Mathematical Analysis of Monotonic Stability of the Amplitude of Forced Oscillations of a String

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Abstract: - The stability of forced oscillations of a finite-length string is considered. The driving force is specified as a known expression containing one harmonic of the time of the string's motion. Monotonic stability of string oscillations is understood as a monotonic decrease in the oscillation amplitude of the modulus of the difference of the solutions describing forced and free oscillations observed at an arbitrary point of the string. In this case, the solutions of the equation of string oscillations in partial derivatives of the second order for free and forced oscillations are assumed to be known. The work aims to analyze three conditions for a monotonic change in the modulus of the difference in the amplitude of forced and free oscillations of a string on a semi-infinite time interval: monotonicity condition, nonlinearity condition, and convergence condition. The analysis of the conditions for monotonic stability of string oscillations is also carried out in the example given in the article.

Key-Words: - mathematical analysis, amplitude stability, monotonic change, string, wave equation, forced oscillations, solution.

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1 Introduction

The mathematical problem of stability analysis of solutions of nonlinear dynamic systems is one of the most important scientific problems. The main approach to studying the stability of solutions of dynamic systems consisting of differential equations is the use of the second Lyapunov method, [1], [2], [3], [4], [5], [6]. A significant drawback of this classical method is the fact that the Lyapunov expression must be known. In this regard, the use of new approaches to studying the stability of particular solutions of dynamic systems is of scientific and practical interest. One of these approaches is the method for studying stability based on the idea of monotonicity of particular solutions of nonlinear ordinary differential equations. In particular, this method is described in the following article, [7]. A distinctive feature of these works is that they consider mathematical models in the form of ordinary differential equations of the first order or their systems. The scientific interest is the study of the issue of applying this method to the analysis of the stability of solutions of dynamic systems modeled by partial differential equations.

A significant number of works [8], [9], [10], [11], etc. are devoted to the study of solutions of several varieties of the wave equation. For example, in the article [8], an exact solution of the inhomogeneous two-speed wave equation was obtained and a criterion for the smoothness of its right-hand side was presented. Another article [9] presents obtaining exact solutions of the quadratic and cubic nonlinear (2+1)-dimensional Klein-Gordon equation by the expressional variable method. This type of equation is a generalization of the classical one-dimensional wave equation to the case of two spatial and one-time variables.

However, in the well-known works on the study of the classical wave equation and its solution, the fulfillment of the conditions of monotonic change in the modulus of the difference in the amplitudes of forced and free oscillations of a string over a semi-infinite time interval is not considered.

The aim of this work is to analyze the conditions of monotonic change of the modulus of the difference in the amplitude of forced and free oscillations of a string on a semi-infinite time interval. The known wave equation of oscillations is used as a mathematical model. This equation is a second-order partial differential equation. The following stability conditions of a particular solution are analyzed in the work: monotonicity conditions, nonlinearity conditions, and convergence conditions.

Let us consider the notation used in the work:

l is the string length,

a is the phase velocity,

$u(0,t), u(l,t)$ are the boundary conditions of the first kind,

$u(x,0)=\varphi(x), u_t(x,0)=\psi(x)$ are the initial conditions,

$u(x,t)$ is the solution of the linear homogeneous wave equation,

$\lambda_n=\left(\pi n x l^{-1}\right)^2$ are the eigenvalues,

α_n, β_n are the coefficients of the Fourier series,

$u_p(x,t)$ is the solution of the linear inhomogeneous wave equation,

$f(t)$ is a known expression with given properties,

$r(x,t)=\sqrt{(u_p(x,t)-u(x,t))^2}$ is the expression of the modulus of the difference between the solutions of the inhomogeneous and homogeneous wave equations,

$q(x,t)=f(t)\left|\sum_{n=1}^{+\infty}(A_n \sin\left(\pi n x l^{-1}\right))\right|$ is the expression describing the envelope of the amplitude of the difference between forced and free oscillations of the string,

$w(x,t)$ is the expression on the right-hand side of the linear inhomogeneous wave equation.

