmatrix} I_n \\ \cdots \\ \tilde{F} \end{bmatrix}\right) \\ &= C_n\left([sI_n - A, B]\right) \cdot C_n\left(\begin{bmatrix} I_n \\ \cdots \\ \tilde{F} \end{bmatrix}\right)\end{aligned} \tag{10}$$

In the above equation we have used the Binet-Cauchy theorem for complex matrices. We notice that the matrix $C_n\left([sI_n - A, B]\right) \in C^{1\times\binom{n+l}{n}}$ has elements polynomials of degree up to n. Moreover, the first element of the matrix is the determinant $\det(sI_n - A)$, in other words the characteristic polynomial of A. So, there is a matrix $P \equiv P(A, B) \in \mathbb{C}^{(n+1)\times\binom{n+l}{n}}$ such that:

$$C_n\left([sI_n - A, B]\right) = e^t(s) \cdot P \tag{11}$$

The columns of matrix P are the coefficients of the polynomials which are the elements of $C_n\left([sI_n - A, B]\right)$. In addition, we set that:

$$g := C_n\left(\begin{bmatrix} I_n \\ \cdots \\ \tilde{F} \end{bmatrix}\right), \in \mathbb{C}^{\binom{n+l}{n}\times 1} \tag{12}$$

Combining equations (7),(9), (10), (11), (12) we get equation $e^t(s)Pg = e^t(s)a$ which holds for every $s \in \mathbb{C}$ if and only if:

$$Pg = a \tag{13}$$

The linear system (13) is always solvable, meaning that for every $a \in \mathbb{C}^{n\times 1}$ that corresponds to a given monic polynomial $a(s)$ of degree n, if and only if the matrix P is full row rank, which is equivalent to the equation $\text{rank}P = n + 1$. It is also known that the last equation is equivalent to the complete controllability of the initial system (1). In this case, the solutions of the equation (13) consist of an algebraic multiplicity V_L of dimension:

$$\dim V_L = k - 1 - (n + 1) = k - n - 2$$

where $k - 1$ is the dimension of the projective space $\mathbb{P}^{k-1}(\mathbb{R}), k = \binom{n+l}{n}$, while $n + 1$ is the rank of the matrix P in the equation (13).

Moreover from equation (12), we understand that we are interested only in solutions of g, that we meet in equation (13), which can be written in the form of:

$$g = C_n\left(\begin{bmatrix} I_n \\ \cdots \\ \tilde{F} \end{bmatrix}\right)$$

for a matrix $\tilde{F}$. Equivalently we are interested in the solutions of g which are simultaneously decomposable vectors, meaning that they belong to the Grassmann variety $\Omega(n, n + l)$. Since

$$\begin{aligned}\dim V_L + dim\Omega(n, n + l) = k - n - 2 + nl &\geq k - 1 \\ &= \dim \mathbb{P}^{k-1}(\mathbb{R})\end{aligned}$$

for every $n, l \geq 1$, the algebraic variety V_L and the Grassmann variety $\Omega(n, n + l)$ have always a non empty intersection and consequently there is always a vector g which is the simultaneous solution of equations (12) and (13).

In order to calculate the matrices $\tilde{F}$ which will satisfy equation (12), we determine among all the solutions of g from the equation (13) only those that belong to the Grassmann variety $\Omega(n, n + l)$, or equivalently those vectors, from the solutions of equation (13) whose coordinates satisfy a set of quadratic Plucker relations which describe the Grassmann variety $\Omega(n, n + l)$. In general, the Grassmann variety, as we have mentioned previously, is described by several relatively complex equations, however in our case a minimum set of Plucker relations can be calculated, significantly limited compared to the initial set, while each one of these relations is much simpler as it is described by only three terms. The whole procedure of the determination of the Plucker relations and then through them the determination of the $\tilde{F}$ matrices is the same as we have already described previously.

Having determined the $\tilde{F}$ matrices which satisfy equation (7), the problem reduces to determining matrix F which satisfies equation (6) and so solves the initial problem of pole placement with output feedback, since equation (5) stands. The system defined in equation (7) with the unknown matrix F, for the given matrices $C \in \mathbb{C}^{m\times n}$ and $\tilde{F} \in \mathbb{C}^{l\times m}$, has solutions if and only if

$$\text{row-span } \tilde{F} \leq \text{ row-span } C$$

This equation has only theoretical importance as a more applicable necessary and sufficient condition derives from the notion of the $\{1\}$-inverse of a matrix,

[5].

