Optimizing the EWMA Control Chart to Detect Changes in the Mean of a Long-Memory Seasonal Fractionally Integrated Moving Average and an Exogenous Variable Process with Exponential White Noise and its Application to Electrical Output Data

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Abstract: - The exponentially weighted moving average (EWMA) control chart is frequently employed to monitor changes in process parameters. We developed a method to efficiently track minor changes sensitively, particularly when the data of the process are correlated. The average run length (ARL) is an essential metric employed to evaluate the efficacy of a control chart. Herein we provide exact formulas for the in-control ARL (ARL_0) and out-of-control ARL (ARL_1) for the mean of a long-memory seasonal fractionally integrated moving average with an exogenous variable model order D, Q, r ($\text{LSFIMAX}(D, Q, K)_S$) process with exponential white noise on an EWMA control chart. The ARL results obtained using the exact formulas method were consistent with those using the classical numerical integral equation (NIE) method. The sensitivity of the EWMA control chart to changes in the ARL of the mean of a $\text{LSFIMAX}(D, Q, K)_S$ process using the proposed and NIE methods with a low ARL_1 value and various change levels was assessed in terms of the percentage difference in the expected ARL obtained using both methods, while the standard deviation of the RL (SDRL) was employed to assess the detected changes. Furthermore, the performances of the methods were evaluated temporally. In contrast, NIE also takes the time to display ARL_1 results in seconds. The extensive simulation-based results indicate that the exact formulas approach performed better than the NIE method for all change levels in the mean of the $\text{LSFIMAX}(D, Q, K)_S$ process in terms of the results delivery time. An illustrative monitoring example using data on electricity production from natural gas is also provided to demonstrate the proposed method's practicability.

Key-Words: - Exponentially weighted moving average (EWMA) control chart, average run length (ARL), long-memory $\text{SFIMAX}(D, Q, K)_S$ or $\text{LSFIMAX}(D, Q, K)_S$ process, standard deviation of the run length (SDRL), exponential white noise, numerical integral equation (NIE) method.

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1 Introduction

Control charts employed for monitoring variations in process parameters to identify ascertainable process improvements comprise a crucial element of statistical process control (SPC), [1], [2]. They can effectively highlight process deviations, thereby aiding in maintaining the underlying process's stability. The Shewhart control chart is effective at tracking significant process changes whereas the cumulative sum (CUSUM) control chart [3] and exponentially weighted moving average (EWMA) control charts [4] are more capable of detecting minor-to-moderate changes in the process mean. In particular, the EWMA control chart, which is highly advantageous for tracking minor changes in the process mean, is of primary interest in the present study.

The design parameters and performance of the EWMA control chart are typically determined using Monte Carlo simulations [4], [5], the Markov chain method [6], the numerical integral equation (NIE) method [7], or exact formulas [8], [9]. All of these are dependent on the run length (RL), which is defined as the average number of sample points plotted on a chart before the first out-of-control (OOC) signal is detected. The average RL (ARL) and standard deviation of the RL (SDRL) are frequently utilized metrics for developing and analyzing the efficacy of control charts. The ARL represents the average number of observations before an OOC signal manifests. In this context, ARL_0 represents the in-control (IC) state while ARL_1 denotes the OOC state, which is the expected number of samples until a control chart signals given that it is IC or OOC, respectively. In this article, we propose exact formulas for both of these criteria based on integral equations to analyze the performance of the EWMA control chart. This approach has been used several times in various scenarios to assess control chart performance, [10], [11].

Observations in SPC adhering to a stochastic process can be complex in terms of trends, cycles, and/or autocorrelation, [12]. Time series models can be autoregressive ($\text{AR}(p)$, where p denotes the AR order), moving-average ($\text{MA}(q)$, where q denotes the MA order) or combinations thereof, such as AR integrated MA ($\text{ARIMA}(p, d, q)$, [13]. The time series in these models is restricted to being either stationary I(0) or integrated I(d), where d, the differencing number, is an integer. Nevertheless, differencing using fractions has been explored in long-memory processes such as AR fractionally integrated MA ($\text{ARFIMA}(p, d, q)$) [14], which is a stationary model with autocorrelation functions that decays more slowly than the short-memory ARMA model.

The ARFIMA model has been further extended to include a seasonal (S) component, giving rise to models such as $\text{SARFIMA}(P, D, Q)_S$ with AR order P, MA order Q, fractional differencing parameter D, and seasonal parameter S. Moreover, The ARFIMA and SARFIMA models are said to be long-memory processes if the fractional differencing parameter is in the $(0.0, 0.5)$ interval, [15]. Some applications of ARIMA, ARFIMA, and SARFIMA models are price forecasting for agriculture commodities such as rice and rubber, network traffic prediction, predicting macroeconomic variables, and financial time series such as the application to the inflation rate, and statistical process control, [16].

The SARFIMA model has been enhanced by introducing an exogenous factor (i.e., SARFIMAX) for econometric modeling and economic forecasting. Government investment plans, currency exchange rates, interest rates, and inflation rates are all examples of exogenous variables that function autonomously from other variables in the system and can affect the econometric model's predictive accuracy. In the present study [17], we were specifically interested in the long-memory ($\text{LSFIMAX}(D, Q, K)_S$) model, which is based on the long-memory SARFIMAX model in which AR order P is restricted by making it 0.

Assessing observational errors (the discrepancy between the actual and estimated values) is essential in model creation. Normally distributed white noise refers to errors in a time-series model characterized by autocorrelated data, [18], [19], [20]. However, white noise can be non-normally distributed, such as exponentially, [21], [22,], [23], [24].

2 Problem Formulation

The objective of this study is to analyze the ARL for various memory patterns in a $\text{LSFIMAX}(D, Q, K)_S$ model with exponential white noise on an EWMA control chart. The following subsections provide the EWMA control scheme, the $\text{LSFIMAX}(D, Q, K)_S$ model with exponential white noise, and the metrics used to evaluate the ARL methods.

2.1 The EWMA Control Chart

The EWMA control chart is an effective tool with memory capabilities. It enables the sensitive detection of minor to moderate shifts in a process parameter, [25]. Study [26] introduced an extension of the EWMA control chart for processes involving time series data. In this study, our interest is the $\text{LSFIMAX}(D, Q, K)_S$ process, the companion EWMA statistic for which is defined recursively as:

$$Z_t = (1-\lambda)Z_{t-1} + \lambda Y_t,\ t = 1, 2, ... \quad (1)$$

where $0 < \lambda \le 1$ is the smoothing parameter constraint and initial value $Z_0 = \varphi$.

The expectation and variance of Z_t can be defined as:

$$E(Z_t) = \mu,\ V(Z_t) = \sigma^2\left(\frac{\lambda}{2-\lambda}\right)\left[1-(1-\lambda)^{2t}\right],$$

respectively. As t approaches infinity, the estimated variance converges to $V(Z_t)=\sigma^2\left(\frac{\lambda}{2-\lambda}\right)$. The target value, which generally equals the mean (μ,), can be represented by the center line (CL) of the EWMA control chart statistic. The upper control limit (UCL) and the lower control limit (LCL) of the EWMA control chart statistic can be expressed as:

$$UCL=\mu+L\sigma\sqrt{\frac{\lambda}{2-\lambda}},$$

and

$$LCL=\mu-L\sigma\sqrt{\frac{\lambda}{2-\lambda}},$$

where L is the width of the control limit. A process is IC when $0<Z_t<\text{UCL}$. and OOC when $Z_t>\text{UCL}$. Thus, the stopping time (τ) for tracking when the process on the upper-sided EWMA control chart becomes OOC is given by:

$$\tau=\inf\left\{t\geq 0;\, Z_t>\text{UCL}\right\},\ \text{for}\ \varphi<\text{UCL}, \tag{2}$$

2.2 The LSFIMAX(*P, D, K*)s Process

Let B be a backward-shift operator (i.e. $B^{kS}Y_t=Y_{t-kS}$, $k\geq 0,$) and let $\nabla^S=1-B^S$ be a seasonal difference operator (i.e. $\nabla^S Y_t=Y_t-Y_{t-S}$.). In addition, let $\{Y_t\}$ be a generalized $\text{LSFIMAX}(D,Q,K)_S$ model with K exogenous variables

For D-multiple difference operator ∇^S, incorporating the original time series Y_t., seasonal fractional differencing operator ∇_S^D can be expanded using a binomial series expansion in the following manner:

$$\nabla_S^D=(1-B^S)^D=\sum_{k=0}^{\infty}\binom{D}{k}(-1)^k B^{12k}$$
$$=1-DB^{12}+\frac{D(D-1)}{2}B^{24}-..., \tag{3}$$

The value of seasonal fractional differencing parameter D being a non-integer when $D\in(-0.5,0.5)$ is essential for characterizing the SFIMA process: D < 0.5 reflects stationarity whereas D > -0.5 indicates invertibility. Y_t is categorized as exhibiting long memory (or long-range dependent) when $0<D<0.5$; intermediate memory when $-0.5<D<0,$ and short memory (or short-range dependent) when D = 0. Our focus in this study is exclusively on the long-memory process, so we aligned the parameters to ensure consistency with this research objective.