2 Problem Statement

Let us consider the classical homogeneous wave equation of string vibrations on a segment $x \in [0,l]$, where l is the string length, which has the form [12], [13], [14]:

$$u_{tt}=a^2 u_{xx}, \tag{1}$$

where a is the phase velocity.

Equation (1) describes the propagation of a wave along the string. This equation is considered under homogeneous boundary conditions of the first kind (both ends of the string are fixed motionless)

$$u(0,t)=0, u(l,t)=0 \tag{2}$$

and under initial conditions

$$u(x,0)=\varphi(x), u_t(x,0)=\psi(x). \tag{3}$$

Here $\varphi(x), \psi(x)$ are the given expressions.

The classical homogeneous wave equation of string vibrations (1) is one of the basic equations of mathematical physics, [13]. It is written in the one-dimensional case and is a linear hyperbolic differential equation in partial derivatives of the second order. In practice, this equation can describe small vibrations of a string or a thin membrane. In addition, this equation is used to describe oscillatory processes in continuous media or in electrodynamics. In this paper, this equation is considered in general terms, outside of specific technical problems.

From a physical point of view, this equation can be interpreted as follows: according to this equation, the force (the second derivative with respect to time) is directly proportional to the curvature of the curve determined from the solution of this equation.

It is assumed that the theoretical problem (1)-(3) has a solution. In this case, the consistency of the boundary conditions (2) and the initial conditions (3) is required, ensured by the simultaneous fulfillment of the equalities $\varphi=0, \psi=0$ at the initial moment of time at the start and end points of the string.

It is known that based on the application of the convergent Fourier series, it is possible to determine the solution to equation (1) (if conditions (2)-(3) are met) in the following form

$$u(x,t)=\sum_{n=1}^{+\infty}(\alpha_n \sin(\sqrt{\lambda_n} t)) \sin\left(\sqrt{\lambda_n}\right) + \sum_{n=1}^{+\infty}(\beta_n \cos(\sqrt{\lambda_n} t)) \sin\left(\sqrt{\lambda_n}\right). \tag{4}$$

Here $\lambda_n=\left(\pi n x l^{-1}\right)^2$ are the eigenvalues, $\alpha_n, \beta_n \in R$ which are the coefficients of the series.

Let us write the solution (4) in a compact form

$$u(x,t)=\sum_{n=1}^{+\infty}(A_n \sin(\omega_n+\sqrt{\lambda_n} t) \sin\left(\sqrt{\lambda_n}\right)). \tag{5}$$

Here $A_n=\sqrt{\alpha_n^2+\beta_n^2}$,
$\sin(\omega_n)=\beta_n(\alpha_n^2+\beta_n^2)^{-0.5}$,
$\cos(\omega_n)=\alpha_n(\alpha_n^2+\beta_n^2)^{-0.5}$.

Let us introduce an external influence into expression (5). As a result, we obtain the following expression:

$$u_p(x,t)=(1+f(t))\sum_{n=1}^{+\infty}(A_n \sin(\omega_n+\sqrt{\lambda_n} t) \sin\left(\sqrt{\lambda_n}\right)). \tag{6}$$

Similar to the solution (5), external influence (6) is a periodic expression of variable t and variable quantity $\sqrt{\lambda_n}$. Here the expression $f(t)$ is defined

on the interval $t \in [0,+\infty)$ and has the following properties:
(i) the expression $f(t)$ is a non-negative, twice continuously differentiable, monotonically decreasing expression on the interval $t \in [0,+\infty)$,
(ii) the expression $f(t)$ has a second derivative with respect to time *t*, which vanishes only at its inflection points on the interval $t \in [0,+\infty)$,
(iii) the expression $f(t)$ has a horizontal asymptote $f(t)$=0.