{1}-Inverse matrices, Moore-Penrose Inverse

Let assume that $A \in \mathbb{C}^{m\times n}$, then a matrix $X \in \mathbb{C}^{n\times m}$ is called the $\{1\}$-inverse of matrix A if

$$AXA = A \tag{14}$$

It can be shown that there is always at least one $\{1\}$-inverse matrix of $A \in \mathbb{C}^{m\times n}$. This can be calculated by using the following proposition, [5], [6]:

Proposition 2

Let $A \in \mathbb{C}^{m\times n}$ and let P and Q be invertible matrices with $P \in \mathbb{C}^{n\times n}$, $Q \in \mathbb{C}^{m\times m}$ such that

$$QAP = \begin{bmatrix} I_r & \vdots & 0 \\ \cdots & \cdot & \cdots \\ 0 & \vdots & 0 \end{bmatrix} \tag{15}$$

where $r = \text{rank}A$. Then every $\{1\}$-inverse matrix X of A can be written in the form:

$$X = P \begin{bmatrix} I_r & \vdots & 0 \\ \cdots & \cdot & \cdots \\ 0 & \vdots & L \end{bmatrix} \cdot Q \tag{16}$$

for an arbitrary matrix $L \in \mathbb{C}^{(n-r)\times(m-r)}$ and P, Q invertible matrices that satisfy equation (15).

We also have that $\text{rank}X = r + \text{rank}L$. It is noted that for an $\{1\}$-inverse matrix of A we use the symbolism $A^{(1)}$. The most important application of $\{1\}$-inverse matrices is the area of systems of linear equations. The following result is derived from [5], [7].

Theorem 1

Let assume that $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$, $D \in \mathbb{C}^{m\times q}$. Then the following matrix equation

$$AXB = D \tag{17}$$

is consistent, if and only if for some matrices $A^{(1)}, B^{(1)}$ it holds that

$$AA^{(1)}DB^{(1)}B = D \tag{18}$$

and in this case the general solution is

$$X = A^{(1)}DB^{(1)} + Y - A^{(1)}AYBB^{(1)} \tag{19}$$

for an arbitrary matrix $Y \in \mathbb{C}^{n\times p}$.

The following characterization of the set of $\{1\}$-inverse matrices of a matrix A, for a given $\{1\}$-inverse matrix of $A^{(1)}$, is attributed to [8].

Corollary 1

Let $A^{(1)}$ be a $\{1\}$-inverse of the matrix $A \in \mathbb{C}^{m\times n}$. Then all $\{1\}$-inverses $\bar{A}^{(1)}$ of A are given by:

$$\bar{A}^{(1)} = A^{(1)} + Z - A^{(1)}AZAA^{(1)} \tag{20}$$

for $Z \in \mathbb{C}^{n\times m}$.

A specific case of the $\{1\}$-inverse matrices are the right/left inverse matrices with full column/row rank, respectively. Let assume first that $A \in \mathbb{C}^{m\times n}$, with $m \geq n$ and $\text{rank}A = n$, meaning that A is full column rank. Then it is proven in [5], that the $\{1\}$-inverse matrices of A identify with its left inverse matrices A_L^{-1}, i.e. $A_L^{-1}A = I_n$.

In this case, if $P \in \mathbb{C}^{m\times m}$ is an invertible row permutator such that

$$PA = \begin{bmatrix} A_1 \\ \cdots \\ A_2 \end{bmatrix}$$

where $A_1 \in \mathbb{C}^{n\times n}$ with $\det A_1 \neq 0$ and $A_2 \in \mathbb{C}^{(m-n)\times n}$, then all left inverses A_L^{-1} of A are given by the equation

$$A_L^{-1} = [A_1^{-1} - BA_2A_1^{-1} \vdots B]P \tag{21}$$

for an arbitrary matrix $B \in \mathbb{C}^{n\times(m-n)}$.

Similarly, if $A \in \mathbb{C}^{m\times n}$, it can be shown that the $\{1\}$-inverse matrices of A identify with its right inverses A_R^{-1} for which $AA_R^{-1} = I_n$.
In this case, if $Q \in \mathbb{C}^{n\times n}$ is an invertible column permutator such that

$$AQ = [A_3 \vdots A_4]$$

with $A_3 \in \mathbb{C}^{m\times m}$, $\det A_3 \neq 0$ and $A_4 \in \mathbb{C}^{m\times(n-m)}$, then all right inverses A_R^{-1} of A are given by the equation

$$A_R^{-1} = Q \begin{bmatrix} A_3^{-1} - A_3^{-1}A_4C \\ \cdots \\ C \end{bmatrix} \tag{22}$$

for an arbitrary matrix $C \in \mathbb{C}^{(n-m)\times m}$.

