On the Dihedral Homology of $\mathcal{A}_\infty$-algebras

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Abstract: - In this paper, we will show and discuss the Steenrod operators and their applications within the framework of the dihedral homology of $\mathcal{A}$-infinity algebras. Steenrod operations have proved to be important tools in developing the study of homological elements and various homological theories, such as the Adams spectral sequence and the Sullivan conjecture, as they were first introduced in algebraic topology. These operators have proven to be invaluable in advancing our understanding of topological and algebraic structures. Therefore, we focus on the generalization of these methods for more general applications, particularly with projective homogeneous varieties over α-characteristic fields. We begin by defining Steenrod operators in dihedral homology of $\mathcal{A}$-infinity algebras, and we will explain the complex relations between these algebraic structures and the homology theory-derived operations.

Key-Words: - Dihedral, Cyclic, Homology, $\mathcal{A}_\infty$-algebras, Steenrod algebra, Adams operations.

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1 Introduction

The Steenrod operations were at first presented in the algebraic topology over the late 1930's. This operation is performed on the topological spaces' modulo α homology.

These operations have been used to verify several conclusions in the algebraic topology. Later, they employed in new ways, for as when discussing the Sullivan conjecture or the Adams spectral sequence. The operations were soon used in the study of projective homogeneous types in algebraic geometry. Although Steenrod operations moduloα, which operate over fields of characteristic α, do not yet exist, this is due to several factors. As a consequence, given the specific values of a characteristic of the basic field, several significant concerns surrounding projective homogeneous varieties remain unanswered. For example, some of the most complex quadratic form theorems are undefined when the base domain of characteristic is two.

The Steenrod operations constructions aimed at Chow groups modulo the major number α designated in [1], [2]. Like Steenrod's initial construction, they all include taking into the action how a cyclic group of order α affects the result of α replicas of a particular scheme. Specifically on a square of the particular projective homogeneous variation, the Steenrod operations are employed as a means of generating new algebraic cycles and offering motives decomposition of this variation. According to the Rost nilpotence theorem, the identical conclusion was obtained using merely reduced Steenrod operations.

Algebraic topology generally aims to offer algebraic techniques to extricate topological spaces. One such diagram that turns out to be very exciting is the infinity homology $H_\bullet(V, \mathcal{A})$ for a space V. In this case, we obtain an additional infinity algebras structure not immediately offered by homology groups, and we can compute this algebra more easily than we can with homotopy groups.

Furthermore, it goes out that if we select the constant $\mathcal{A}_\infty$-algebras $\mathcal{M}$ to be one of infinite fields with the form F_α, for a prime α, we have even additional structure. In this instance, Steenrod presented the stable homology operations, which are natural transformations $\psi: H_n(-, F_\alpha) \to H_m(-, F_\alpha)$ with specific properties that it turns out combine to produce an infinity algebras.

Without ever taking into account topological spaces, it is possible to study the Steenrod algebra, as it is known. A further restriction on the existence and behavior of such spaces is imposed by the fact that homology operations with these abilities can be explicitly created for any topological space.

The aim of [2] has examined how generalized the Steenrod operations can be constructed in relation to multiplicative spectral sequences. [3], [4] have been used to study Adams operations in Hochschild, cyclic homology of the de-Rham algebra, and allowed loop spaces.

We study Adams operations on the $\mathcal{A}_\infty$-algebras by the rational coefficients are developed and proved to descend to the universal relating to their group law oriented cohomology theories. We introduce and study some basic statements of the dihedral homology theory of $\mathcal{A}_\infty$-algebras, and we define Adams and Steenrod operators in algebras. The main study of this paper is the form of the Adam`s and Steenrod`s of the dihedral homology of an $\mathcal{A}_\infty$-algebras. We introduce the Steenrod operator in dihedral homology on $\mathcal{A}_\infty$-algebras.

2 The Dihedral Homology of $\mathcal{A}_\infty$-Algebras

$\mathcal{A}_\infty$-algebras is one of several branches of algebras, and it has certain unique properties. It is described as a graded algebra with graded maps, which satisfies some conditions. Stasheff introduced infinity algebras in the 1960's and provides the properties of topological algebras. Its homological theory was also studied in (2013) by Alaa. H., Y. Gouda. In view of this, we will show some previous studies of some definitions, theorems and algebraic properties of $\mathcal{A}_\infty$-algebras and its homological properties by using the references, [5], [6].

2.1 Definition, [7]

By considering a differential module $(\mathcal{C},\delta)$ such as $\delta: \mathcal{C}_p \to \mathcal{C}_{p-1}$, then we can define a simplicial faces as $\partial_\iota: \mathcal{C}_p \to \mathcal{C}_{p-1}, 0 \le \iota \le n$, where $\partial_\iota \partial_j = \partial_{j-1}\partial_\iota,\ \iota < j$, additionally ∂_ι refers to be the $(\mathcal{C},\delta)$-simplicial faces. Let the permutation σ of a symmetrical group Σ_q of q-elements of permutations, in which its components are $(\sigma(\iota_1), \dots, \sigma(\iota_q))$ that operates on $(\iota_1, \dots, \iota_q)$ where $\iota_1 < \cdots < \iota_q$, then $(\widehat{\sigma(\iota_1)}, \dots, \widehat{\sigma(\iota_q)})$ write as:

$$\widehat{\sigma(\iota_k)} = \sigma(\iota_k) - \gamma(\sigma(\iota_k)),\ 1 \le k \le q,$$

while, $\gamma(\sigma(\iota_k))$ is a number of $(\sigma(\iota_1), \dots, \sigma(\iota_k), \dots, \sigma(\iota_q))$.