By following [25], the $\text{LSFIMAX}(D,Q,K)_S$ model can be written as:

$$\nabla_S^D Y_t=\varepsilon_t-\Theta_1\varepsilon_{t-S}-\Theta_2\varepsilon_{t-2S}-...-\Theta_Q\varepsilon_{t-QS}$$
$$+\omega_1 X_{1t}+\omega_2 X_{2t}+...+\omega_K X_{Kt}, \tag{4}$$

Substituting ∇_S^D in Equation (3) into Equation (4) gives us:

$$(1-DY_{t-S}+\frac{D(D-1)}{2}Y_{t-2S}-...)Y_t=\varepsilon_t-\Theta_1\varepsilon_{t-S}-\Theta_2\varepsilon_{t-2S}$$
$$-...-\Theta_Q\varepsilon_{t-QS}+\omega_1 X_{1t}+\omega_2 X_{2t}+...+\omega_K X_{Kt}. \tag{5}$$

Simplifying Equation (5) and the generalized $\text{LSFIMAX}(D,Q,K)_S$ model enables tracking changes in the process mean on the EWMA control as follows:

$$Y_t=\varepsilon_t-\sum_{i=1}^{Q}\Theta_i\varepsilon_{t-Si}+\sum_{j=1}^{K}\omega_j X_{jt}+DY_{t-S}-\frac{D(D-1)}{2}Y_{t-2S}+...,\tag{6}$$

such that $|\Theta_i|<1$, where $\Theta_i, i=1,2,...,Q$ are the coefficients for the seasonal MA; $\omega_j, j=1,2,...,K$ are coefficients that are influenced by K exogenous variables; t is the time; $X_{jt}, j=1,2,...,K$ are exogenous variables; ε_t is the white noise following exponential distribution $Exp(\nu)$; $E(\varepsilon_t)=\nu$ is the mean for the exponential white noise. It is important to note that process mean $\nu=\nu_0$= 1 when process $\{Y_t\}$ is IC whereas $(\nu=\nu_1)$. when process $\{Y_t\}$ is OOC. The process mean is articulated as $\nu_1=(1+\delta)\nu_0$, where δ represents the magnitude of the shift in the mean, [27].

3 Problem Solution

$\text{L}(\varphi)$ representing the ARL for the $\text{LSFIMAX}(D,Q,K)_S$ process with initial value φ. can be defined as:

3.1 The Proposed ARL Method

$\mathrm{L}(\varphi)$ representing the ARL for the $\mathrm{LSFIMAX}(D, Q, K)_S$ process with initial value φ. can be defined as

$$\mathrm{ARL} = \mathrm{L}(\varphi) = \mathrm{E}_{\infty}(\tau) \geq \gamma \quad , Z_0 = \varphi \qquad (7)$$

In the context of the first observation, Y_1 only has two possible outcomes:

Case 1: When Y_1 is in the OOC state for Z_1, then

$$(1-\lambda)Z_0 + Y_t > b \text{ or } (1-\lambda)Z_0 + \lambda Y_t < 0.$$

The RL will be 1 in this case.

Case 2: When Y_1 is in the IC state for Z_1, then

$$0 < (1-\lambda)Z_0 + \lambda Y_t < b.$$

Before the OOC signs occur, υ observations will be performed.

Similarly, for $Z_0 = \varphi$ as the initial value:

$$\left(-(1-\lambda)\varphi - \lambda Y_t\right)/\lambda < \upsilon < \left(b - (1-\lambda)\varphi - \lambda Y_t\right)/\lambda,$$

or $$\ell < \upsilon < \hbar$$

where

$$\ell = \left(-(1-\lambda)\varphi - \lambda\varepsilon_1 + \lambda\sum_{i=1}^{Q}\Theta_i\varepsilon_{t-Si-1} - \lambda\sum_{j=1}^{K}\omega_j X_{jt-1} - D\lambda Y_{t-S-1} + \ldots\right)/\lambda,$$

$$\hbar = \left(b - (1-\lambda)\varphi - \lambda\varepsilon_1 + \lambda\sum_{i=1}^{Q}\Theta_i\varepsilon_{t-Si-1} - \lambda\sum_{j=1}^{K}\omega_j X_{jt-1} - D\lambda Y_{t-S-1} + \ldots\right)/\lambda.$$

Random variable υ, which is within the lower and upper bounds of the EWMA control chart, can be expressed as:

$$P(\ell < \upsilon < \hbar) = \int_{\ell}^{\hbar} f(\upsilon)d\upsilon$$

where $f(\upsilon)$ is the probability density function (pdf) of random variable υ; $f(\upsilon) = \dfrac{1}{\nu}\exp\left\{\dfrac{-\upsilon}{\nu}\right\}$.

Thus, the expression of $\mathrm{L}(\varphi)$ can be reformulated using the method established by [7] as follows:

$$\mathrm{L}(\varphi) = \left\{1 - P\left(\frac{-(1-\lambda)\varphi - \lambda Y_t}{\lambda} < \varepsilon_1 < \frac{b-(1-\lambda)\varphi - \lambda Y_t}{\lambda}\right)\right\} + \int_{\ell}^{\hbar}(1 + \mathrm{L}((1-\lambda)\varphi + \lambda Y_t)f(\varepsilon)d\varepsilon$$

$$\mathrm{L}(\varphi) = 1 + \int_{\ell}^{\hbar}\mathrm{L}((1-\lambda)\varphi + \lambda Y_t))f(\varepsilon)d\varepsilon.$$

The integral equation can thus be derived by considering only the upper control limit of the EWMA control chart and $u = (1-\lambda)\varphi + \lambda Y_t$:

$$\mathrm{L}(\varphi) = 1 + \frac{1}{\lambda}\int_0^b \mathrm{L}(u) f\left(\frac{u-(1-\lambda)\varphi}{\lambda} - Y_t\right)du$$

$$= 1 + \frac{1}{\lambda}\int_0^b \mathrm{L}(u)\left(\frac{1}{\nu}\exp\left\{-\frac{u}{\lambda\nu}\right\}.\exp\left\{\frac{(1-\lambda)\varphi}{\lambda\nu} + \frac{Y_t}{\nu}\right\}\right)du.$$

Thus, we can use Fredholm's integral equation of the second kind to solve for the ARL as follows:

$$\mathrm{L}(\varphi) = 1 + \frac{1}{\lambda\nu}\int_0^b \mathrm{L}(u)\left(\exp\left\{-\frac{u}{\lambda\nu}\right\}.\exp\left\{\frac{(1-\lambda)\varphi}{\lambda\nu} + \frac{Y_t}{\nu}\right\}\right)du. \qquad (8)$$

Using the solution for the integral equation in Equation 8, $\mathrm{T}\left(\mathrm{L}(\varphi)\right) = \mathrm{L}(\varphi)$, we can illustrate the existence and uniqueness of the analytical ARL by using Banach's fixed point theorem (the details of which are elaborated in the appendix):

$$\mathrm{L}(\varphi) = 1 + \frac{\exp\left\{\frac{(1-\lambda)\varphi}{\lambda\nu} + \frac{Y_t}{\nu}\right\}}{\lambda\nu}\int_0^b \mathrm{L}(u)\left(\exp\left\{-\frac{u}{\lambda\nu}\right\}\right)du. \quad (9)$$

Since $g = \int_0^b \mathrm{L}(u)\left(\exp\left\{-\frac{u}{\lambda\nu}\right\}\right)du$, we can be compute:

$$g = \int_0^b\left(\exp\left\{-\frac{u}{\lambda\nu}\right\}\right)du + \frac{\exp\left\{\frac{(1-\lambda)\varphi}{\lambda\nu} + \frac{Y_t}{\nu}\right\}}{\lambda\nu}.\, g\left(\exp\left\{-\frac{u}{\lambda\nu}\right\}\right)du$$

$$= \int_0^b \exp\left\{-\frac{u}{\lambda\nu}\right\}du + \frac{g}{\lambda\nu}\exp\left\{\frac{Y_t}{\nu}\right\}.\int_0^b \exp\left\{-\frac{u}{\lambda}\right\}du$$

$$= \lambda\nu\left(1 - \exp\left\{-\frac{b}{\lambda\nu}\right\}\right) + \frac{g}{\lambda\nu}\exp\left\{\frac{Y_t}{\nu}\right\}\left(\nu\left(1 - \exp\left\{-\frac{b}{\lambda\nu}\right\}\right)\right)$$

Thus,

$$g = \frac{\lambda\nu\left(1-\exp\left\{-\frac{b}{\lambda\nu}\right\}\right)}{1-\frac{1}{\lambda}\left(1-\exp\left\{-\frac{b}{\nu}\right\}\right)\exp\left\{\frac{Y_t}{\nu}\right\}}.$$