Finally, a special case of $\{1\}$-inverses is the Moore-Penrose inverse matrix $A^{(\dagger)}$ of A. This is the unique $\{1\}$-inverse matrix of A which satisfies in addition to equation (14) also the following equations:

$$XAX = X,\ (AX)^* = AX,\ (XA)^* = XA$$

The Moore-Penrose inverse matrix $A^{(\dagger)}$ of $A \in \mathbb{C}^{m\times n}$, for which rank$A = r, r \geq 1$, can be determined if we can obtain matrices $F \in \mathbb{C}^{m\times r}$ and $G \in \mathbb{C}^{r\times n}$ such that $A = FG$. Then, $A^{(\dagger)}$ is given by:

$$A^{(\dagger)} = G^*(GG^*)^{-1}(F^*F)^{-1}F^* \tag{23}$$

Now, we can go back to relation (6), i.e. the relation $\tilde{F} = FC$. According to the Theorem 1, this system of linear equations is consistent if and only if

$$\tilde{F}C^{(1)}C = \tilde{F} \tag{24}$$

for a $\{1\}$-inverse matrix $C^{(1)}$ of C. If equation (24) holds for a $C^{(1)}$ matrix, then it is easily proven using equation (20) that it holds for all of them. So, if $\tilde{C}^{(1)}$ is a different $\{1\}$-inverse matrix of C then,

$$\tilde{F}\tilde{C}^{(1)}C = \tilde{F}C^{(1)}C + \tilde{F}ZC - \tilde{F}C^{(1)}CZCC^{(1)}C = \tilde{F}C^{(1)}C + \tilde{F}ZC - \tilde{F}ZC = \tilde{F}$$

For a given matrix $\tilde{F}$ which satisfies equation (24), the solutions F of the initial pole placement problem with output feedback are given by the equation:

$$F = \tilde{F}C^{(1)} + Y - YCC^{(1)} \tag{25}$$

where $Y \in \mathbb{C}^{l\times m}$ is an arbitrary matrix. Thus, we have the following theorem:

Theorem 2 [5]

For a system of the form (1) and for a choice of poles corresponding to the monic polynomial $a(s)$ of degree n, the problem of pole assignment with output feedback of the form (2) has a solution if and only is there is a matrix $\tilde{F} \in \mathbb{C}^{l\times n}$ such that the following equations are simultaneously satisfied:

$$\det(sI_n - A + B\tilde{F}) = a(s) \tag{26}$$

and the equation (24)

$$\tilde{F}C^{(1)}C = \tilde{F}$$

with $C^{(1)}$ being any $\{1\}$-inverse matrix of matrix C. In this case, the F solutions of the problem are given by the equation (25)

$$F = \tilde{F}C^{(1)} + Y - YCC^{(1)}$$

Moreover, if the system (1) is completely controllable, then equation (26) has always a solution for $\tilde{F}$ for every polynomial $a(s)$.

As matrix $C \in \mathbb{C}^{mxn}$ is usually full row rank, meaning that $m \leq n$ and rank$C = m$, we can find the form of solutions of the problem in this case, if they exist. In this case, a matrix $C^{(1)}$ is a right inverse of matrix C, meaning that

$$CC^{(1)} = I_m \tag{27}$$

and correspondingly for the relation (25) we have that

$$F = \tilde{F}C^{(1)} + Y - YI_m = \tilde{F}C^{(1)} \tag{28}$$

Corollary 2 [5]

If the matrix C is full row rank then the solution of the form (28) of our system (1) is independent from the choice of matrix $C^{(1)}$.

Proof: If $\tilde{C}^{(1)}$ is a different right inverse matrix of C, then from the relation (20), there exists a matrix $Z \in \mathbb{C}^{nxm}$ such that

$$\tilde{C}^{(1)} = C^{(1)} + Z - C^{(1)}CZCC^{(1)} = C^{(1)} + Z - C^{(1)}CZ$$

and so,

$$\tilde{F}C^{(1)} = \tilde{F}C^{(1)} + \tilde{F}Z - \tilde{F}C^{(1)}CZ = \tilde{F}C^{(1)} + \tilde{F}Z - \tilde{F}Z = \tilde{F}C^{(1)} = F$$

In the example at the end of the paper we present a complete application of Theorem 2 and Corollary 1 and 2.