As of right now, the differential module $(\mathcal{C},\delta)$ with the family map:
$\tilde{\partial} = \partial_{(\iota_1,\dots,\iota_q)}: \mathcal{C}_p \to \mathcal{C}_{p-q},\ 1 \le q \le p,\ 0 \le \iota_1 < \cdots < \iota_q \le p,\ \iota_1, \dots, \iota_q \in \mathbb{Z}$, can be used to define the $\mathcal{F}_\infty$-module $(\mathcal{C},\delta,\tilde{\partial})$, which satisfy that:

$$\delta\left(\partial_{(\iota_1,\dots,\iota_q)}\right) = \sum_{\sigma\in\Sigma_q}\sum_{I_\sigma}(-1)^{1+sign(\sigma)}\partial_{(\widehat{\sigma(\iota_1)},\dots,\widehat{\sigma(\iota_\ell)})}\partial_{(\widehat{\sigma(\iota_{\ell+1})},\dots,\widehat{\sigma(\iota_q)})}. \quad (1)$$

Since I_σ indicates the permutations of $(\widehat{\sigma(\iota_1)}, \dots, \widehat{\sigma(\iota_q)})$ such as:
$\widehat{\sigma(\iota_1)} < \cdots < \widehat{\sigma(\iota_\ell)} < \widehat{\sigma(\iota_{\ell+1})} < \cdots < \widehat{\sigma(\iota_q)}$.

Then, $\tilde{\partial} = \partial_{(\iota_1,\dots,\iota_q)}$ is the $\mathcal{F}_\infty$-differential of $(\mathcal{C},\delta)$ is the ∞-simplicial of faces of $\mathcal{F}_\infty$-module.

Therefore, let $q = 1$, then $\delta(\partial_{(\iota_1)}) = 0,\ \iota_1 \ge 0$,
let $q = 2$, then:
$\delta(\partial_{(\iota_1,\iota_2)}) = \partial_{(\iota_2-1)}\partial_{(\iota_1)} - \partial_{(\iota_1)}\partial_{(\iota_2)},\ \iota_1 < \iota_2$,
let $q = 3$, then:

$$\begin{aligned}\delta(\partial_{(\iota_1,\iota_2,\iota_3)}) = &-\partial_{(\iota_1)}\partial_{(\iota_2,\iota_3)} - \partial_{(\iota_1,\iota_2)}\partial_{(\iota_3)} \\ &- \partial_{(\iota_3-2)}\partial_{(\iota_1,\iota_2)} - \partial_{(\iota_2-1,\iota_3-1)}\partial_{(\iota_1)} \\ &+ \partial_{(\iota_2-1)}\partial_{(\iota_1,\iota_3)} \\ &+ \partial_{(\iota_1,\iota_3-1)}\partial_{(\iota_2)},\quad \iota_1 < \iota_2 < \iota_3,\end{aligned}$$

We can define a module of cyclic differential $(\mathcal{C},\delta,t)$ by using define of differential module $(\mathcal{C},\delta)$ and the map $t = \{t_p: \mathcal{C}_p \to \mathcal{C}_p\}$, $\forall p \ge 0$, $t_p^{p+1} = I_{\mathcal{C}_p},\ \delta t_p = t_p\delta$.

Similarly, we can get a module of dihedral differential $(\mathcal{C},\delta,t,r)$ by define the map:
$r = \{r_p: \mathcal{C}_p \to \mathcal{C}_p\},\ \forall p \ge 0,\ r_p^2 = I_{\mathcal{C}_p}$,

Then we have:

$$r_p t_p = t_p^{-1} r_p, \quad \delta r_p = r_p \delta.$$

For the cyclic and the dihedral modules, we have:

$$\partial_\iota t_p = t_{p-1}\partial_{\iota-1}, \quad 0 < \iota \le p$$
$$\partial_0 t_p = \partial_p, \quad \partial_\iota r_p = r_{p-1}\partial_{\iota-1}, \quad 0 \le \iota \le p.$$

Then, the $\mathcal{DF}_\infty$-module $(\mathcal{C},\delta,t,r,\tilde{\partial})$ can be identified as the dihedral module, such that it is through ∞-simplicial of faces, subsequently $(\mathcal{C},\delta,t,r)$ denotes a module of dihedral differential and fulfils that:

$$\partial_{(\iota_1,\dots,\iota_q)} t_p = \begin{cases} t_{p-q}\,\partial_{(\iota_1-1,\dots,\iota_q-1)}, & \iota_1 > 0 \\ (-1)^{q-1}\,\partial_{(\iota_2-1,\dots,\iota_q-1,p)}, & \iota_1 = 0 \end{cases}, \quad (2)$$

$$\partial_{(\iota_1,\dots,\iota_q)} r_p = (-1)^{\frac{q(q-1)}{2}} r_{p-q}\partial_{(p-\iota_q,\dots,p-\iota_1)},\ \iota_1 = 0. \quad (3)$$

2.2 Definition, [5]

Assume that $\mathcal{M} = \{\mathcal{M}_p\}$, $\forall p \in \mathbb{Z}, p > 0$, is a unital $\mathcal{A}_\infty$-algebra where the $\mathcal{A}_\infty$-algebra

$(\mathcal{M}, \delta, \varphi_{p})$ be any differential module $(\mathcal{M}, \delta)$, as $\delta: \mathcal{M}_{\star} \to \mathcal{M}_{\star-1}$, prepared through a family of functions $\{\varphi_{p}: (\mathcal{M}^{\otimes(p+2)})_{\star} \to \mathcal{M}_{\star+p}\}$, fulfilling the preceding relations for each integer $p > 1$, since $\delta(\varphi_{p-1}) = \delta\varphi_{p-1} + (-1)^{p}\varphi_{p-1}\delta$,

$$\delta(\varphi_{p-1}) = \sum_{q=1}^{p-1}\sum_{s=1}^{q+1}(-1)^{s(p-q)+p+1}\pi_{q-1}\left(\underbrace{1\otimes\ldots\otimes 1}_{s-1}\otimes \varphi_{p-q-1}\otimes\underbrace{1\otimes\ldots\otimes 1}_{q-s-1}\right), \quad (4)$$