By substituting constant g into Equation (9), $\mathrm{L}(\varphi)$ becomes:

$$\mathrm{L}(\varphi) = 1 + \frac{\lambda\left(1-\exp\left\{-\frac{b}{\lambda\nu}\right\}\right)\exp\left\{\frac{(1-\lambda)\varphi}{\lambda\nu}\right\}\exp\left\{\frac{Y_t}{\nu}\right\}}{\lambda-\left(1-\exp\left\{-\frac{b}{\nu}\right\}\right)\exp\left\{\frac{Y_t}{\nu}\right\}}.$$

As a result, the analytical ARL derived from exact formulas by solving the IE becomes:

$$\mathrm{L}(\varphi) = 1 - \frac{\lambda\left(1-\exp\left\{-\frac{b}{\lambda\nu}\right\}\right)\exp\left\{\frac{(1-\lambda)\varphi}{\lambda\nu}\right\}}{\left(1-\exp\left\{-\frac{b}{\nu}\right\}\right)-\lambda\exp\left\{-\frac{Y_t}{\nu}\right\}}, \text{ where}$$

$$Y_t = \varepsilon_t - \sum_{i=1}^{Q}\Theta_i\varepsilon_{t-Si} + \sum_{j=1}^{K}\omega_j X_{jt} + DY_{t-S} - \frac{D(D-1)}{2}Y_{t-2S} + \dots,$$

Therefore, when the process is in the IC state with exponential parameter $\nu = \nu_0$, the exact formula for ARL_0 becomes:

$$\mathrm{L}_0(\varphi) = 1 - \frac{\lambda\left(1-\exp\left\{-\frac{b}{\lambda\nu_0}\right\}\right)\exp\left\{\frac{(1-\lambda)\varphi}{\lambda\nu_0}\right\}}{\left(1-\exp\left\{-\frac{b}{\nu_0}\right\}\right)-\lambda\exp\left\{-\frac{Y_t}{\nu_0}\right\}}. \quad (10)$$

Similarly, when the process is in the OOC state, exponential parameter ν_1 is defined as $(1+\delta)\nu_0$, where δ represents the shift size. Thus, the exact formula for ARL_1 can be expressed as:

$$\mathrm{L}_1(\varphi) = 1 - \frac{\lambda\left(1-\exp\left\{-\frac{b}{\lambda\nu_1}\right\}\right)\exp\left\{\frac{(1-\lambda)\varphi}{\lambda\nu_1}\right\}}{\left(1-\exp\left\{-\frac{b}{\nu_1}\right\}\right)-\lambda\exp\left\{-\frac{Y_t}{\nu_1}\right\}}. \quad (11)$$

3.2 The NIE Method

Let $\hat{L}(\varphi)$ represent the NIE method. In this case, we use the composite midpoint rule to calculate the ARL, the integral equation for which is defined as

$$\mathrm{L}(\varphi) = 1 + \frac{1}{\lambda}\int_0^b \mathrm{L}(u) f\left(\frac{u-(1-\lambda)\varphi}{\lambda} - Y_t\right)du.$$

Integral $\int_0^b \mathrm{L}(u) f\left(\frac{u-(1-\lambda)\varphi}{\lambda} - Y_t\right)du$ represents the sum of the areas of m equivalent rectangles or intervals, each having a base of $b = \frac{h-0}{m}$ and heights determined by the values of the integrand at the midpoints of intervals of length b, commencing at zero. Using the composite midpoint rule, interval $[0, h]$ is divided into sub-grids $[u_{r-1}, u_r]$ with midpoint point $u_r = b\left(r - \frac{1}{2}\right); r = 1, 2, \dots, m$ and a set of constant weights $w_r = \frac{h}{m}; r = 1, 2, \dots, m.$

The integral can be approximated as

$$\int_0^h \mathrm{L}(u) f(u) du \approx \sum_{r=1}^{m} w_r f(u_r). \quad (12)$$

Let $\hat{\mathrm{L}}(u_l); l = 1, 2, \dots, m$ be an approximation for the integral equation evaluated by the solution of a system of algebraic linear equations as follows:

$$\hat{\mathrm{L}}(u_l) \approx 1 + \frac{1}{\lambda}\sum_{r=1}^{m} w_r \tilde{L}(u_r) f\left(\frac{u_r-(1-\lambda)u_l}{\lambda} - Y_t\right). \quad (13)$$

Substituting u_l with φ in Equation (13) yields

$$\hat{\mathrm{L}}(\varphi) \approx 1 + \frac{1}{\lambda}\sum_{r=1}^{m} w_r \tilde{L}(u_r) f\left(\frac{u_r-(1-\lambda)\varphi}{\lambda} - Y_t\right). \quad (14)$$

This represents the approximation of ARL using the NIE method, which was employed to assess the accuracy of the proposed method.

3 .3 Metrics for Evaluating the Efficacy of the EWMA Control Chart

3.3.1 SDRL

The RL distribution properties can be utilized to monitor the sensitivity to variations in the mean of the analyzed process. For the IC state, we can compute:

$$\mathrm{ARL}_0 = \frac{1}{v_0} \text{ and } \mathrm{SDRL}_0 = \sqrt{\frac{1-v_0}{{v_0}^2}}, \quad (15)$$

where v_0 denotes the Type I error, which indicates that the mean of a long-memory process has not changed. In our case, setting the ARL_0 to 370 gives us v_0 as 0.0027. Similarly, for the OOC state:

$$\mathrm{ARL}_1 = \frac{1}{v_1} \text{ and } \mathrm{SDRL}_1 = \sqrt{\frac{1-v_1}{{v_1}^2}}, \quad (16)$$

where v_1 is referred to as the Type II error, which indicates that the mean of the process has changed.

When testing, the lowest ARL_1 or SDRL_1 value is the most sensitive to detect each situation, it is employed to detect changes in the mean.

3.3.2 The Expected ARL (EARL) and Expected SDRL (ESDRL)

These are criteria that can be used to evaluate the range of change values, specifically $\delta_{\min}$ and $\delta_{\max}$, which represent the lower and upper bounds of δ, respectively. The pdf of the shift size is $f_\delta(\delta)$. Thereby, we can evaluate the performance of the proposed method for a defined range of shifts by identifying the lowest EARL or ESDRL values. The EARL and ESDRL are respectively defined as:

$$\mathrm{EARL} = \int_{\delta_{\min}}^{\delta_{\max}} \mathrm{ARL}(\delta).f_\delta(\delta)d\delta \text{ and}$$

$$\mathrm{ESDRL} = \int_{\delta_{\min}}^{\delta_{\max}} \mathrm{SDRL}(\delta).f_\delta(\delta)d\delta, \quad (17)$$

where $f_\delta(\delta)$ is the pdf of the shift size and $\mathrm{ARL}(\delta)$ and $\mathrm{SDRL}(\delta)$ represent the ARL and SDRL, respectively, as a function of shift size. Thus, Equation (17) can be rewritten as:

$$\mathrm{EARL} = \frac{1}{\Delta}\sum_{\delta=\delta_{\min}}^{\delta_{\max}} \mathrm{ARL}(\delta) \text{ and}$$

$$\mathrm{ESDRL} = \frac{1}{\Delta}\sum_{\delta=\delta_{\min}}^{\delta_{\max}} \mathrm{SDRL}(\delta), \quad (18)$$

where Δ represents the incrementing value from $\delta_{\min}$ to $\delta_{\max}$, by applying the mathematical technique for approximating the definite integral of a function through the Riemann sum approach.

3.4 Performance Evaluation of the ARL Methods Via a Simulation Study

Here, we present the details and results of a comparative study analyzing the performances of the exact formulas alongside the NIE method applied to detect changes in the mean of a $\mathrm{LSFIMAX}(D, Q, K)_S$ process on the EWMA control chart. We considered positive and negative $\mathrm{LSFIMAX}(D, Q, K)_S$ processes, so both the MA values of the parameters and the corresponding OOC state (ARL_1) for $\mathrm{LSFIMAX}(D, 1, 1)_{12}$ and $\mathrm{LSFIMAX}(D, 1, 2)_{12}$ processes were of interest. Although positive MA is the most common in manufacturing processes, negative MA is equally interesting. To this end, the ARL values based on the exact formulas and NIE methods were computed for several parameter combinations using Mathematica 8.

Table 1 (Appendix) provides the chart parameters with various combinations of (λ, b) and corresponding OOC state (ARL_1) for $\mathrm{LSFIMAX}(D, 1, 1)_{12}$ and $\mathrm{LSFIMAX}(D, 1, 2)_{12}$ processes: $D =$ 0.125, 0.25, 0.375, $\Theta_1 = \pm 0.1$, $\omega_1 = 0.1$, and $\omega_2 = 0.2$ when the IC ARL (ARL_0) was 370. The computed values for b for a fixed value of the smoothing constant (λ) can be seen in columns 6 and 7. For illustration, when $\mathrm{LSFIMAX}(0.125, 1, 1)_{12}$ for $\Theta_1 = 0.1$, $\omega_1 = 0.1$, $(\lambda, b) =$ (0.05, 0.0.0000000834678) attained the desired $\mathrm{ARL}_0 = 370$.