4 Solution of the Problem when the System is Observable

We assume that the system (1) is completely observable, meaning that

$$\text{rank}\begin{bmatrix} sI_n - A \\ \dots \\ C \end{bmatrix} = n, \forall s \in \mathbb{C}$$

or equivalently,

$$\text{rank}\begin{bmatrix} C \\ CA \\ CA^2 \\ \dots \\ CA^{n-1} \end{bmatrix} = n$$

Setting

$$\tilde{F} := BF \tag{29}$$

equation (5) can be written by the form

$$\varphi_{CL}(s) = \det(sI_n - A + \tilde{F}C) = a(s) \qquad (30)$$

and the initial problem can be formulated as the problem of finding a matrix $\tilde{F} \in \mathbb{C}^{nxm}$ which satisfies equation (30) for a given polynomial $a(s)$ and then solving equation (29) in order to find the matrix F. The above problem is dual with the problem when the system is controllable.
Write:

$$a(s) = a^t \cdot e^t(s) \qquad (31)$$

Moreover,

$$\begin{aligned}\det(sI_n - A + \tilde{F}C) &= C_n(sI_n - A + \tilde{F}C)\\ &= C_n\left([I_n, \tilde{F}] \cdot \begin{bmatrix} sI_n - A \\ \dots \\ C \end{bmatrix}\right)\\ &= C_n\left([I_n, \tilde{F}]\right) \cdot C_n\left(\begin{bmatrix} sI_n - A \\ \dots \\ C \end{bmatrix}\right)\\ &=: g^t P(A, C) e(s)\end{aligned}$$

and consequently (30) holds for every $s \in \mathbb{C}$ if and only if $g^t P(A, C) = a^t$, which is equivalent to:

$$P^t(A, C) \cdot g = a \qquad (32)$$

The system in (32) always has a solution. More precisely, the system is solvable for any $a \in \mathbb{C}^{n+1}$ if and only if the matrix $P(A, C)$ is full rank, meaning that rank$P(A, C) = n + 1$. It is also known that relation, [9], is equivalent to complete observability of the initial system (1). In this case, everything regarding the solutions of g and $\tilde{F}$ of equations (32) and (30) follows from the discussion of a previous section in which the system was assumed to be controllable. Having calculated the matrices F which satisfy equation (30), it follows that the system of linear equations $\tilde{F} = BF$ is solvable if and only if

$$\text{col-span}\tilde{F} \leq \text{col-span}B$$

A more useful necessary and sufficient condition is obtained by using the $\{1\}$-inverse matrices of B. According to Theorem 1, the system (29), where matrix B is unknown, is consistent if and only if

$$BB^{(1)}\tilde{F} = \tilde{F} \qquad (33)$$

for an $\{1\}$-inverse matrix $B^{(1)}$ of B. Equation (33) holds for all $\{1\}$-inverse matrices of B, if it holds for just one of them. So, for a given matrix $\tilde{F}$ which satisfies equation (33), the solutions for matrix F of the initial pole-assignment problem with output feedback, are given by:

$$F = B^{(1)}\tilde{F} + W - B^{(1)}BW \qquad (34)$$

where $W \in \mathbb{C}^{lxm}$ is an arbitrary matrix.

As a consequence, we have the following dual theorem of the Theorem 2.

Theorem 3
Let assume system (1) and a set of desired closed-loop poles corresponding to the arbitrary monic polynomial $a(s)$ of degree n. Then the pole assignment problem with output feedback of the form (2) has a solution if and only if there exists a matrix $\tilde{F} \in \mathbb{C}^{nxm}$ which satisfies simultaneously the equation (30)

$$\det(sI_n - A + \tilde{F}C) = a(s)$$

and the equation (33)

$$BB^{(1)}\tilde{F} = \tilde{F}$$

where matrix $B^{(1)}$ is any $\{1\}$-inverse of the matrix B. In this case the solutions for matrix F are given by the equation (34) where

$$F = B^{(1)}\tilde{F} + W - B^{(1)}BW$$

with $W \in \mathbb{C}^{lxm}$ being an arbitrary matrix.

If the system of the form (1) is completely observable, then the equation (30) is always solvable for $\tilde{F}$, for every polynomial $a(s)$.

Usually, matrix B is full column rank which means that $l \leq n$ and rank$B = l$. Then $B^{(1)}$ is a left-inverse of the matrix B, i.e.

$$B^{(1)}B = I_m \qquad (35)$$

and the solution for matrix F is given from the equation (34) and it is:

$$F = B^{(1)}\tilde{F} \qquad (36)$$

In this case also, the solution of system (1), which is given by equation (36), is independent of the choice of the matrix $B^{(1)}$.