For example, the relations (4) have the following forms:

- for $p = 1$: then $\delta(\varphi_0) = 0$,
- for $p = 2$: $\delta(\varphi_1) = \varphi_0(\varphi_0 \otimes 1) - \varphi_0(1 \otimes \varphi_0)$,
- for $p = 3$: $\delta(\varphi_2) = \varphi_0(\varphi_1 \otimes 1 + 1 \otimes \varphi_1) - \varphi_1(\varphi_0 \otimes 1^{\otimes 2} - 1 \otimes \varphi_0 \otimes 1 + 1^{\otimes 2} \otimes \varphi_0)$.

Since $(\mathcal{M}, \delta, \varphi_{p})$ is the $\mathcal{A}_{\infty}$-algebra and given by auto-morphism $\star: \mathcal{M}_{p} \to \mathcal{M}_{p}$, the involutive $\mathcal{A}_{\infty}$-algebra also can be identifying as the complex $(\mathcal{M}, \delta, \varphi_{p}, \star): \mathcal{M}_{p} \to \mathcal{M}_{p}$ such as $\forall m \in \mathcal{M}$, $\star(m) = m^{\star}$ and the conditions are fulfilled as follows:

$$(m^{\star})^{\star} = m, \qquad \delta(m^{\star}) = \delta(m)^{*},$$
$$\varphi_n(m_0 \otimes m_1 \otimes \ldots \otimes m_{p} \otimes m_{p+1})^{\star} = (-1)^{\xi}\varphi_{p}(m_{p+1}^{*} \otimes m_{p}^{*} \otimes \ldots \otimes m_1^{*} \otimes m_0^{*}).$$

Such that $\xi = \frac{p(p-1)}{2} + \sum_{0 \le \iota < j \le p} |m_{\iota}||m_j|$, $p \ge 0$. Therefore, a module of a dihedral differential remains the complex $({}^{\varrho}\mathcal{M}(\mathcal{M}), t, r, \delta)$, such as $\varrho = \pm 1$, also

$$t_{p}(m_0 \otimes \ldots \otimes m_{p}) = (-1)^{\beta} m_{p} \otimes m_0 \otimes m_1 \otimes \ldots \otimes m_{p-1},$$
$$r_{p}(m_0 \otimes \ldots \otimes m_{p}) = \varrho(-1)^{\gamma} m_0^{*} \otimes m_{p}^{*} \otimes m_{p-1}^{*} \otimes \ldots \otimes m_1^{*},$$
$$\delta(m_0 \otimes \ldots \otimes m_{p}) = \sum_{\iota=0}^{p}(-1)^{\mu} m_0 \otimes \ldots \otimes m_{\iota-1} \otimes \delta m_{\iota} \otimes \ldots \otimes m_{p}.$$

2.3 Theorem, [5]

The dihedral module ($\mathcal{DF}_{\infty}$-module) is defined as $({}^{\varrho}\mathcal{M}(\mathcal{M}), t, r, \delta)$, if $(\mathcal{M}, \delta, \varphi_{p}, \star)$ seems to be the involutive $\mathcal{A}_{\infty}$-algebra.

2.4 Definition, [5]

For the field $\mathcal{K}$ of a characteristic zero, the one-dimensional vector spaces of degrees -1 and 1 with 0-differential, respectively, are denoted by the notations $\Sigma\mathcal{K}$ and $\Sigma^{-1}\mathcal{K}$. The free formalized augment differential of the graded associative algebra denoted by $\hat{\mathcal{T}}\mathcal{V}$, which is produced by $\mathcal{V}$ and given by:

$$\hat{\mathcal{T}}\mathcal{V} = \prod_{p=0}^{\infty} \mathcal{V}^{\otimes p} = \mathcal{K} \times \mathcal{V} \times (\mathcal{V} \otimes \mathcal{V}) \times \ldots .$$

Over $\hat{\mathcal{T}}_{\ge\iota}\mathcal{V}$, we refer to the sub-algebra with element orders equal to or greater than ι.

2.5 Definition, [6]

Suppose that $(\mathcal{M}, \mathcal{Q})$ and $(\mathcal{N}, \mathcal{Q}')$ are $\mathcal{A}_{\infty}$-algebras. Then the $\mathcal{A}_{\infty}$-morphism of $\mathcal{A}_{\infty}$–algebras are a map f of associative algebras:

$$f: \hat{T}_{\ge 1}\Sigma^{-1}\mathcal{N}^{*} \to \hat{T}_{\ge 1}\Sigma^{-1}\mathcal{M}^{\star},$$

such that $\mathcal{Q} \circ f = f \circ \mathcal{Q}'$ and f preserves the involution: $f(m^{\star}) = f(m)^{\star}$.