The ARL results for $0.125 \le \delta \le 1$ are reported in Appendix in Table 2, Table 3, Table 4 and Table 5; the decrease in both ARL and SDRL as δ increased is evident, which implies that tracking more significant shifts can be expedited, thereby reducing ARL dispersion. Furthermore, the results indicate that the ARL and SDRL values for the EWMA control chart decreased as the smoothing value was increased ($\lambda = 0.05,\ 0.1,$ and 0.3) for the same mean shift value δ.

The ARL_1 results using both methods for $\mathrm{LSFIMAX}(D, Q, K)_S$ processes on a one-sided EWMA control chart were similar for all process mean shifts. Moreover, the sensitivity of the EWMA control chart to issue an OOC signal was better when tracking minor process mean changes $(0.125 \le \delta < 0.375)$ compared to significant changes

$(0.375 \le \delta \le 1)$. This means that its ability to detect small changes is excellent. When considering $SDRL_1$ as a performance metric, it yielded lower results than ARL_1.

EARL and ESDRL were used to measure the performance of the EWMA control chart for shift sizes that fall in the interval $(\delta_{\min} = 0, \delta_{\max} = 1)$. The EARL values obtained using the exact formulas method were larger than the corresponding EARL values using the NIE method for all $\text{LSFIMAX}(D, Q, K)_S$ processes with combinations of D, Q, K_1 or K_2 and S, as reported in Table 6 (Appendix); $(\lambda) = 0.05$ provided the smallest EARL and SDRL values (boldfaced in the table). The results enabled us to ascertain the optimal parameters for the $\text{LSFIMAX}(D, Q, K)_S$ processes on the EWMA control chart based on the ARL from the exact formulas.

As can be seen in Table 6 (Appendix), the computational time for calculating the OOC ARL results only took a fraction of a second when using the exact formulas, while it took around 12–13 seconds using the NIE method. Thus, the shorter computation time using the proposed ARL method makes it superior to using the NIE method.

3.5 Application of the Exact Formulas Approach to Real Data

The first dataset comprising monthly electricity production from natural gas from January 1, 1987, to December 1, 1995 (exogenous variable (X_1) was found fit well to a $\text{LSFIMAX}(D, 1, 1)_{12}$ process, the estimated coefficients for which were $\hat{D} = 0.169122$, $\hat{\Theta}_1 = 0.470586$, and $\hat{\omega}_1 = 0.102217$. The white noise from the process was subsequently analyzed using the Kolmogorov-Smirnov test, revealing that it followed an exponential distribution (KS = 0.865; p-value = 0.443 > 0.05). The exponential parameter $(\nu = \nu_0)$ was 76.0421 (Table 7 and Table 8).

The second dataset comprising electricity exports from April 1, 1994, to August 1, 2004 (exogenous variable $(X_2))$ was found to fit well to a $\text{LSFIMAX}(D, 1, 2)_{12}$ model with coefficients $\hat{D} =$ 0.1645, $\hat{\Theta}_1 = 0.5756$, and $\hat{\omega}_1 = 0.0985$, $\hat{\omega}_2 =.45.4324$. The parameter value for the exponential white noise was 157.0983. The appropriateness of the fitted models was then evaluated by plotting the graph shown in Figure 1. The values obtained from the estimation were similar to the actual values.

Table 7. Parameter estimation for the $\text{LSFIMAX}(D, Q, K)_S$ processes based on the two real datasets

Parameters	Estimate	Std.Error	p-value
First dataset: $\text{LSFIMAX}(D, 1, 1)_{12}$			
D	0.1691	0.0464	0.0004*
SMA 12	0.4706	0.1013	0.0000*
X_1: Natural gas quantitie	0.1022	0.0013	
R^2	0.9221		
Adjusted R^2	0.9206		
Second dataset: $\text{LSFIMAX}(D, 1, 2)_{12}$			
D	0.1645	0.0487	0.0009*
SMA 12	0.5756	0.0769	0.0000*
X_1: Natural gas quantitie	0.0985	0.0027	0.0000*
X_2: Electricity export	45.4324	8.7513	0.0000*
R^2	0.9662		
Adjusted R^2	0.9655		

** significance level of 0.05*
ns non-significance level of 0.05.

Table 8. Testing the distributions of the white noise in the real datasets

	Residuals
First dataset: $\text{SFIMAX}(0.1691, 1, 1)_{12}$	
Exponential parameter	76.0421
Kolmogorov-Smirnov	0.865
Asymptotic Significance (2-Sided)	0.443^{ns}
Second dataset: $\text{SFIMAX}(0.1645, 1, 2)_{12}$	
Exponential parameter	157.0983
Kolmogorov-Smirnov	1.319
Asymptotic Significance (2-Sided)	0.062^{ns}

** significance level of 0.05*
ns non-significance level of 0.05.

For the first dataset, the $\text{LSFIMAX}(D, 1, 1)_{12}$ process is:

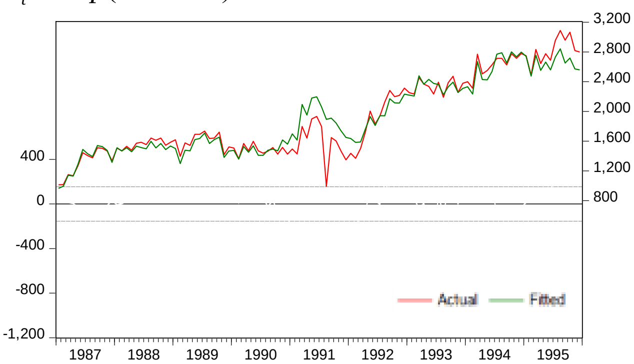
$$Y_t = \varepsilon_t - 0.4706\varepsilon_{t-12} + 0.1691Y_{t-12} + 0.0703Y_{t-24} + \ldots + 0.1022X_1,$$

where $\varepsilon_t \sim Exp(76.0421)$.

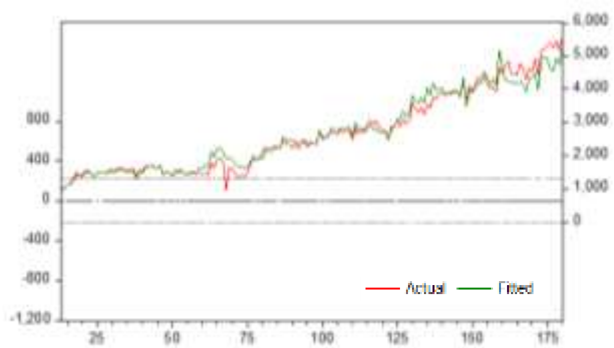
For the second dataset, the $\text{LSFIMAX}(D, 1, 2)_{12}$ process is:

$$Y_t = \varepsilon_t - 0.5756\varepsilon_{t-12} + 0.1645Y_{t-12} + 0.6872Y_{t-24} + \ldots + 0.0985X_1 + 45.4324X_2,$$

$\varepsilon_t \sim Exp(157.0983)$.



(a) $\text{LSFIMAX}(D, 1, 1)_{12}$ process



(b) $\text{LSFIMAX}(D, 1, 2)_{12}$ process

Fig. 1: Graphs of the fitted models to the real datasets.

From Equations (10) and (11), we derived the ARL values using the exact formulas in Equations (19) and (20), specifically for the IC state predefined as $\text{ARL}_0 = 370$ to calculate the $b = 3.8772974$, 11.3758767 values when $\lambda = 0.05$, (the optimal EWMA control chart parameter values from the simulation study results).

For the two real datasets, one must respectively substitute $v = v_0$ for IC and $v = v_1$ for OOC in the following manner:

$$\text{L}(\varphi) = 1 - \lambda\left(1 - \exp\left\{-\frac{b}{\lambda v}\right\}\right)\exp\left\{\frac{(1-\lambda)\varphi}{\lambda v}\right\} \Big/ \left(\left(1 - \exp\left\{-\frac{b}{v}\right\}\right)\right.$$
$$\left.-\lambda\exp\left\{\frac{0.4706\varepsilon_{t-12} - 0.1691Y_{t-12} - 0.0703Y_{t-24} - \ldots - 0.1022X_1}{v}\right\}\right) \quad (19)$$

and

$$\text{L}(\varphi) = 1 - \lambda\left(1 - \exp\left\{-\frac{b}{\lambda v}\right\}\right)\exp\left\{\frac{(1-\lambda)\varphi}{\lambda v}\right\} \Big/ \left(\left(1 - \exp\left\{-\frac{b}{v}\right\}\right)\right.$$
$$-\lambda\exp\left\{\frac{0.5756\varepsilon_{t-12} - 0.1645Y_{t-12} - 0.6872Y_{t-24} - \ldots}{v}\right.$$
$$\left.\left.\frac{-0.0985X_1 - 45.4324X_{21}}{v}\right\}\right). \quad (20)$$

The results in Table 9 (Appendix) provide information on the ARL_1 and SDRL_1 for a range of shifts (δ) in the process mean. The results derived using the exact formulas and NIE methods were identical when applied to electricity production from natural gas using one or two exogenous variables. Moreover, SDRL_1 yielded results similar to ARL_1, exhibiting a decreasing pattern as the mean shift size increased in these real-world scenarios. However, in terms of computational time, the exact formulas method required mere fractions of a second in contrast to the NIE method, which required a waiting period of approximately 1–2 seconds. These results are the same as those from the simulation study.