5 Example

In order to clarify the proposed method a numerical example is presented. We consider a system of the form (1) with matrices A, B and C as follows, [10]

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix},\ B = \begin{bmatrix} -1 & -2 \\ 0 & -1 \\ 0 & -1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Suppose we wish to transfer the set of poles of $\{1, 2, 3\}$ which are the eigenvalues of the open-loop system to the set of poles $\{-1, -2, -3\}$ by using output feedback of the form $u(t) = -Fy(t) + v(t)$. Then, the problem is reduced to the calculation of a matrix $F \in \mathbb{C}^{2x2}$ such that for the closed loop system of the form

$$x'(t) = (A - BFC)x(t) + Bv(t)$$

its characteristic polynomial will be

$$\varphi_{CL}(s) := det(sI_3 - A + BFC) = a(s) \qquad (37)$$

with

$$a(s) = 6 + 11s + 6s^2 + s^3 = [1, s, s^2, s^3] \cdot \begin{bmatrix} 6 \\ 11 \\ 6 \\ 1 \end{bmatrix}$$

Setting

$$\tilde{F} = FC \qquad (38)$$

and applying the methodology corresponding to a controllable initial system and reference, [10], we calculate the solutions $\tilde{F}$ of $\det(sI_3 - A + B\tilde{F}) = a(s)$. In order to confirm that the system is controllable we calculate the matric $\mathcal{C}$ which is equal to

$$\mathcal{C} = [B\ AB\ A^2B] = \begin{bmatrix} -1 & -2 & \vdots & -1 & -2 & \vdots & -2 & -2 \\ 0 & -1 & \vdots & 0 & -2 & \vdots & 0 & -4 \\ 0 & -1 & \vdots & 0 & -3 & \vdots & 0 & -9 \end{bmatrix}$$

and by computing its compound matrix of 3 we see that $C_3(\mathcal{C}) \neq \bar{0}^t$, so rank$\mathcal{C} = 3$. We obtain according to reference, [10], that

$$\tilde{F} = \begin{bmatrix} a_1 & -\frac{ba_1 + a_3}{c} & \frac{a_2 + a_1 a}{c} \\ c & -b & a \end{bmatrix} \qquad (39)$$

where

$$a_1 := -12 - a + b - 2c,\ a_2 := -120 - 2a,\ a_3 := -60 - b$$

with $a, b, c \in \mathbb{R}$ are arbitrarily assigned apart from the constraints $c \neq 0$ and $b \neq -60$. In order to find the F solutions of relation (38), matrix $\tilde{F}$ has to satisfy:

$$\tilde{F}C^{(1)}C = \tilde{F} \qquad (40)$$

where $C^{(1)}$ is a $\{1\}$-inverse of C. Because rank$C = 2$ matrix C is full row rank, so matrix $C^{(1)}$ will be one of the right-inverse matrices of C and according to our approach when the initial system is controllable, we can easily calculate a (non-unique) right inverse of the matrix C as

$$C^{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

According to what we have already mentioned, $C^{(1)}$ can be chosen to satisfy equation $\tilde{F}C^{(1)}C = \tilde{F}$. It turns out that the parameters a, b, c have to satisfy the conditions: $b \in \mathbb{R}$ and $a = c = -60$. So, the solutions of the relation (39) are limited to:

$$\tilde{F} = \begin{bmatrix} 168 + b & \frac{b^2 + 167b - 60}{60} & 168 + b \\ -60 & -b & -60 \end{bmatrix}$$

and hence the initial problem has the following solutions:

$$F = \tilde{F}C^{(1)} = \begin{bmatrix} 168 + b & \frac{b^2 + 167b - 60}{60} \\ -60 & -b \end{bmatrix}$$

for an arbitrary $b \in \mathbb{R}$. The same result is obtained if we set $\tilde{F} := BF$ and we apply the methodology corresponding to the case when the initial system is observable.

6 Conclusion

The paper has presented a methodology for designing static output feedback controllers of linear time-invariant systems. Two dual procedures are presented corresponding to the cases when the system matrix pairs (A, B) and (A, C) are controllable and observable, respectively. The algorithm relies on multi-linear algebra, algebraic geometry and generalized-inverse matrix theory and avoids altogether the calculation of the transfer function as an intermediate step, which may be numerically ill-conditioned. An example of a minimal system (both controllable and observable) illustrates our approach and shows that in this case, under the assumption of identical closed-loop poles, the two sets of output-feedback matrices, which are obtained when either method is applied (corresponding to the controllability of the pair (A, B) or the observability of the pair (A, C)), are identical.

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