2.6 Definition, [6]

If the space of derivations $Der(\hat{T}_{\ge 1}\Sigma^{-1}\mathcal{M}^{*})$ for the $\mathcal{A}_{\infty}$-algebra $(\mathcal{M}, \mathcal{Q})$, then the differential graded vector space $\mathcal{M}$ is the Hochschild homology complex $\mathcal{HH}_{\blacksquare}(\mathcal{M}, \mathcal{M})$ of $\mathcal{M}$ with coefficients in itself:

$$\mathcal{CH}_{\blacksquare}(\mathcal{M}, \mathcal{M}) = \Sigma^{-1}Der(\hat{T}_{\ge 1}\Sigma^{-1}\mathcal{M}^{\star}). \quad (5)$$

2.7 Definition, [5]

Let $(\mathcal{M}, \mathcal{Q})$ be the involutive $\mathcal{A}_{\infty}$-algebras. So the cyclic homology of $\mathcal{A}_{\infty}$-algebras $\mathcal{HC}_{\blacksquare}(\mathcal{M})$ is a differential graded vector space $\mathcal{CC}_{\blacksquare}(\mathcal{M})$, which is represented by:

$$\mathcal{CC}_{\blacksquare}(\mathcal{M}) = \sum \prod_{\iota=1}^{\infty} [(\Sigma^{-1}\mathcal{M}^{\star})^{\otimes \iota}]_{\mathcal{Z}_{\iota}}, \quad (6)$$

Such as, $\mathcal{Z}_{\iota}$ is the cyclic group of order ι.

2.8 Definition, [8]

By considering $\mathcal{M}$ is $\mathcal{A}_{\infty}$-algebras $(\mathcal{M}, \delta, \varphi_n)$, then the cyclic differential module $(\mathcal{C}(\mathcal{M}), t, \delta)$ is denoted by:

$$\mathcal{C}(\mathcal{M}) = \{\mathcal{C}(\mathcal{M})_{s,n}\},$$
$$since\ \mathcal{C}(\mathcal{M})_{s,n} = (\mathcal{M}^{\otimes(n+2)})_{s}, \quad \forall n, s \ge 0,$$
$$t_n(m_0 \otimes \ldots \otimes m_s) = (-1)^{|m_n|(|m_0|+\cdots+|m_{s-1}|} m_n \otimes m_0 \otimes \ldots \otimes m_{s-1},$$

$$\delta_n(m_0 \otimes \dots \otimes m_s) = \sum_{k=0}^{n} (-1)^{|m_0|+\dots+|m_{k-1}|} m_0 \otimes \dots \otimes m_{k-1} \otimes \delta m_k \otimes m_{k+1} \otimes m_n,$$

such $|m| = *$ means that, $m \in \mathcal{M}_*$. Also, suppose that the family of maps:

$$\mathscr{b}' = \{\mathscr{b}_{(k_1,\dots,k_s)} : \mathcal{C}(\mathcal{M})_{n,p} \to \mathcal{C}(\mathcal{M})_{n-s,p+s-1}\}, \quad 0 \le k_1 < \dots < k_s < n, \quad n,p \ge 0,$$

denoted by:

$$\mathscr{b}_{(k_1,\dots,k_s)} = \begin{cases} (-1)^{s(p-1)} 1^{\otimes j} \otimes \varphi_{s-1} \otimes 1^{\otimes(n-s-j)}, & \text{if } 0 \le j \le n-s, (k_1,\dots,k_s) = (j, j+1, \dots, j+s-1) \\ (-1)^{q(s-1)} \mathscr{b}_{(0,1,\dots,s-1)} t_n^q, & \text{if } 1 \le q \le s, \text{ and } (k_1,\dots,k_s) = (0,1,\dots,s-q-1, n-q+1, n-q+2, \dots, n) \\ 0 & \text{otherwise.} \end{cases}$$

The quadruple $(\mathcal{C}(\mathcal{M}), t, \partial, \mathscr{b}')$ is the cyclic modules of ∞-simplicial sides, for each $\mathcal{A}_\infty$-algebras $(\mathcal{M}, \delta, \varphi_n)$.

Also, we can define a cyclic homology $\mathcal{HC}(\mathcal{M})$ of $\mathcal{A}_\infty$-algebra by the cyclic homology $\mathcal{HC}(\mathcal{C}(\mathcal{M}))$ of the cyclic modules of ∞-simplicial sides $(\mathcal{C}(\mathcal{M}), t, \partial, \mathscr{b}')$.

$$\mathcal{HC}(\mathcal{M}) = \left(Tot\mathcal{C}\left(\overline{\mathcal{C}(\mathcal{M})}\right), D\right), \; D = D_1 + D_2, \quad (7)$$

Consequently, the cyclic homology $\mathcal{HC}(\mathcal{M})$ of $\mathcal{A}_\infty$-algebra is the homology of the chain complex

$$\mathcal{HC}(\mathcal{M}) = \left(Tot\mathcal{C}\left(\overline{\mathcal{C}(\mathcal{M})}\right), D\right), \; D = D_1 + D_2,$$

associated with the chain bicomplex $(\mathcal{C}(\overline{\mathcal{C}(\mathcal{M})}), D_1, D_2)$.

Note that if an $\mathcal{A}_\infty$-algebra is a differential associative algebra $(\mathcal{M}, \delta, \varphi_n)$, where $\varphi_0 = \varphi$ and $\varphi_n = 0$, $n > 0$, then the chain bicomplex $(\mathcal{C}(\overline{\mathcal{C}(\mathcal{M})}), D_1, D_2)$ coincides with the Tsygan chain bicomplex for the differential associative algebra $(\mathcal{M}, \delta, \varphi_n)$.

2.9 Definition, [6]

For a graded vector space $\mathcal{M}$ be with an involution. Then the dihedral group of order $2p$ symbolized by $\mathcal{D}_p$, since $\mathcal{D}_p = [r, s \mid r^p = s^2 = 1, \; srs^{-1} = r^{-1}]$. Then there are the following two actions of $\mathcal{D}_p$ on $\mathcal{M}^{\otimes p}$, $\forall m_i \in \mathcal{M}$.