For electricity production from natural gas in conjunction with natural gas quantities, a change in the electricity production process can be detected efficiently by using the EWMA control chart and applying the exact formulas method.

4 Conclusion

We proposed a method using exact formulas for the ARL to evaluate the efficacy of the EWMA control chart to detect changes in the mean of a long-memory $\text{LSFIMAX}(D, Q, K)_S$ process with exponential white noise. We validated the exact formulas approach by comparing its performance with that of the standard NIE method. We found that both methods produced comparable results but the proposed method significantly reduced the CPU time. Subsequently, the ARL derived using exact formulas was compared with the SDRL for various magnitudes of changes in the process mean. Moreover, every range of control chart changes was evaluated using the EARL and the ESDRL.

The proposed EWMA control chart design parameters were computed for various $\text{LSFIMAX}(D, Q, K)_S$ processes, and the optimum was 0.05. The numerical findings based on the performance evaluation revealed that the exact formulas method is excellent for detecting minor shifts in the process mean on the EWMA control chart.

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Declaration of Generative AI and AI-assisted technologies in the writing process

During the preparation of this work, the author used QuillBot in order to study the source and importance of research.. After using this tool/service, the author reviewed and edited the content as needed and took full responsibility for the content of the publication.

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APPENDIX

Theorem 1 (Banach's fixed-point theorem).
Suppose that (M, d) is a complete metric space, then mapping $\mathrm{T}:\mathrm{M}\to\mathrm{M}$ represents a contraction mapping on M if there exists real number $\eta;\ 0\le\eta<1$ such that

$$d\left(\mathrm{T}(\mathrm{L}_1),\mathrm{T}(\mathrm{L}_2)\right)\le \eta d\left(\mathrm{L}_1,\mathrm{L}_2\right) \text{ for } \mathrm{L}_1,\ \mathrm{L}_2\in\mathrm{M}.$$

Consequently, T has a fixed point that is precisely unique, such as a unique $\mathrm{L}(.)\in\mathrm{M}$ that satisfies $\mathrm{T}(\mathrm{L})=\mathrm{L}..$

Equation (9) presents the $\mathrm{L}(\varphi)$ of Fredholm's integral equation of the second kind for various memory patterns in a $\mathrm{LSFIMAX}(D,Q,K)_S$ model with exponential white noise on an EWMA control chart.

$$\mathrm{L}(\varphi)-\frac{1}{\lambda\nu}\int_0^b \mathrm{L}(u)\left(\exp\left\{-\frac{u}{\lambda\nu}\right\}.\exp\left\{\frac{(1-\lambda)\varphi}{\lambda\nu}+\frac{Y_t}{\nu}\right\}\right)du=1,$$

where $k(\varphi,u)=\frac{1}{\lambda\nu}\exp\left\{-\frac{u}{\lambda\nu}\right\}.\exp\left\{\frac{(1-\lambda)\varphi}{\lambda\nu}+\frac{Y_t}{\nu}\right\}$ denotes a kernel function, $\mathrm{L}(\varphi):[0,b]\to[0,b]$ represents an unknown function, and the mapping T is defined as

$$\mathrm{T}\left(\mathrm{L}(\varphi)\right)=1+\frac{\exp\left\{\frac{(1-\lambda)\varphi}{\lambda\nu}+\frac{Y_t}{\nu}\right\}}{\lambda\nu}\int_0^b \mathrm{L}(u)\left(\exp\left\{-\frac{u}{\lambda\nu}\right\}\right)du.$$

Theorem 2 $\mathrm{L}(\varphi)$ representing the ARL for the $\mathrm{LSFIMAX}(D,Q,K)_S$ model with exponential white noise on a EWMA control chart has existence and uniqueness.

***Proof*:** To prove the existence of the ARL.
Let T be a contraction in complete metric space (M, d), $\mathbf{C}[0,b]$ be a set of continuous functions of the $\mathrm{L}(\varphi)$ on $[0,b]$, $\mathrm{L}_0(\varphi)\in\mathbf{C}[0,b]$, and $(\mathrm{L}_n)_{n\ge0}$ be a Cauchy's sequence of $\mathrm{L}(\varphi)$ that satisfies

$$\mathrm{L}_{n+1}=\mathrm{T}\left(\mathrm{L}_{n+1}\right).$$

Hence,

$$d\left(\mathrm{L}_{n+1},\mathrm{L}_n\right)=d\left(T\left(\mathrm{L}_n\right),T\left(\mathrm{L}_{n-1}\right)\right)\le\eta d\left(T\left(\mathrm{L}_n\right),T\left(\mathrm{L}_{n-1}\right)\right),$$

where $0\le\eta<1$.

Iteratively $d\left(\mathrm{L}_{n+1},\mathrm{L}_n\right)\le\eta^n d\left(T\left(\mathrm{L}_n\right),T\left(\mathrm{L}_{n-1}\right)\right)$, for $n\ge0$.

Applying the triangle inequality to this formula repeatedly when $n\ge m$ implies that

$$d\left(\mathrm{L}_n,\mathrm{L}_m\right)\le d(\mathrm{L}_n,\mathrm{L}_{n-1})+...+d(\mathrm{L}_{m+1},\mathrm{L}_m)$$
$$\le(\eta^{n-1}+...+\eta^m)d(\mathrm{L}_1,\mathrm{L}_0)$$

By applying the property of the sum of a geometric series in the set of η, we derive

$$d\left(\mathrm{L}_n,\mathrm{L}_m\right)\le\frac{\eta^m}{1-\eta}d(\mathrm{L}_1,\mathrm{L}_0),$$

Therefore, $(\mathrm{L}_n)_{n\ge0}$ is a Cauchy sequence, and $\lim_{n\to\infty}\mathrm{T}^n\left(\mathrm{L}_n\right)\to\mathrm{L}$. Hence, there exists a unique point $\mathrm{L}\in\mathrm{M}$ such that $\mathrm{T}\left(\mathrm{L}\right)=\mathrm{L}..$.
This completes the proof.

***Proof*:** To prove the uniqueness of the ARL.
Let T be a contraction mapping on the complete metric space $\left(\mathbf{C}[0,b],\|.\|_\infty\right)$. A set of continuous functions of the ARL defined on $[0,b]$, and $\mathbf{C}[0,b]$ becomes a normed space if we define

$$\|\mathrm{L}\|_\infty=\sup_{\varphi\in[0,b]}\left|\int_0^b k(\varphi,u)du\right|,$$

where $k(\varphi,u)=\frac{1}{\lambda\nu}\exp\left\{-\frac{u}{\lambda\nu}\right\}.\exp\left\{\frac{(1-\lambda)\varphi}{\lambda\nu}+\frac{Y_t}{\nu}\right\}$ is a continuous functions of the $\mathrm{L}(\varphi)$ on $[0,b]$, where

$$Y_t=\varepsilon_t-\sum_{i=1}^{Q}\Theta_i\varepsilon_{t-Si}+\sum_{j=1}^{K}\omega_j X_{jt}+DY_{t-S}-\frac{D(D-1)}{2}Y_{t-2S}+...,$$

$$\|\mathrm{T}(\mathrm{L}_1)-\mathrm{T}(\mathrm{L}_2)\|_\infty=\sup_{\varphi\in[0,b]}\left|\int_0^b k(\varphi,u)\left[\mathrm{L}_1(u)-\mathrm{L}_2(u)\right]du\right|$$
$$\le\sup_{\varphi\in[0,b]}\int_0^b\left|k(\varphi,u)\right|\left[\mathrm{L}_1(u)-\mathrm{L}_2(u)\right]du$$
$$\le\sup_{\varphi\in[0,b]}\int_0^b\left|k(\varphi,u)\right|du\,\|\mathrm{L}_1(u)-\mathrm{L}_2(u)\|$$

where $\eta=\sup_{\varphi\in[0,b]}\int_0^b\left|k(\varphi,u)\right|du<1$. Hence, T is a contraction mapping.
This completes the proof.

Consequently, analytical ARL using the exact formula for various memory patterns in an FF model with exponential white noise on an EWMA control chart is both existent and unique.

