A Homing Problem for a Geometric Brownian Motion and Its Integral

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Abstract: Let $\{Y(t), t \geq 0\}$ be a controlled geometric Brownian motion and $X(t)$ be the integral of $Y(t)$. The problem of minimizing the expected time that the ratio $X(t)/Y(t)$ will spend between two constants is considered. The optimal control is obtained explicitly in terms of special functions. A risk-sensitive version of the cost criterion is also proposed.

Key-Words: Stochastic optimal control, first-passage time, dynamic programming, special functions, risk-sensitivity.

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1 Introduction

We consider the two-dimensional controlled degenerate diffusion process $(X(t), Y(t))$ defined by the system of stochastic differential equations

$$\begin{aligned} \mathrm{d}X(t) &= Y(t)\,\mathrm{d}t, & (1)\\ \mathrm{d}Y(t) &= b_0\,Y(t)\,u[X(t),Y(t)]\,\mathrm{d}t + Y(t)\,\mathrm{d}t \\ &\quad + \{2Y^2(t)\}^{1/2}\,\mathrm{d}W(t), & (2) \end{aligned}$$

where $\{W(t), t \geq 0\}$ is a standard Brownian motion, $b_0 \neq 0$ is a constant and the control $u(\cdot,\cdot)$ is assumed to be continuous. The stochastic process $\{Y(t), t \geq 0\}$ is a controlled geometric Brownian motion, and $\{X(t), t \geq 0\}$ is therefore an integrated (controlled) geometric Brownian motion. Geometric Brownian motion is a very important diffusion process for applications, notably in financial mathematics. Because $Y(t) > 0$ for all values of t (when $Y(0) > 0$), the two-dimensional process $(X(t), Y(t))$ could also serve as a model for the wear of a machine, [1].

Let $(X(0), Y(0)) = (x, y)$ and define

$$\tau(x,y) = \inf\{t > 0 : X(t) = v_1 Y(t) \text{ or } v_2 Y(t)\}, \quad (3)$$

where x, y, v_1 and v_2 are all positive and are such that $v_1 < x/y < v_2$. The random variable $\tau(x,y)$ is called a *first-passage time* in probability theory.

The aim is to find the control $u^*[X(t), Y(t)]$ that minimizes the expected value of the cost function

$$J(x,y) = \int_0^{\tau(x,y)} \left\{\frac{1}{2}\,q_0\,u^2[X(t),Y(t)] + \lambda\right\}\mathrm{d}t + c\,\frac{X(\tau)}{Y(\tau)}, \quad (4)$$

where q_0, λ and c are positive constants.

The above problem is a particular *homing problem*, [2]. In this type of problem, the optimizer controls a stochastic process until a given event occurs. Therefore, the final time in the cost function $J(x,y)$ is neither fixed nor infinite, as is the case in other papers on optimal control of diffusion processes. In, [2], the controlled process is an n-dimensional diffusion process. The author has extended homing problems to the case of other stochastic processes, such as jump-diffusion processes, [3]. Other papers on homing problems are, [4], [5] and, [6].

Next, we define the *value function*

$$F(x,y) = \inf_{u[X(t),Y(t)],\ 0 \leq t < \tau(x,y)} E[J(x,y)]. \quad (5)$$

In the next section, we will give the *dynamic programming equation* (DPE) satisfied by the function $F(x,y)$. We will also express the optimal control in terms of $F(x,y)$.