1- The dihedral action can be described as:

$r(m_1 \otimes m_2 \otimes \dots \otimes m_p) = (-1)^\varepsilon m_p \otimes m_1 \otimes \dots \otimes m_{p-1}$,

$s(m_1 \otimes m_2 \otimes \dots \otimes m_p) = (m_1 \otimes m_2 \otimes \dots \otimes m_p)^\star$.

2- The skew-dihedral action can be described as:

$r(m_1 \otimes m_2 \otimes \dots \otimes m_p) = (-1)^\varepsilon m_p \otimes m_1 \otimes \dots \otimes m_{p-1}$,

$s(m_1 \otimes m_2 \otimes \dots \otimes m_p) = -(m_1 \otimes m_2 \otimes \dots \otimes m_p)^\star$.

3 Problem Solution

Through this section, we discuss and study the Steenrod's operator for the dihedral homology of $\mathcal{A}_\infty$-algebras. By using [9], [10] and [11] let us assume that $\mathcal{K}$ be the field with characteristic zero, where $\mathcal{M}$ is the commutative $\mathcal{K}$-infinity algebras. Suppose that $[\mathcal{D}_p]$ is a dihedral category, then $\mathcal{K}[\mathcal{D}_p]$ is an $\mathcal{A}_\infty$-algebras related with $[\mathcal{D}_p]$ over $\mathcal{K}$ [1], [4], [7]. Also, ${}^\varepsilon\mathcal{A}_\mathcal{D}$ is describing on the $\mathcal{K}[\mathcal{D}_p]$-module, the construction of the commutative $\mathcal{K}[\mathcal{D}_p]$-algebra as determined by:

$${}^\varepsilon(\mathcal{M}^{\otimes n})^D \xrightarrow{\Delta} {}^\varepsilon(\mathcal{M}^{\otimes n} \otimes \mathcal{M}) \xrightarrow{f} {}^\varepsilon(\mathcal{M}^{\otimes n})^D \otimes {}^\varepsilon(\mathcal{M})^D \quad (8)$$

where Δ is the homomorphism of $\mathcal{K}[\mathcal{D}_p]$, and f is defined by:

$$f((m_0 \otimes n_0) \otimes (m_1 \otimes n_1) \otimes \dots \otimes (m_s \otimes n_s)) = (m_0 \otimes m_1 \otimes \dots \otimes m_s) \otimes (n_0 \otimes n_1 \otimes \dots \otimes n_s).$$

Assume that $f \circ \Delta = {}^\varepsilon\Delta_\mathcal{D}$ provides the commutative multiplication in ${}^\varepsilon\mathcal{M}_\mathcal{D}$. We indicate that ${}^\varepsilon\Delta_\mathcal{D}$ is the $\mathcal{K}[\mathcal{D}_p]$-homomorphism known on the $\mathcal{A}_\infty$-algebras $\mathcal{K}[\mathcal{D}_p]$, the multiplication $\mathcal{K}[\mathcal{D}_p] \to \mathcal{K}[\mathcal{D}_p] \otimes_k \mathcal{K}[\mathcal{D}_p]$, like that $\kappa \to \kappa \otimes \kappa$, $\kappa \in \mathcal{K}[\mathcal{D}_p]$.

As $({}^\varepsilon\mathcal{M}_\mathcal{D} \otimes_k {}^\varepsilon\mathcal{M}_\mathcal{D})$ is $(\mathcal{K}[\mathcal{D}_p] \otimes_k \mathcal{K}[\mathcal{D}_p])$ module, then through the multiplication on $({}^\varepsilon\mathcal{M}_\mathcal{D} \underset{k}{\otimes} {}^\varepsilon\mathcal{M}_\mathcal{D})$ one can describe $\mathcal{K}[\mathcal{D}_p]$-module construction and $\mathcal{K}[\mathcal{D}_p]$-module homomorphism f, subsequently:

$$\begin{aligned} f\left(\kappa((m_0 \otimes n_0) \otimes (m_1 \otimes n_1) \otimes \dots \otimes (m_s \otimes n_s))\right) &= \kappa(m_0 \otimes m_1 \otimes \dots \otimes m_s) \otimes \kappa(n_0 \otimes n_1 \otimes \dots \otimes n_s) \\ &= \kappa((m_0 \otimes m_1 \otimes \dots \otimes m_s) \otimes (n_0 \otimes n_1 \otimes \dots \otimes n_s)) \\ &= \kappa f((m_0 \otimes n_0) \otimes (m_1 \otimes n_1) \otimes \dots \otimes (m_s \otimes n_s)), \\ &, \kappa \in \mathcal{K}[\mathcal{D}_p] \end{aligned} \quad (9)$$

Therefore, the morphism ${}^{\varepsilon}\Delta_{\mathcal{D}}$ is the $\mathcal{K}[\mathcal{D}_p]$-module homomorphism. Then the dihedral homology $Ext^m_{\mathcal{K}[\mathcal{D}_p]}({}^{\varepsilon}\mathcal{M}_{\mathcal{D}}, (\mathcal{K}_{\mathcal{D}})_\star)$ can be determined by applying the normalized bar construction $\beta(\mathcal{L})$, [3]. By assuming that $\mathcal{L}$ be the triples$({}^{\varepsilon}\mathcal{M}_{\mathcal{D}}, \mathcal{K}[\mathcal{D}_p], \mathcal{K}_{\mathcal{D}})$, $(\mathcal{K}[\mathcal{D}_p], \mathcal{K}[\mathcal{D}_p], \mathcal{K}_{\mathcal{D}})$, and also let $\mathcal{JK}[\mathcal{D}_p]$ be the kernel identity $\mathscr{k} \to \mathcal{K}[\mathcal{D}_p]$.

