New Stability Analysis of a Time-Delay System Subject to Bounded Distributed Delay

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Abstract: – Time-delay systems model various types of real systems. The problem of the time delay effect on the system performance, particularly its stability, is a recurring challenge. The presence of the time delay can lead to complex and disruptive behaviors, such as oscillations, poor performance, and instability. The Lyapunov method is widely used to analyze the stability of these systems by inferring stability conditions. In this paper, a simplified methodological approach for the stability analysis of distributed delay systems is proposed. This work aims to significantly reduce the complexity of calculations while preserving the rigor of classical methods based on the Lyapunov-Krasovskii functional, by utilizing Leibniz integrals. The study focuses on a linear system with state-distributed delay three numerical examples are used to verify the superiority of the stability conditions obtained in this work.

Key-Words: - Time-delay system, linear system, state-delay, bounded distributed delay, stability analysis, Krasovskii Lyapunov functional, Leibniz integrals.

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1 Introduction

Time delay is common in many control systems, including robotic systems, networked systems, chemical processes, biological systems, and more [1], [2]. Ignoring time delay can lead to reduced system performance, especially its stability. There are various types of delay systems, including neutral delay systems, which have a delay in the derivative of their state [3], uncertain delay systems [4], singular delay systems which are characterized by their singular matrix on the state derivative [5], [6], and distributed delay systems [7]. In the case of distributed delay systems, the distributed delay can either be limited (also known as bounded or finite) [8], [9] or infinite (unbounded) [10], [11], [12].
The stability of time-delay systems has been one of the most widely studied problems. Various methods have been developed. The Lyapunov method is classified as the most commonly used and well-known method in this sense.

Conducting stability analyses for delayed systems is crucial. It can predict and mitigate potential problems caused by delays, as well as provide critical information for effective control strategies and system design, [13].

In the case of time-delay systems, there are two main method categories to analyze the stability analysis: the input-output method and the Lyapunov method, [14]. The efficient method is the Lyapunov method. For TDSs, two main Lyapunov methods exist: the Krasovskii method of Lyapunov functionals (1956) and the Razumikhin method of Lyapunov functions (1956). The Krasovskii method applies to a wider range of problems, and it usually leads to less conservative results than the Razumikhin method, [15], [16]. By evaluating the sign of the derivative of these functions (respectively functional), which are generally defined as positive real value functions related to the state of the system, the conditions of stability of the system can be deduced. The absence of a universal framework for their construction encourages researchers to develop new approaches aimed at reducing the conservatism of stability criteria (e.g., an augmented Lyapunov function for variable delays and Lyapunov functionals for periodic delays where proposed [13], a novel Lyapunov Krasovskii functional consisting of integral terms based on the second-order derivative is constructed to enhance the feasible region of delay-dependent stability [17], etc.), as well as various methods for calculating and estimating integral terms (e.g., an integral inequality based on fragmented components is developed using matrix separation and mixed estimation of the augmented integral term, furthermore, a linearized transformation method that deals with non-linear terms related to a low complexity delay is also proposed [18], an improved free-matrix-based double integral inequality is proposed based on free-matrix-based integral inequality [14], a generalized multiple-integral inequality is put forward based on Schur complement lemma is proposed [19], and so on).

Even though these approaches are effective, their practical application is sometimes limited by the complexity of the required calculations. In this stage, this paper aims to provide a new methodological perspective by clarifying and simplifying these calculations while maintaining the theoretical rigor of classical approaches, using Leibniz integrals in stability analysis based on Lyapunov-Krasovskii functionals. The given method, which improves the accessibility of stability analyses, is particularly useful for complex systems, which makes it useful for systems with nonlinear characteristics or more varied delays.

In this work, we apply this method to a linear system with a bounded distributed state delay confined between -h and 0, assuming the system is autonomous. By using Leibniz's laws and Linear Matrix Inequalities (LMIs), we demonstrate that the results obtained are equivalent to those from traditional methods but with significantly clear and simplified calculations.