Table 1. The parameter values for $\text{LSFIMAX}(D, Q, K)_S$ processes for an EWMA control chart

$\text{LSFIMAX}(D, Q, K)_S$	Coefficient parameters			In-control ARL value equal to 370		
	Θ_1	ω_1	ω_2	$\lambda = 0.05$	$\lambda = 0.10$	$\lambda = 0.3$
$(0.125, 1, 1)_{12}$	0.1	0.1		0.0000000834678	0.003577620	0.27082758
	-0.1	0.1		0.0000000683377	0.002919921	0.21570576
$(0.250, 1, 1)_{12}$	0.1	0.1		0.0000000694004	0.002965980	0.21947581
	-0.1	0.1		0.0000000568202	0.002422030	0.17576348
$(0.375, 1, 1)_{12}$	0.1	0.1		0.0000000593033	0.002529172	0.18423551
	-0.1	0.1		0.0000000485534	0.002066130	0.14808628
$(0.125, 1, 2)_{12}$	0.1	0.1	0.3	0.0000000618345	0.002638508	0.19294930
	-0.1	0.1	0.3	0.0000000506258	0.002155240	0.15494992
$(0.250, 1, 2)_{12}$	0.1	0.1	0.3	0.0000000514130	0.002189103	0.15757003
	-0.1	0.1	0.3	0.0000000420934	0.001788856	0.12700185
$(0.375, 1, 2)_{12}$	0.1	0.1	0.3	0.0000000439330	0.001867740	0.13295859
	-0.1	0.1	0.3	0.0000000359692	0.001526680	0.10743461

Table 2. Comparison of the ARL_1 values derived using the exact formulas and NIE methods for $LSFIMAX(D, Q, K)_S$ processes on an EWMA control chart when $\lambda = 0.05$.

Models	LSFIMAX $(D, 1, 1)_{12}$ with $\Theta_1 = 0.1$, $\omega_1 = 0.1$								
	$D = 0.125$			$D = 0.250$			$D = 0.375$		
δ	ARL_{Exact}	ARL_{NIE}	SDRL	ARL_{Exact}	ARL_{NIE}	SDRL	ARL_{Exact}	ARL_{NIE}	SDRL
0.125	39.789	39.789	39.286	39.002	39.002	38.499	38.344	38.344	37.841
0.250	7.327	7.327	6.809	7.098	7.098	6.579	6.909	6.909	6.389
0.375	2.422	2.422	1.856	2.352	2.352	1.783	2.296	2.296	1.725
0.500	1.407	1.407	0.757	1.383	1.383	0.728	1.363	1.363	0.703
0.625	1.140	1.140	0.399	1.131	1.131	0.385	1.123	1.123	0.372
0.750	1.056	1.056	0.243	1.052	1.052	0.234	1.048	1.048	0.224
0.875	1.025	1.025	0.160	1.023	1.023	0.153	1.021	1.021	0.146
1.000	1.012	1.012	0.110	1.011	1.011	0.105	1.010	1.01	0.100
Expected	441.42		396.96	432.42		387.73	424.91		380.00
Models	LSFIMAX $(D, 1, 1)_{12}$ with $\Theta_1 = -0.1$, $\omega_1 = 0.1$								
	$D = 0.125$			$D = 0.250$			$D = 0.375$		
δ	ARL_{Exact}	ARL_{NIE}	SDRL	ARL_{Exact}	ARL_{NIE}	SDRL	ARL_{Exact}	ARL_{NIE}	SDRL
0.125	38.937	38.937	38.434	38.167	38.167	37.664	37.523	37.523	37.020
0.250	7.079	7.079	6.560	6.859	6.859	6.339	6.678	6.678	6.158
0.375	2.347	2.347	1.778	2.281	2.281	1.709	2.227	2.227	1.653
0.500	1.381	1.381	0.725	1.358	1.358	0.697	1.340	1.340	0.675
0.625	1.130	1.130	0.383	1.121	1.121	0.368	1.114	1.114	0.356
0.750	1.051	1.051	0.232	1.047	1.047	0.222	1.044	1.044	0.214
0.875	1.023	1.023	0.153	1.021	1.021	0.146	1.02	1.02	0.143
1.000	1.011	1.011	0.105	1.01	1.01	0.100	1.009	1.009	0.095
Expected	431.67		386.96	422.91		377.96	415.64		370.51
Models	LSFIMAX $(D, 1, 2)_{12}$ with $\Theta_1 = 0.1$, $\omega_1 = 0.1$, $\omega_2 = 0.3$								
	$D = 0.125$			$D = 0.250$			$D = 0.375$		
δ	ARL_{Exact}	ARL_{NIE}	SDRL	ARL_{Exact}	ARL_{NIE}	SDRL	ARL_{Exact}	ARL_{NIE}	SDRL
0.125	38.517	38.517	38.014	37.756	37.756	37.253	37.119	37.119	36.616
0.250	6.959	6.959	6.440	6.743	6.743	6.223	6.565	6.565	6.044
0.375	2.310	2.310	1.740	2.246	2.246	1.673	2.194	2.194	1.619
0.500	1.368	1.368	0.710	1.346	1.346	0.682	1.329	1.329	0.661
0.625	1.125	1.125	0.375	1.116	1.116	0.360	1.11	1.11	0.349
0.750	1.049	1.049	0.227	1.045	1.045	0.217	1.042	1.042	0.209
0.875	1.022	1.022	0.150	1.02	1.02	0.143	1.019	1.019	0.139
1.000	1.011	1.011	0.105	1.01	1.01	0.100	1.009	1.009	0.095
Expected	426.89		382.09	418.26		373.21	411.10		365.86
Models	LSFIMAX $(D, 1, 2)_{12}$ with $\Theta_1 = -0.1$, $\omega_1 = 0.1$, $\omega_2 = 0.3$								
	$D = 0.125$			$D = 0.250$			$D = 0.375$		
δ	ARL_{Exact}	ARL_{NIE}	SDRL	ARL_{Exact}	ARL_{NIE}	SDRL	ARL_{Exact}	ARL_{NIE}	SDRL
0.125	37.693	37.693	37.190	36.948	36.948	36.445	36.326	36.326	35.823
0.250	6.725	6.725	6.205	6.518	6.518	5.997	6.347	6.347	5.826
0.375	2.241	2.241	1.668	2.18	2.18	1.604	2.130	2.13	1.551
0.500	1.344	1.344	0.680	1.324	1.324	0.655	1.307	1.307	0.633
0.625	1.116	1.116	0.360	1.108	1.108	0.346	1.101	1.101	0.333
0.750	1.045	1.045	0.217	1.042	1.042	0.209	1.039	1.039	0.201
0.875	1.020	1.020	0.143	1.018	1.018	0.135	1.017	1.017	0.131
1.000	1.010	1.010	0.100	1.009	1.009	0.095	1.008	1.008	0.090
Expected	417.55		372.50	409.18		363.89	402.20		356.70

Table 3. Comparison of the ARL_1 values derived using the exact formulas and NIE methods for LSFIMAX$(D, Q, K)_S$ processes on an EWMA control chart when $\lambda = 0.10$.