We establish the identity of the normalized bar structure $\beta(\mathcal{L})$ with the $\mathscr{k}$-module: $\beta(\mathcal{L}) = {}^{\varepsilon}\mathcal{M}_{\mathcal{D}} \otimes_{\mathcal{K}[\mathcal{D}_p]} \mathcal{T}(\mathcal{JK}[\mathcal{D}_p]) \otimes_{\mathcal{K}[\mathcal{D}_p]} \mathcal{K}_{\mathcal{D}}$, where $\mathcal{T}(\mathcal{JK}[\mathcal{D}_p])$ be the algebraic tensor of $\mathcal{JK}[\mathcal{D}_p]$. Obviously the $\mathcal{K}$-module $\beta(\mathcal{L})$ can be graded. Then the elements of $\mathcal{K}$-module $\beta(\mathcal{L})$ is possible to write: $\mathscr{m}[v_1, v_2, \dots, v_s]\mathscr{k} \in \beta(\mathcal{L})_s$, $\mathscr{m} \in {}^{\varepsilon}\mathcal{M}$, $v_i \in \mathcal{K}[\mathcal{D}_p]$, $\mathscr{k} \in \mathcal{K}_{\mathcal{D}}$.

The differential $\delta: \beta(\mathcal{L})_s \longrightarrow \beta(\mathcal{L})_{s-1}$ and the argument $f: \beta(\mathcal{L}) \longrightarrow {}^{\varepsilon}\mathcal{M}_{\mathcal{D}} \otimes_{\mathcal{K}[\mathcal{D}_p]} \mathcal{K}_{\mathcal{D}}$ written as:

$$\delta[\mathscr{m}[v_1|v_2 \dots |v_s]\mathscr{k}] = \mathscr{m}v_1[v_2|v_3| \dots |\mathscr{k} + \sum_{i=1}^{s-1}(-1)^i \mathscr{m}[v_1| \dots |v_{i-1}|v_i v_{i+1}|v_{i+2}| \dots |v_s]\mathscr{k} + (-1)^s \mathscr{m}[v_1| \dots |v_{s-1}v_s]\mathscr{k}. \quad (10)$$

and $f[v_1| \dots |v_s]\mathscr{k} = 0$, $\quad f(\mathscr{m}[]\mathscr{k}) = 0$.

Also, the maps δ and f can be defined for ζ in the similar way.

As a reminder, the differential δ for $\mathcal{L}$ seems to be the left $\mathcal{K}[\mathcal{D}_p]$-module homomorphism, with $\delta S + S\delta = 1 - \sigma f$, where σ is the homomorphism as: $\sigma: \mathcal{K}_{\mathcal{D}} \longrightarrow \beta(\zeta)$, and $S: \beta(\zeta)_s \longrightarrow \beta(\zeta)_{s+1}$, is assumed by the forms:

$$\sigma(\mathscr{k}) = []\mathscr{k} \otimes [], \quad S(v[v_1| \dots |v_s]\mathscr{k}) = v[v_1| \dots |v_s]\mathscr{k}.$$

Obviously, in the complex $\beta(\ell)_s \longrightarrow {}^{\varepsilon}\mathcal{M}_{\mathcal{D}} \underset{\mathcal{K}[\mathcal{D}_p]}{\otimes} \beta(\zeta)$, there is the differential $\delta = 1 \otimes_{\mathcal{K}[\mathcal{D}_p]} \delta$.

From [9], we have the next:

$$Hom_{\mathcal{K}[\mathcal{D}_p]}(\beta(\zeta), ({}^{\varepsilon}\mathcal{M}_{\mathcal{D}}))_\star = (\beta(\zeta))_\star = Hom_{\mathcal{K}[\mathcal{D}_p]}(\beta({}^{\varepsilon}\mathcal{M}_{\mathcal{D}}), \mathcal{K}[\mathcal{D}_p], (\mathcal{K}_{\mathcal{D}})_\star)$$

Then,

$$_{\varepsilon}\mathcal{HD}_m(\mathcal{A}) = Ext^m_{\mathcal{K}[\mathcal{D}_p]}({}^{\varepsilon}\mathcal{M}_{\mathcal{D}}, (\mathcal{K}_{\mathcal{D}})_\star) = \mathcal{H}_m\beta(\mathcal{L})_\star). \quad (11)$$

Assume the triples $\mathcal{L} = \left(({}^{\varepsilon}\mathcal{M}_{\mathcal{D}}), \mathcal{K}[\mathcal{D}_p], \mathcal{K}_{\mathcal{D}}\right)$ and $\hat{\zeta} = \left(({}^{\varepsilon}\widehat{\mathcal{M}}_{\mathcal{D}}), \widehat{\mathcal{K}}[\mathcal{D}_p], \widehat{\mathcal{K}}_{\mathcal{D}}\right)$ and ruminate the product $\Gamma: \beta(\mathcal{L} \otimes \hat{\zeta}) \longrightarrow \beta(\mathcal{L}) \otimes \beta(\hat{\zeta})$.

Describe on $\beta(\mathcal{L})$ such that a construction of associative algebra through multiplication $\tilde{\Delta} = \Gamma\beta\left({}^{\varepsilon}\Delta_{\mathcal{D}}, \Delta_{\mathcal{K}[\mathcal{D}_p]}, \Delta_{\mathcal{K}_{\mathcal{D}}}\right): \beta(\mathcal{L}) \longrightarrow \beta(\mathcal{L}) \otimes \beta(\mathcal{L})$ and on a complex $(\beta(\mathcal{L}))_\star$ the next multiplication:

$$(\beta(\mathcal{L}))_\star \otimes (\beta(\mathcal{L}))_\star \to \left(\beta(\mathcal{L}) \otimes \beta(\mathcal{L})\right)_\star \xrightarrow{(\tilde{\Delta})_\star} (\beta(\mathcal{L}))_\star. \quad (12)$$

The following lemma is simply confirmed by applying the standard methods of the homological $\mathcal{A}_\infty$-algebras.