2 Dynamic Programming Equation

Using Bellman's principle of optimality, [7] and standard arguments, we can show that the value function satisfies the following equation:

$$\begin{aligned} 0 \;=\; \inf_{u(x,y)} \Big\{ &\frac{1}{2}\,q_0\,u^2(x,y) + \lambda + y\,F_x(x,y) \\ &+ b_0\,y\,u(x,y)\,F_y(x,y) + y\,F_y(x,y) \\ &+ y^2\,F_{yy}(x,y) \Big\}, \end{aligned} \quad (6)$$

where $F_x(x,y) = \partial F(x,y)/\partial x$, etc. Hence, differentiating with respect to $u(x,y)$, we can state that the optimal control is given by

$$u^*(x,y) = -\frac{b_0\, y}{q_0} F_y(x,y). \tag{7}$$

Next, substituting the above expression for the optimal control into the DPE 6, we find that $F(x,y)$ satisfies the second-order non-linear partial differential equation (PDE)

$$y^2 F_{yy} + y F_y + y F_x - \frac{1}{2}\frac{b_0^2 y^2}{q_0} F_y^2 + \lambda = 0. \tag{8}$$

Moreover, we have the boundary conditions

$$F(x,y) = c v_i \quad \text{if } x/y = v_i, \text{ for } i = 1,2. \tag{9}$$

Now, Eq. 8 can be linearized. Indeed, let

$$M(x,y) := \exp\{-F(x,y)/k\}, \tag{10}$$

where

$$k := \frac{2 q_0}{b_0^2} \tag{11}$$

is a positive constant. We then find that the function $M(x,y)$ satisfies the PDE

$$y^2 M_{yy} + y M_y + y M_x = \alpha M, \tag{12}$$

where

$$\alpha := \frac{\lambda}{k}, \tag{13}$$

subject to the boundary conditions

$$M(x,y) = e^{-c v_i/k} \quad \text{if } x/y = v_i, \text{ for } i = 1,2. \tag{14}$$

Remark. The function $M(x,y)$ can be interpreted as follows:

$$M(x,y) = E_{(x,y)}\left[\exp\left\{-\alpha\tau_0 - \frac{c}{k}\frac{X(\tau_0)}{Y(\tau_0)}\right\}\right], \tag{15}$$

where $\tau_0(x,y)$ is the random variable that corresponds to $\tau(x,y)$ in the case of the *uncontrolled* process $(X_0(t), Y_0(t))$ obtained by setting $u[X(t), Y(t)] \equiv 0$ in Eq. (2). Moreover, Eq. (12) is the Kolmogorov backward equation satisfied by $M(x,y)$.

In the next section, Eq. (12 will be solved explicitly by making use of the method of similarity solutions.

3 Explicit Solution

Let $z := x/y$ and define

$$N(z) = M(x/y). \tag{16}$$

The function N satisfies the *ordinary* differential equation (ODE)

$$z^2 N''(z) + (1+z) N'(z) = \alpha N(z), \tag{17}$$

subject to the boundary conditions

$$N(v_i) = e^{-c v_i/k} \quad \text{for } i = 1,2. \tag{18}$$

The general solution of Eq. (17) is in terms of the modified Bessel functions $I_\nu(\cdot)$ and $K_\nu(\cdot)$, [8, p. 374]:

$$N(z) = c_1 \frac{e^{1/(2z)}}{\sqrt{z}}\left[I_{\sqrt{\alpha}+\frac{1}{2}}\left(-\frac{1}{2z}\right) + I_{\sqrt{\alpha}-\frac{1}{2}}\left(-\frac{1}{2z}\right)\right] + c_2 \frac{e^{1/(2z)}}{\sqrt{z}}\left[K_{\sqrt{\alpha}+\frac{1}{2}}\left(-\frac{1}{2z}\right) - K_{\sqrt{\alpha}-\frac{1}{2}}\left(-\frac{1}{2z}\right)\right]. \tag{19}$$

When $\alpha = 1$, the solution can be expressed as elementary functions:

$$N(z) = c_1(1+z) + c_2\, z\, e^{1/z}. \tag{20}$$

When $\lambda = 0$, the function $N(z)$ becomes

$$N_0(z) = c_1 + c_2 \,\mathrm{Ei}_1(-1/z), \tag{21}$$

where $\mathrm{Ei}_1(z)$ is an exponential integral function defined by

$$\mathrm{Ei}_1(z) = \int_1^\infty \frac{e^{-wz}}{w}\,\mathrm{d}w. \tag{22}$$

Once the function $N(z)$ has been computed explicitly, the value function $F(x,y)$ is given by

$$F(x,y) = -k \ln[N(z)] = -k \ln[M(x/y)], \tag{23}$$

and the optimal control is deduced from Eq. (7).