In this case, expression (6) is a solution to the linear inhomogeneous wave equation

$$u_{tt} = a^2 u_{xx} + w(x,t). \quad (7)$$

The expression $w(x,t)$ on the right side of equation (7) is equal to:

$$w(x,t) = 2a^2(1+f(t))\sum_{n=1}^{+\infty}(A_n|\lambda_n|(x)^{-2} sin(\omega_n + \sqrt{\lambda_n}t) sin(\sqrt{\lambda_n}))$$
$$-(1+f(t))\sum_{n=1}^{+\infty}(A_n|\lambda_n| sin(\omega_n + \sqrt{\lambda_n}t) sin(\sqrt{\lambda_n}))$$
$$-2f(t)\sum_{n=1}^{+\infty}(A_n|\lambda_n| cos(\omega_n + \sqrt{\lambda_n}t) sin(\sqrt{\lambda_n}))$$
$$+f(t)\sum_{n=1}^{+\infty}(A_n sin(\omega_n + \sqrt{\lambda_n}t) sin(\sqrt{\lambda_n})).$$

Let us introduce the modulus expression of the difference between solutions (6) and (5) in the form of $r(x,t) = \sqrt{(u_p(x,t) - u(x,t))^2}$. At a result we obtain

$$r(x,t) = \left| f(t)\sum_{n=1}^{+\infty}(A_n sin(\omega_n + \sqrt{\lambda_n}t) sin(\sqrt{\lambda_n})) \right|. \quad (8)$$

At each specific point of the string *x*, the expression describing the envelope of the amplitude of the difference between forced and free vibrations of the string is determined by the quantity

$$q(x,t) = f(t)\left|\sum_{n=1}^{+\infty}(A_n sin(\pi n x l^{-1}))\right|. \quad (9)$$

Here $\left|\sum_{n=1}^{+\infty}(A_n sin(\pi n x l^{-1}))\right| \neq 0$.

Let us consider the following definition.

Definition. Monotonic stability of the amplitude of string oscillations is a monotonic decrease in the modulus of the amplitude of the difference between forced, and free oscillations of the string over the semi-infinite time interval $t \in [0,+\infty)$, in which the difference expressions $r(x,t) \to 0$ at $t \to +\infty$.

Let us formulate a sufficient condition for the monotonic stability of string vibrations.

Theorem. If an expression $f(t)$ of constant sign within a specific coordinate quarter satisfies conditions (i)-(iii) and the continuous derivative $\frac{df}{dt}$ is negative on a semi-infinite interval $t \in [0,+\infty)$ (the expression $\frac{df}{dt}$ reaches zero only at $t \to +\infty$), then the amplitude of the string vibrations is monotonically stable on the interval.

Proof. Let a non-negative expression $f(t)$ have a continuous expression $\frac{df}{dt}$ on the interval $t \in [0,+\infty)$. Let the first derivative of this expression be negative on the interval $t \in [0,+\infty)$, i.e. the condition $\frac{df}{dt} < 0$ is satisfied (except for the zero value $\frac{df}{dt} = 0$ at $t \to +\infty$). We apply the fundamental sufficient condition for the decrease of a real expression of one variable. In this case, the expression $f(t)$ of constant sign within a specific coordinate quarter satisfies conditions (i)–(iii) and this expression decreases monotonically on the interval $t \in [0,+\infty)$ (except for the limiting value $\frac{df}{dt} = 0$ at $t \to +\infty$). In this case, the envelope $q(x,t)$ of the amplitude of the difference between forced and free oscillations of the string also tends to zero at $t \to +\infty$ for any values of *x* belonging to the string. At the same time, the modulus expression of the difference between solutions (6) and (5) in the form (8) also tends to zero, i.e. $r(x,t) \to 0$ at $t \to +\infty$ for any values of *x* belonging to the string. Consequently, according to the Definition, the amplitude of the string oscillations has the property of monotonous stability in the interval $t \in [0,+\infty)$. The theorem is proven.

Note. The presented Theorem is applicable to the analysis of the stability of forced oscillations of a string with two fixed ends, having boundary and initial conditions identical to conditions (2) and (3). In this case, the string must have the same rigidity along its entire length.

Note. The expression $f(t)$ determines the nature of the damping of forced oscillations of the string relative to its free oscillations.

Note. Different strings may correspond to different expressions $f(t)$, different rigidity, and initial tension.

3 Analysis of Conditions of Monotonic Stability of Oscillations

The analysis of monotonic stability of the amplitude of the difference between forced and free oscillations of a string consists of the analysis of three conditions: the condition of monotonic stability, the condition of nonlinearity, and the condition of convergence.