Models	LSFIMAX $(D, 1, 1)_{12}$ with $\Theta_1 = 0.1$, $\omega_1 = 0.1$								
	D = 0.125			D = 0.250			D = 0.375		
δ	ARL_{Exact}	ARL_{NIE}	SDRL	ARL_{Exact}	ARL_{NIE}	SDRL	ARL_{Exact}	ARL_{NIE}	SDRL
0.125	118.348	118.348	117.847	115.836	115.836	115.335	113.746	113.746	113.245
0.250	47.419	47.419	46.916	45.654	45.654	45.151	44.209	44.209	43.706
0.375	22.54	22.54	22.034	21.435	21.435	20.929	20.542	20.542	20.036
0.500	12.275	12.275	11.764	11.574	11.574	11.063	11.014	11.014	10.502
0.625	7.479	7.479	6.961	7.017	7.017	6.498	6.652	6.652	6.132
0.750	5.007	5.007	4.479	4.691	4.691	4.161	4.443	4.443	3.911
0.875	3.630	3.630	3.090	3.406	3.406	2.863	3.230	3.230	2.684
1.000	2.812	2.812	2.257	2.647	2.647	2.088	2.519	2.519	1.956
Expected	1,756.08		1,722.78	1,698.08		1,664.70	1,650.84		1,617.38
Models	LSFIMAX $(D, 1, 1)_{12}$ with $\Theta_1 = -0.1$, $\omega_1 = 0.1$								
	D = 0.125			D = 0.250			D = 0.375		
δ	ARL_{Exact}	ARL_{NIE}	SDRL	ARL_{Exact}	ARL_{NIE}	SDRL	ARL_{Exact}	ARL_{NIE}	SDRL
0.125	115.629	115.629	115.128	113.185	113.185	112.684	111.149	11.149	110.648
0.250	45.51	45.51	45.007	43.825	43.825	43.322	42.444	42.444	41.941
0.375	21.346	21.346	20.840	20.307	20.307	19.801	19.466	19.466	18.959
0.500	11.518	11.518	11.007	10.867	10.867	10.355	10.346	10.346	9.833
0.625	6.980	6.980	6.461	6.557	6.557	6.036	6.220	6.220	5.698
0.750	4.666	4.666	4.136	4.378	4.378	3.846	4.152	4.152	3.618
0.875	3.388	3.388	2.844	3.185	3.185	2.638	3.026	3.026	2.476
1.000	2.634	2.634	2.075	2.486	2.486	1.922	2.371	2.371	1.803
Expected	1,693.37		1,659.98	1,638.32		1,604.83	1,593.39		1,559.81
Models	LSFIMAX $(D, 1, 2)_{12}$ with $\Theta_1 = 0.1$, $\omega_1 = 0.1$, $\omega_2 = 0.3$								
	D = 0.125			D = 0.250			D = 0.375		
δ	ARL_{Exact}	ARL_{NIE}	SDRL	ARL_{Exact}	ARL_{NIE}	SDRL	ARL_{Exact}	ARL_{NIE}	SDRL
0.125	114.298	114.298	113.797	111.885	111.885	111.384	109.876	109.876	109.375
0.250	44.588	44.588	44.085	42.941	42.941	42.438	41.591	41.591	41.088
0.375	20.775	20.775	20.269	19.767	19.767	19.261	18.952	18.952	18.445
0.500	11.160	11.160	10.648	10.532	10.532	10.020	10.03	10.03	9.517
0.625	6.747	6.747	6.227	6.340	6.340	5.819	6.018	6.018	5.495
0.750	4.507	4.507	3.976	4.232	4.232	3.698	4.016	4.016	3.480
0.875	3.275	3.275	2.730	3.082	3.082	2.533	2.931	2.931	2.379
1.000	2.552	2.552	1.990	2.411	2.411	1.844	2.302	2.302	1.731
Expected	1,663.22		1,629.78	1,609.52		1,575.98	1,565.73		1,532.08
Models	LSFIMAX$(D, 1, 2)_{12}$ with $\Theta_1 = -0.1$, $\omega_1 = 0.1$, $\omega_2 = 0.3$								
	D = 0.125			D = 0.250			D = 0.375		
δ	ARL_{Exact}	ARL_{NIE}	SDRL	ARL_{Exact}	ARL_{NIE}	SDRL	ARL_{Exact}	ARL_{NIE}	SDRL
0.125	111.686	111.686	111.185	109.337	109.337	108.836	107.379	107.379	106.878
0.250	42.806	42.806	42.303	41.231	41.231	40.728	39.940	39.940	39.437
0.375	19.686	19.686	19.179	18.737	18.737	18.230	17.968	17.968	17.461
0.500	10.482	10.482	9.969	9.898	9.898	9.385	9.431	9.431	8.917
0.625	6.308	6.308	5.786	5.934	5.934	5.411	5.637	5.637	5.113
0.750	4.211	4.211	3.677	3.960	3.960	3.424	3.762	3.762	3.223
0.875	3.067	3.067	2.518	2.892	2.892	2.339	2.755	2.755	2.199
1.000	2.400	2.400	1.833	2.273	2.273	1.701	2.176	2.176	1.600
Expected	1,605.17		1,571.60	1,554.10		1,520.43	1,512.38		1,478.62

Table 4. Comparison of the ARL_1 values derived using the exact formulas and NIE methods for $LSFIMAX(D, Q, K)_S$ processes on an EWMA control chart when $\lambda = 0.3$.

Models	LSFIMAX $(D, 1, 1)_{12}$ with $\Theta_1 = 0.1$, $\omega_1 = 0.1$								
	D = 0.125			D = 0.250			D = 0.375		
δ	ARL_{Exact}	ARL_{NIE}	SDRL	ARL_{Exact}	ARL_{NIE}	SDRL	ARL_{Exact}	ARL_{NIE}	SDRL
0.125	39.614	39.614	39.111	35.286	35.286	34.782	32.372	32.372	31.868
0.250	18.196	18.196	17.689	16.056	16.056	15.548	14.618	14.618	14.109
0.375	11.103	11.103	10.591	9.773	9.773	9.260	8.877	8.877	8.362
0.500	7.769	7.769	7.252	6.842	6.842	6.322	6.216	6.216	5.694
0.625	5.909	5.909	5.386	5.216	5.216	4.689	4.746	4.746	4.216
0.750	4.756	4.756	4.227	4.212	4.212	3.678	3.843	3.843	3.305
0.875	3.988	3.988	3.452	3.547	3.547	3.006	3.246	3.246	2.700
1.000	3.448	3.448	2.905	3.080	3.08	2.531	2.829	2.829	2.275
Expected	758.26		724.90	672.10		638.53	613.98		580.23

Models	LSFIMAX $(D, 1, 1)_{12}$ with $\Theta_1 = -0.1$, $\omega_1 = 0.1$								
	D = 0.125			D = 0.250			D = 0.375		
δ	ARL_{Exact}	ARL_{NIE}	SDRL	ARL_{Exact}	ARL_{NIE}	SDRL	ARL_{Exact}	ARL_{NIE}	SDRL
0.125	34.973	34.973	34.469	31.669	31.669	31.165	29.342	29.342	28.838
0.250	15.902	15.902	15.394	14.272	14.272	13.763	13.13	13.13	12.620
0.375	9.677	9.677	9.163	8.661	8.661	8.146	7.950	7.950	7.433
0.500	6.775	6.775	6.255	6.065	6.065	5.542	5.567	5.567	5.042
0.625	5.166	5.166	4.639	4.633	4.633	4.103	4.26	4.26	3.727
0.750	4.173	4.173	3.639	3.754	3.754	3.215	3.461	3.461	2.918
0.875	3.514	3.514	2.972	3.174	3.174	2.627	2.935	2.935	2.383
1.000	3.053	3.053	2.504	2.769	2.769	2.213	2.57	2.571	2.009
Expected	665.86		632.28	599.98		566.19	553.72		519.76

Models	LSFIMAX $(D, 1, 2)_{12}$ with $\Theta_1 = 0.1$, $\omega_1 = 0.1$, $\omega_2 = 0.3$								
	D = 0.125			D = 0.250			D = 0.375		
δ	ARL_{Exact}	ARL_{NIE}	SDRL	ARL_{Exact}	ARL_{NIE}	SDRL	ARL_{Exact}	ARL_{NIE}	SDRL
0.125	33.093	33.093	32.589	30.146	30.146	29.642	28.038	28.038	27.533
0.250	14.974	14.974	14.465	13.524	13.524	13.014	12.492	12.492	11.982
0.375	9.099	9.099	8.584	8.195	8.195	7.679	7.553	7.553	7.035
0.500	6.371	6.371	5.850	5.739	5.739	5.215	5.29	5.29	4.764
0.625	4.863	4.863	4.334	4.389	4.389	3.857	4.052	4.052	3.517
0.750	3.935	3.935	3.398	3.562	3.562	3.021	3.298	3.298	2.753
0.875	3.321	3.321	2.776	3.017	3.017	2.467	2.802	2.802	2.247
1.000	2.892	2.892	2.339	2.638	2.638	2.079	2.459	2.459	1.894
Expected	628.38		594.68	569.68		535.79	527.87		493.80

Models	LSFIMAX$(D, 1, 2)_{12}$ with $\Theta_1 = -0.1$, $\omega_1 = 0.1$, $\omega_2 = 0.3$								
	D = 0.125			D = 0.250			D = 0.375		
δ	ARL_{Exact}	ARL_{NIE}	SDRL	ARL_{Exact}	ARL_{NIE}	SDRL	ARL_{Exact}	ARL_{NIE}	SDRL
0.125	29.925	29.925	29.421	27.515	27.515	27.010	25.746	25.746	25.241
0.250	13.416	13.416	12.906	12.237	12.237	11.726	11.377	11.377	10.866
0.375	8.128	8.128	7.612	7.394	7.394	6.876	6.861	6.861	6.341
0.500	5.692	5.692	5.168	5.179	5.179	4.652	4.807	4.807	4.278
0.625	4.353	4.353	3.820	3.969	3.969	3.433	3.691	3.691	3.152
0.750	3.534	3.534	2.993	3.233	3.233	2.687	3.014	3.014	2.464
0.875	2.995	2.995	2.444	2.749	2.749	2.193	2.572	2.572	2.011
1.000	2.619	2.619	2.059	2.415	2.415	1.849	2.267	2.267	1.695
Expected	565.30		531.38	517.53		483.41	482.68		448.38

Table 5. The ARL_1 computation time using the exact formulas and NIE methods.