Let us consider a particular case: assume that $b_0 = q_0 = 1$, so that $k = 2$. Moreover, let us choose $c = \lambda = 2$. It follows that $\alpha = 1$.

Assume that $v_1 = 1$ and $v_2 = 2$. Then, $N(z)$ becomes

$$N(z) = \frac{e^{-5/2}}{3e^{1/2} - 4}\left[\left(e - 2e^{3/2}\right)(1+z) + (3e-2)\, z\, e^{1/z}\right]. \tag{24}$$

It is now possible to compute the value function $F(x,y)$ and hence the optimal control $u^*(x,y)$. We obtain that

$$F(x,y) = 5 + 2\ln\left(3e^{1/2} - 4\right) - 2\ln\left\{(3e-2)(x/y)\, e^{y/x} + \left(e - 2e^{3/2}\right)[1 + (x/y)]\right\} \tag{25}$$

and

$$u^*(x,y) = -y\,F_y(x,y) = -2\,\frac{(3e-2)\,(x-y)\,e^{y/x} + \left(e - 2\,e^{3/2}\right)x}{(3e-2)\,x\,e^{y/x} + (e - 2\,e^{3/2})\,(x+y)}. \quad (26)$$

Now, let $F_0(x,y)$ denote the value function when $\lambda = 0$. We find that

$$\begin{aligned} F_0(x,y) &= 2 + 2\,\ln\left[\mathrm{Ei}_1(-1/2) - \mathrm{Ei}_1(-1)\right] \\ &\quad - 2\,\ln[\left(e^{-1} - 1\right)\,\mathrm{Ei}_1(-y/x) \\ &\quad - e^{-1}\,\mathrm{Ei}_1(-1) + \mathrm{Ei}_1(-1/2)]. \quad (27) \end{aligned}$$

It follows that the corresponding optimal control is

$$u_0^*(x,y) = \frac{2\,(1-e)\,e^{y/x}}{(e-1)\,\mathrm{Ei}_1(-y/x) + \mathrm{Ei}_1(-1) - e\,\mathrm{Ei}_1(-1/2)}. \quad (28)$$

The functions $F(x,y)$ and $F_0(x,y)$ are shown in Figure 1 in terms of x/y, while the optimal controls $u^*(x,y)$ and $u_0^*(x,y)$ are presented in Figure 2 when $x = 1$.

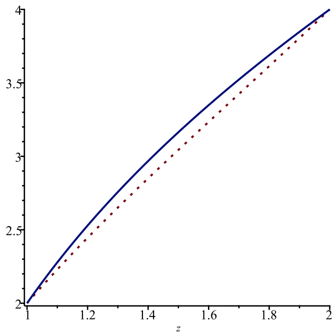

Figure 1: Functions $F(x,y)$ (solid line) and $F_0(x,y)$ for $z := x/y \in [1,2]$.

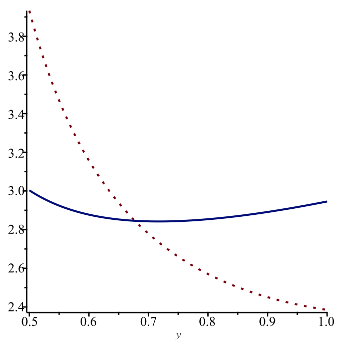

Figure 2: Functions $u(1,y)$ (solid line) and $u_0(1,y)$ for $y \in [0.5,1]$.