3.1 Lemma

By assuming that η is a subgroup of a symmetrical group ξ_r and $\mathcal{E}$ seems to be the $\mathcal{K}[\eta]$-free resolution $\mathcal{K}[\eta]$-module $\mathcal{K}$ such $\mathcal{E}_0 = \mathcal{K}[\eta]$ through $\mathscr{v}_0$ the generator of $\mathcal{K}[m]$, since $[\mathcal{E} \otimes \beta(\mathcal{L})]_s = \sum_{i+j=s} \mathcal{E}_i \otimes \beta_j(\mathcal{L})$, the module $\mathcal{E} \otimes \beta(\mathcal{L})$ is graded.

Then the graded $\mathcal{K}[m]$complexes exist with the next conditions of the homomorphism $\Lambda: \mathcal{E} \otimes \beta(\mathcal{L}) \longrightarrow \beta(\mathcal{L})^{\otimes r}$ such as:

(i) $\Lambda(e \otimes \mathscr{b}) = 0$, $\mathscr{b} \in \beta(\mathcal{L})_0$ and $e \in \mathcal{E}_i$, $i > 0$.
(ii) $\Lambda(\mathscr{v}_0 \otimes \mathscr{b}) = \tilde{\Delta}^{\otimes r}(\mathscr{b})$, $if\ \mathscr{b} \in \beta(\mathcal{L})$, $\tilde{\Delta}^{\otimes r}: \beta(\mathcal{L}) \longrightarrow \beta(\mathcal{L})^{\otimes r}$.
(iii) The map Λ for $\beta(\mathcal{L})$ is the homomorphism of the left $\mathcal{K}[\mathcal{D}_p]$-module, as $\mathcal{K}[\mathcal{D}_p]$ works on $\mathcal{E} \otimes \beta(\mathcal{L})$ through the relation $\mathcal{K}(e \otimes \mathscr{b})e \otimes \mathscr{kb}$.
(iv) $\Lambda(e_i \otimes \beta(\mathcal{L})_s) = 0$ When $i > (r - 1)$.

Additionally, each pair of homomorphisms with similar properties has $\mathcal{K}[\eta]$-homotopy. Now, give the $\mathcal{K}[\mathcal{D}_p]$-homomorphism Ω the following definition:

$\Omega: \mathcal{E} \otimes (\beta(\mathcal{L})_\star)^{\otimes r} \longrightarrow \beta(\mathcal{L})_\star$, since;
$\Omega(e \otimes x)(\ell) = \mathfrak{B}(x)\Lambda(e \otimes \ell), e \in \mathcal{E}, x \in (\beta(\mathcal{L})_\star)^{\otimes r} and\ \ell \in \beta(\mathcal{L})$.
$\mathfrak{B}: (\beta(\mathcal{L})_\star)^{\otimes r} \longrightarrow (\beta(\mathcal{L})^{\otimes r})_\star$, is a homomorphism that is trivial.

Proof: Now, the operator in $\mathcal{H}(\beta(\mathcal{L})_\star)$ will be defined. In lemma (3.1), considering $\mathcal{K} = \mathcal{Z}/\mathcal{P}$. Assume that $\mathcal{E}$ has the normal $\mathcal{K}(\mathcal{Z}/\mathcal{P})$-free resolution. Here, the free $\mathcal{K}(\mathcal{Z}/\mathcal{P})$-module using the generator $\mathscr{v}_i$, denoted as $\mathcal{E}_i$ for $i \geq 0$. Assuming that the graded $\mathcal{E}_i = \mathcal{E}^{-i}$ remains the free $\mathcal{K}(\mathcal{Z}/\mathcal{P})$-module using the generator $\mathscr{v}^{-i}$.

Letting $a \in \mathcal{H}^{q}(\beta(\mathcal{L})_{\star}$ and define the homomorphism: $\mathfrak{N}_i: \mathcal{H}_q(\beta(\mathcal{L})_{\star}) \longrightarrow \mathcal{H}_{pq-i}(\beta(\mathcal{L})_{\star}$ as $\mathfrak{N}_i(a) = \Omega_{\star}(v^{-i} \otimes a_p), i \geq 0$.

Now, The Steenrod operator $\mathcal{P}_i$ defined with the operator $\mathcal{R}_i$, as follows:

1) If $p = 2$ then, $p_s(a) = \mathfrak{N}_{q-s}(x) \in \mathcal{H}_{q+s}(\beta(\mathcal{L})_{\star})$, since $\mathfrak{N}_i = 0$ if $i < 0$.
2) If $p > 2$ then,

$p_s(a) = (-1)^s \gamma(-q)\mathfrak{N}_{(q-2s)(p-1)}(a) \in \mathcal{H}_{q+2s(p-1)}(\beta(\mathcal{L})_{\star})$,
$\mathfrak{B}p_s(a) = (-1)^s \gamma(-q)\mathfrak{N}_{(q-2s)(p-1)-1}(a) \in \mathcal{H}_{q+2s(p-1)+1}(\beta(\mathcal{L})_{\star})$,
where $\mathfrak{N}_i = 0$ and $\ell = 0$ $or 1$ and $\gamma(-q) = (-1)^{j}(mI)_{\mathcal{L}}, m = \frac{p-1}{2}$ if $q = 2J - \ell$, $i < 0$.