EWMA	δ	ARL_{Exact}	ARL_{NIE}			ARL_{Exact}	ARL_{NIE}		
		LSFIMAX $(D, 1, 1)_{12}$ with $\Theta_1 = 0.1$, $\omega_1 = 0.1$				LSFIMAX $(D, 1, 2)_{12}$ with $\Theta_1 = 0.1$, $\omega_1 = 0.1$, $\omega_2 = 0.3$			
		0.125, 0.250,0.375	0.125	0.250	0.375	0.125, 0.250,0.375	0.125	0.250	0.375
$\lambda = 0.05$	0.125	<0.01	12.235	12.564	12.463	<0.01	13.015	12.546	12.865
	0.250	<0.01	11.721	12.795	12.353	<0.01	12.064	12.793	12.546
	0.375	<0.01	12.151	12.953	12.564	<0.01	12.355	12.635	12.793
	0.500	<0.01	12.138	13.017	12.785	<0.01	11.946	12.454	12.872
	0.625	<0.01	12.065	12.653	12.135	<0.01	12.035	12.335	11.897
	0.750	<0.01	13.578	11.976	11.985	<0.01	12.765	12.986	12.035
	0.875	<0.01	12.796	12.356	12.032	<0.01	12.119	12.565	12.465
	1.000	<0.01	12.235	12.956	12.563	<0.01	12.265	12.135	12.597
$\lambda = 0.1$	0.125	<0.01	11.985	11.946	12.855	<0.01	12.796	12.562	12.035
	0.250	<0.01	12.387	12.035	13.006	<0.01	11.974	12.253	12.765
	0.375	<0.01	12.065	12.765	12.865	<0.01	12.465	12.462	12.119
	0.500	<0.01	11.786	12.119	12.446	<0.01	12.635	13.015	12.635
	0.625	<0.01	12.974	12.265	12.945	<0.01	12.454	12.356	12.454
	0.750	<0.01	12.864	12.546	12.468	<0.01	12.335	12.103	12.335
	0.875	<0.01	12.554	12.466	12.846	<0.01	12.986	11.953	12.986
	1.000	<0.01	12.015	12.478	12.795	<0.01	12.201	12.036	12.565
$\lambda = 0.3$	0.125	<0.01	12.560	13.012	12.466	<0.01	12.633	12.462	12.265
	0.250	<0.01	13.012	12.984	12.643	<0.01	12.465	13.015	12.065
	0.375	<0.01	12.896	12.544	12.568	<0.01	12.765	12.356	12.793
	0.500	<0.01	12.703	12.597	12.865	<0.01	13.011	11.956	12.872
	0.625	<0.01	11.985	12.492	12.656	<0.01	12.466	12.235	11.897
	0.750	<0.01	12.065	12.765	12.458	<0.01	12.855	12.564	12.035
	0.875	<0.01	12.564	12.545	12.235	<0.01	13.006	12.795	12.465
	1.000	<0.01	12.358	12.354	12.186	<0.01	12.865	12.865	12.597
EWMA	δ	LSFIMAX $(D, 1, 1)_{12}$ with $\Theta_1 = -0.1$, $\omega_1 = 0.1$				LSFIMAX $(D, 1, 2)_{12}$ with $\Theta_1 = -0.1$, $\omega_1 = 0.1$, $\omega_2 = 0.3$			
		0.125, 0.250,0.375	0.125	0.250	0.375	0.125, 0.250,0.375	0.125	0.250	0.375
$\lambda = 0.05$	0.125	<0.01	12.864	12.154	13.013	<0.01	12.103	12.235	12.531
	0.250	<0.01	12.132	12.785	12.565	<0.01	12.465	12.465	12.152
	0.375	<0.01	11.985	13.013	12.456	<0.01	12.635	12.866	12.235
	0.500	<0.01	12.031	11.956	12.865	<0.01	12.454	13.013	12.465
	0.625	<0.01	12.598	12.103	12.546	<0.01	12.335	12.565	12.866
	0.750	<0.01	12.687	12.465	12.793	<0.01	12.986	12.456	13.013
	0.875	<0.01	13.015	12.635	12.872	<0.01	12.562	12.865	12.565
	1.000	<0.01	12.981	12.454	11.897	<0.01	12.253	12.346	12.456
$\lambda = 0.1$	0.125	<0.01	12.355	12.531	12.046	<0.01	12.567	12.896	12.981
	0.250	<0.01	11.958	12.152	11.596	<0.01	12.262	12.703	11.978
	0.375	<0.01	13.044	12.235	12.201	<0.01	13.051	12.465	12.746
	0.500	<0.01	12.983	12.465	12.022	<0.01	12.945	12.597	13.015
	0.625	<0.01	12.546	12.866	12.559	<0.01	12.359	11.975	12.843
	0.750	<0.01	12.793	13.013	13.012	<0.01	12.795	13.154	12.987
	0.875	<0.01	12.872	12.565	12.896	<0.01	12.496	12.562	12.752
	1.000	<0.01	11.897	12.456	12.703	<0.01	12.667	12.015	11.866
$\lambda = 0.3$	0.125	<0.01	12.358	12.456	12.598	<0.01	12.865	12.335	12.562
	0.250	<0.01	12.125	12.356	12.687	<0.01	12.381	12.986	12.253
	0.375	<0.01	13.023	12.567	13.015	<0.01	12.569	12.562	12.462
	0.500	<0.01	12.866	12.262	12.981	<0.01	12.896	12.253	13.015
	0.625	<0.01	12.985	13.051	11.978	<0.01	11.055	12.462	12.356
	0.750	<0.01	12.468	12.945	12.746	<0.01	11.985	13.015	11.956
	0.875	<0.01	12.789	12.359	13.015	<0.01	12.387	12.356	12.235
	1.000	<0.01	13.022	12.795	12.843	<0.01	12.065	11.956	12.564

The CPU time is in seconds.

Table 6. The EARL and ESDRL values for $\text{LSFIMAX}(D, Q, K)_S$ processes on an EWMA control chart when $\delta_{\min} = 0$ and $\delta_{\max} = 1$.

$\text{LSFIMAX}(D, Q, K)_S$	Coefficient parameters			EWMA					
				$\lambda = 0.05$		$\lambda = 0.10$		$\lambda = 0.3$	
	Θ_1	ω_1	ω_2	EARL	ESDRL	EARL	ESDRL	EARL	ESDRL
$(0.125, 1, 1)_{12}$	0.1	0.1		**441.42**	**396.96**	1,756.08	1,722.78	758.26	724.90
	-0.1	0.1		**431.67**	**386.96**	1,693.37	1,659.98	665.86	632.28
$(0.250, 1, 1)_{12}$	0.1	0.1		**432.42**	**387.73**	1,698.08	1,664.70	672.10	638.53
	-0.1	0.1		**422.91**	**377.96**	1,638.32	1,604.83	599.98	566.19
$(0.375, 1, 1)_{12}$	0.1	0.1		**424.91**	**380.00**	1,650.84	1,617.38	613.98	580.23
	-0.1	0.1		**415.64**	**370.51**	1,593.39	1,559.81	553.72	519.76
$(0.125, 1, 2)_{12}$	0.1	0.1	0.3	**426.89**	**382.09**	1,663.22	1,629.78	628.38	594.68
	-0.1	0.1	0.3	**417.55**	**372.50**	1,605.17	1,571.60	565.30	531.38
$(0.250, 1, 2)_{12}$	0.1	0.1	0.3	**418.26**	**373.21**	1,609.52	1,575.98	569.68	535.79
	-0.1	0.1	0.3	**409.18**	**363.89**	1,554.10	1,520.43	517.53	483.41
$(0.375, 1, 2)_{12}$	0.1	0.1	0.3	**411.10**	**365.86**	1,565.73	1,532.08	527.87	493.80
	-0.1	0.1	0.3	**402.20**	**356.70**	1,512.38	1,478.62	482.68	448.38

Table 9. The ARL_1 SDRL_1, EARL and ESDRL values are derived using the exact formulas and NIE methods for the two processes on a EWMA chart when $\lambda = 0.05$.

δ	***First dataset with b = 3.8772974***					***Second dataset with b =11.3758767***				
	Exact formulas		NIE method		SDRL	Exact formulas		NIE method		SDRL
	ARL	CPU time	ARL	CPU time		ARL	CPU time	ARL	CPU time	
0.125	7.729	<0.01	7.729	1.908	7.212	8.956	<0.01	8.956	1.781	8.441
0.250	4.445	<0.01	4.445	1.856	3.913	5.135	<0.01	5.135	1.751	4.608
0.375	3.332	<0.01	3.332	1.832	2.788	3.835	<0.01	3.835	1.889	3.297
0.500	2.770	<0.01	2.770	1.912	2.214	3.175	<0.01	3.175	1.735	2.628
0.625	2.430	<0.01	2.430	1.795	1.864	2.775	<0.01	2.775	1.735	2.219
0.750	2.202	<0.01	2.202	1.803	1.627	2.504	<0.01	2.504	1.875	1.941
0.875	2.038	<0.01	2.038	1.778	1.454	2.309	<0.01	2.309	1.766	1.739
1.00	1.915	<0.01	1.915	1.901	1.324	2.160	<0.01	2.160	1.843	1.583

The CPU time is in seconds.

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Conflict of Interest

The authors declare no conflict of interest.