4 Risk-sensitive Formulation

The results presented in the previous section can be generalized by considering a *risk-sensitive* version of the cost function $J(x,y)$ defined in Eq. (4). Let θ be a real constant and define

$$C(x,y) = -\frac{1}{\theta}\,\ln\left(E\left[\exp\left\{-\theta\,J(x,y)\right\}\right]\right). \quad (29)$$

If θ is positive, the optimizer is said to be risk-seeking, while when θ is negative, the optimizer is risk-averse, [9] and, [10]. The case when θ tends to zero corresponds to the risk-neutral case considered in the preceding sections.

We can show that the value function then satisfies the DPE

$$\begin{aligned} 0 \;=\; \inf_{u(x,y)} \Big\{ &\frac{1}{2}\,q_0\,u^2(x,y) + \lambda + y\,F_x(x,y) \\ &+ b_0\,y\,u(x,y)\,F_y(x,y) + y\,F_y(x,y) \\ &- \theta\,y^2\,F_y^2(x,y) + y^2\,F_{yy}(x,y) \Big\}. \quad (30) \end{aligned}$$

Thus, the optimal control is still given by Eq. (7).

Let

$$\beta := \frac{b_0}{2\,q_0} + \theta. \quad (31)$$

We assume that $\beta \neq 0$ and we define

$$H(x,y) = \exp\left\{-\beta\,F(x,y)\right\}. \quad (32)$$

We find, [9], that the function $H(x,y)$ satisfies the linear PDE

$$y^2\,H_{yy} + y\,H_y + y\,H_x = \beta\,\lambda\,H. \quad (33)$$

The boundary conditions are

$$H(x,y) = e^{-c\beta v_i} \quad \text{if } x/y = v_i, \text{ for } i = 1,2. \quad (34)$$

Proceeding as in Section 3, we can find the function $H(x,y)$, and hence the value function $F(x,y)$. Then, we can compute the optimal control $u^*(x,y)$.

In Figure 3, we present the optimal control when $x = 1$ and $y \in [-0.5, 1]$ with the choices made for the various constants in Section 3, for $\theta = 0, -1$ and 1. We can clearly see the effect of the risk parameter θ on the optimal control.

5 Conclusion

In this note, a new homing problem for a (degenerate) two-dimensional diffusion process has been solved explicitly and exactly. To do so, we had first to linearize the DPE satisfied by the value function, and then make use of the method of similarity solutions to reduce the linear PDE to an ODE. The solution of the ODE was expressed in terms of modified Bessel

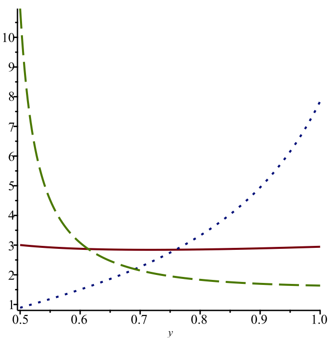

Figure 3: Optimal control $u(1, y)$ for $y \in [0.5, 1]$ when $\theta = 0$ (solid line), $\theta = -1$ (dotted line) and $\theta = 1$ (dashed line).

functions. In particular cases, these Bessel functions become elementary functions.

The two-dimensional diffusion process considered was a geometric Brownian motion and its integral. As mentioned in the Introduction, geometric Brownian motions are among the most important diffusion processes for applications. Moreover, its integral $X(t)$ can be used as a model for the wear of a machine (or for its remaining lifetime when the derivative of $X(t)$ is multiplied by a negative constant).

In Section 4, the cost function $J(x, y)$ defined in Eq. (4) was generalized by considering a cost criterion that takes the risk-sensitivity of the optimizer into account. In an example, we saw the effect of the risk parameter on the optimal control.

Finally, in cases when it is not possible to linearize the PDE satisfied by the value function, we can still try to use the method of similarity solutions to transform this non-linear PDE into an ODE. This ODE will generally be non-linear as well. Therefore, obtaining explicit solutions to such problems is a difficult task. However, it is at least possible to use numerical methods to solve any particular problem.

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