3.2 Theorem

Given that $\mathcal{K} = \mathcal{Z}/\mathcal{P}$ and $\mathcal{M}$ is the commutative $\mathcal{K}$-infinity algebras, so the next homomorphisms "Steenrod map" defined for the dihedral homology group ${}_{\varepsilon}\mathcal{HD}(\mathcal{M})$ as:

(i) $\mathcal{P}_i: {}_{\varepsilon}\mathcal{HD}_s(\mathcal{M}) \to {}_{\varepsilon}\mathcal{HD}_{s+i}(\mathcal{M})$, if $p = 2$,
(ii) $\mathcal{P}_i: {}_{\varepsilon}\mathcal{HD}_s(\mathcal{M}) \to {}_{\varepsilon}\mathcal{HD}_{s+2i(p-1)}(\mathcal{M})$, and $\mathfrak{B}\mathcal{P}_i: {}_{\varepsilon}\mathcal{HD}_s(\mathcal{M}) \to {}_{\varepsilon}\mathcal{HD}_{s+i+2i(p-1)}(\mathcal{M})$ if $p > 2$.

The following characteristics apply to the operators $\mathcal{P}_i$ and $\mathfrak{B}\mathcal{P}_i$:

1) $\begin{cases} \mathcal{P}_i|_{{}_{\varepsilon}\mathcal{HD}_s(\mathcal{M})} = 0, if\ p = 2,\ i > s, \\ \mathcal{P}_i|_{{}_{\varepsilon}\mathcal{HD}_s(\mathcal{M})} = 0, if\ p > 2,\ 2i > s, \\ \mathfrak{B}\mathcal{P}_i|_{{}_{\varepsilon}\mathcal{HD}_s(\mathcal{M})} = 0, if\ p > 2,\ 2i \geq s, \end{cases}$
2) $\mathcal{P}_i(a) = a_p, if\ p = 2,\ i = s,\ or\ p > 2\ and\ 2i = s,$
3) $\mathcal{P}_j = \sum \mathcal{P}_i \otimes \mathcal{P}_{j-i}$ and $\mathfrak{B}\mathcal{P}_j = \sum \mathfrak{B}\mathcal{P}_{j-i} + \mathcal{P}_i \otimes \mathcal{P}_{j-i}$.
4) The next relations of Adam are satisfied by the operators $\mathcal{P}_i$ and $\mathfrak{B}\mathcal{P}_i$:

(a) If $y < pb$ and $p \geq 2$, we have:

$$\mathfrak{B}_{\gamma}\mathcal{P}_{y}\mathcal{P}_{b} \sum_i (-1)^{y+i}(y - p_i, (p-1)b - y + i - 1).\mathfrak{B}_{\gamma}\mathcal{P}_{y+b-i}\mathcal{P}_i$$

Since $\gamma = 0\ or\ 1\ for\ p = 2$, also $\gamma = 1\ for\ p > 2$ and for any two integers i and j, there exist:

$$(i, j) = \begin{bmatrix} \frac{(i,j)!}{i!, j!}, & if \quad i \geq 0, j \geq 0, \\ 0 & if \quad i < 0, j < 0, \end{bmatrix}$$

(b) If $y < pb, p = 2\ and\ \gamma = 0\ or\ 1$, then:

$$\mathfrak{B}_{\gamma}\mathcal{P}_{y}\mathcal{P}_{b} = (1-\gamma)\sum_i (-1)^{y+i}(y - p_i, (p-1)b - y + i - 1).\mathfrak{B}\mathcal{P}_{y+b-i}\mathcal{P}_i - \sum_i (-1)^{y+i}(y - p_i - 1, (p-1)b - y + i).\mathfrak{B}_{\gamma}\mathcal{P}_{y+b-i}\mathfrak{B}\mathcal{P}_i.$$

By noted that, the operators $\mathfrak{B}_0\mathcal{P}_s$ and $\mathfrak{B}_1\mathcal{P}_s$ represent, respectively, $\mathcal{P}_s$ and $\mathfrak{B}\mathcal{P}_s$.

Proof: By assuming that the triple $\mathfrak{C} = (\mathcal{R}, \mathcal{M}, \mathcal{T})$, when $\mathcal{M}$ is the commutative $\mathcal{A}_{\infty}$-algebras for $\mathcal{K} = \mathcal{Z}/\mathcal{P}$, such $\mathcal{T}$ and $\mathcal{R}$ are the left and the right commutative $\mathcal{M}$-algebras, respectively. Because of the explanation above and taking into consideration the triple $\mathcal{L} = \left(({}^{\varepsilon}\mathcal{M}_{\mathcal{D}}), \mathcal{K}[\mathcal{D}_p], \mathcal{K}_{\mathcal{D}}\right)$, we get that $\mathcal{K}[\mathcal{D}_p]$ is the commutative $\mathcal{A}_{\infty}$-algebras over $\mathcal{K} = \mathcal{Z}/\mathcal{P}$, ${}^{\varepsilon}\mathcal{M}_{\mathcal{D}}$ and $\mathcal{K}_{\mathcal{D}}$, respectively, the left and right commutative $\mathcal{K}[\mathcal{D}_p]$-algebras and hence $\mathcal{H}((\beta(\mathcal{L})_{\star}) = {}_{\varepsilon}\mathcal{HD}(\mathcal{M})$.

4 Conclusion

In our paper, we introduced and studied another definition for the Steenrod operator for dihedral homology of $\mathcal{A}_{\infty}$-algebras. It is shown by applying a tensor product of the group of symmetry-free resolution, besides the normal $\mathcal{A}_{\infty}$-algebras resolution by the dihedral group and does sporadic Steenrod operation computations and obtains a little approximation.

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