Let us consider the fulfillment of these conditions.

The condition of monotonicity of stability of the amplitude of the difference between forced and free oscillations of a string implies the negativity of the first partial derivative of the expression $q(x,t)$ with respect to time t at all points of the string at $t \in [0,+\infty)$. It has the following form:

$$\frac{\partial q(x,t)}{\partial t} < 0 . \quad (10)$$

Taking into account expression (9), we obtain that condition (10) is satisfied at all points x of the string at $t \in [0,+\infty)$.

In this case, the amplitude of the difference between forced and free oscillations of the string decreases monotonically to zero at $t \to +\infty$. Consequently, in this case, forced oscillations do not differ from free oscillations. Indeed, the amplitudes and frequencies of these oscillations are equal at $t \to +\infty$. It should be noted that condition (10) is satisfied when the condition $\frac{df}{dt} < 0$ is fulfilled, which is fulfilled according to the Theorem. In this case, the expression $\frac{df}{dt}$ should reach zero value only at $t \to +\infty$. Thus, according to the definition, the amplitude of string oscillations is monotonically stable on this semi-infinite interval.

The condition of nonlinearity of the amplitude of the difference between forced and free oscillations of a string assumes the absence of linear sections on the graph of the expression $q(x,t)$ as it changes over the interval $t \in [0,+\infty)$.

Let us determine the sign of the second partial derivative of the expression $q(x,t)$ with respect to time *t*. Since the second partial derivative of this expression is equal to $\frac{\partial^2 q(x,t)}{\partial t^2} = \frac{d^2 f}{dt^2}\left|\sum_{n=1}^{+\infty}\left(A_n \sin\left(\pi n x l^{-1}\right)\right)\right|$, this partial derivative is nonzero at all points of the string (except for points where $\left|\sum_{n=1}^{+\infty}\left(A_n \sin\left(\pi n x l^{-1}\right)\right)\right| = 0$) at $t \in [0,+\infty)$. Indeed, the expression $\left|\sum_{n=1}^{+\infty}\left(A_n \sin\left(\pi n x l^{-1}\right)\right)\right|$ can vanish only at individual points. Consequently, the condition of nonlinearity of the amplitude of the difference between forced and free oscillations of the string is satisfied. As time *t* increases, we find that the envelope $q(x,t)$ of the amplitude of the string oscillations for all x of the string (except for values satisfying the equality $\left|\sum_{n=1}^{+\infty}\left(A_n \sin\left(\pi n x l^{-1}\right)\right)\right| = 0$) is a curvilinear line. Moreover, with an increase in time *t*, we obtain that the envelope of the string oscillation amplitude $q(x,t)$ for all points of the string is a curvilinear line (except for points where the equality $\left|\sum_{n=1}^{+\infty}\left(A_n \sin\left(\pi n x l^{-1}\right)\right)\right| = 0$ is satisfied).

The convergence condition assumes that the magnitude of the envelope of the amplitude of the difference between forced and free oscillations (9) reaches zero ($q(x,t) \to 0$) at all points of the string at $t \to +\infty$.

Calculating the limit of expression (9) for an arbitrary value of x, we obtain that $\lim_{t \to +\infty} q(x,t) = 0$. Indeed, this limit is equal to zero, since the expression $f(t)$ has a horizontal asymptote $f(t) = 0$, and when calculating the expression $\left|\sum_{n=1}^{+\infty}\left(A_n \sin\left(\pi n x l^{-1}\right)\right)\right|$ for each value x, we obtain a certain number. Consequently, the limit equality $\lim_{t \to +\infty} f(t) = 0$ is satisfied. Thus, the condition of convergence of the amplitude envelope is satisfied at $t \to +\infty$.

4 The Example of Analysis of Monotonic Stability of the Amplitude of String Vibrations

Let us consider a specific mathematical model of a forced string with two fixed ends, having boundary and initial conditions identical to conditions (2) and

(3). Let the external influence on this string have the following form:

$$u_p(x,t)=(1+e^{-t})\sum_{n=1}^{+\infty}(A_n\,sin(\omega_n+\sqrt{\lambda_n}t)sin(\sqrt{\lambda_n})). \quad (11)$$

The dependence describes forced vibrations of a string, occurring with the amplitude that decreases exponentially to values $u(x,t)=\sum_{n=1}^{+\infty}(A_n\,sin(\omega_n+\sqrt{\lambda_n}t)sin(\sqrt{\lambda_n}))$. Here the damping characteristic of forced oscillations relative to free oscillations is equal to the expression $f(t)=e^{-t}$.

Expression (11) is a solution of the equation:

$$u_{tt}=a^2u_{xx}+v(x,t). \quad (12)$$

Here

$$v(x,t)=2a^2(1+e^{-t})\sum_{n=1}^{+\infty}(A_n|\lambda_n|(x)^{-2}\,sin(\omega_n+\sqrt{\lambda_n}t)sin(\sqrt{\lambda_n}))$$
$$-(1+e^{-t})\sum_{n=1}^{+\infty}(A_n|\lambda_n|sin(\omega_n+\sqrt{\lambda_n}t)sin(\sqrt{\lambda_n}))$$
$$-2e^{-t}\sum_{n=1}^{+\infty}(A_n|\lambda_n|cos(\omega_n+\sqrt{\lambda_n}t)sin(\sqrt{\lambda_n}))$$
$$+e^{-t}\sum_{n=1}^{+\infty}(A_n\,sin(\omega_n+\sqrt{\lambda_n}t)sin(\sqrt{\lambda_n})).$$

The equation (12) is an inhomogeneous wave equation. Let us write the modulus expression of the difference between solutions (11) and (5) in the form $r(x,t)=\sqrt{(u_p(x,t)-u(x,t))^2}$. As a result, we obtain:

$$r(x,t)=e^{-t}\left|\sum_{n=1}^{+\infty}(A_n\,sin(\omega_n+(\pi nxl^{-1})t)sin(\pi nxl^{-1}))\right|. \quad (13)$$

From expression (13) it follows that at each specific point of the string x the envelope of the difference between forced and free vibrations is determined by the expression

$$q(x,t)=e^{-t}\left|\sum_{n=1}^{+\infty}(A_n\,sin(\pi nxl^{-1}))\right|. \quad (14)$$

Let us consider the fulfillment of these stability conditions.

The condition of monotonicity of stability of the amplitude of the difference between forced and free oscillations is fulfilled $\frac{\partial q(x,t)}{\partial t}<0$, since the first derivative of the expression $f(t)=e^{-t}$ is negative $(\frac{de^{-t}}{dt}=-e^{-t}<0)$.

The condition of nonlinearity of the amplitude of the difference between forced and free oscillations of the string is also fulfilled, since the expression $\frac{\partial^2 q(x,t)}{\partial t^2}=e^{-t}\left|\sum_{n=1}^{+\infty}(A_n\,sin(\pi nxl^{-1}))\right|$ is positive at all points of the interval $t\in[0,+\infty)$, except for individual points where the expression $\left|\sum_{n=1}^{+\infty}(A_n\,sin(\pi nxl^{-1}))\right|$ can turn to zero.

The convergence condition is also satisfied. Indeed, the magnitude of the envelope of the amplitude of the difference between forced and free oscillations (14) tends to zero at $t\to+\infty$, since the limit equality is satisfied $\lim_{t\to+\infty} e^{-t}=0$.

5 Conclusion

Thus, in the considered formulation of the problem of forced string vibrations, all three conditions of monotonic stability of the modulus of the difference between the solutions for forced and free vibrations can be satisfied. In this case, the modulus of the difference between the solution for forced vibrations and the solution for free vibrations decreases monotonically, reaching zero at $t\to+\infty$. In this case, the forced vibrations of the string become identical to its free vibrations. It should be noted that with increasing time t, the envelope of the amplitude of the difference between the forced and free vibrations of the string on the interval $t\in[0,+\infty)$ is a curvilinear line. As a prospect for further research, further analysis of the monotonic stability of the difference between the solution for forced oscillations and the solution for free oscillations, described by equations of a more general form than the classical one-dimensional wave equation, is possible.

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