2 Problem Formulation

Consider the autonomous linear system as described below, with a bounded distributed state delay:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d \int_{-h}^{0} x(t+s)ds \\ x(t) = \varphi(t) \qquad\qquad \forall t \epsilon [-h, 0] \end{cases} \qquad (1)$$

where $x(t) \in R^n$ is the state vector, $A, A_d \in R^{n\times m}$ are constants matrices such that A and $A + A_d$ are Hurwitz matrices, the delay h is assumed to be known and the initial condition $\varphi(t)$ is a continuous vector-valued function in $t \in [-h, 0]$.

Lemma 1. (Leibniz's generalized differentiation product rule, [20]). If the functions f and g are n-fold differentiable on a certain interval, then their product is also n-fold differentiable when $n \epsilon \mathbb{Z}^+$, Leibniz's rule is as follows:

$$D^n\{f(x)g(x)\} = \sum_{k=0}^{n} \binom{n}{k} \{D^k f(x)\}\{D^{n-k} g(x)\}$$

Lemma 2. (Leibniz's integral rule for differentiation under the sign, [21]). Let a(t) and b(t) two functions dependent on time, Leibniz's integral rule for differentiation under the sign of the integral for any integral of the form $\int_{a(t)}^{b(t)} f(t,x)dx$ is written as follows, [21]:

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(t,x)dx = \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(t,x)dx + \frac{db(t)}{dt} f\big(t, b(t)\big) - \frac{da(t)}{dt} f\big(t, a(t)\big)$$

3 Main Results

The following stability criterion is based on the new integral inequality.

Theorem 1. Given a constant $h \in [-h, 0]$, the system (1) is asymptotically stable if there exists $P > 0$, $R \geq 0$ with dimensions appropriate, such that the following LMI is feasible:

$$\begin{bmatrix} A^T P + PA + R & 0 & PA_d \\ 0 & -R & 0 \\ A_d{}^T P & 0 & R \end{bmatrix} < 0 \qquad \square$$

Proof. Choose the following Lyapunov–Krasovskii functional candidate:

$V(x_t) = \sum_{i=1}^{2} V_i(x(t), t)$, such that:

$$V_1(x(t)) = x^T(t)Px(t) \qquad (2a)$$

$$V_2\big(x(t)\big) = \mathrm{h}\int_{-h}^{0} x^T(t+s)Rx(t+s)ds. \qquad (2b)$$

where P and R are constant symmetrical matrix $\in R^{\,nxn}$ satisfying $R = R^T > 0$ and $P = P^T > 0$.

Differentiating, in the first time, V_1 along (1) by applying the Lemma 1. of Leibniz rule. In our case we are interested in the first derivative of the product of the functions $f(t) = x^T(t)$ and $g(t) = x(t)$, which is equal to the sum of the product of the first derivative of the function $f(t)$, and the function $g(t)$, and the product of the first derivative of the function $g(t)$, and the function $f(t)$ as described bellow :

$$\dot{V}_1\big(x(t)\big) = \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t)$$

Replace the derivative of x(t) by its formula we have:

$$\begin{aligned}\dot{V}_1\big(x(t)\big) &= (Ax(t) + A_\mathrm{d}\int_{-h}^{0} x(t+s)ds)^T Px(t) \\ &\quad + x^T(t)P(Ax(t) \\ &\quad + A_\mathrm{d} \int_{-h}^{0} x(t+s)ds) \\ &= (x^T(t)A^T + \int_{-h}^{0} x^T(t+s)A_\mathrm{d}^T ds)\, Px(t) \\ &\quad + x^T(t)PAx(t) \\ &\quad + x^T(t)PA_\mathrm{d}\int_{-h}^{0} x(t+s)ds)\end{aligned}$$

$$= x^T(t)A^TPx(t) + x^T(t)PAx(t) + \int_{-h}^{0} x^T(t+s){A_d}^T Px(t)ds + x^T(t)PA_{\mathrm{d}}\int_{-h}^{0} x(t+s)ds)$$

We factorize the first two terms having the same variables:

$$\dot{V}_1\big(x(t)\big) = x^T(t)(A^TP + PA)x(t) + \int_{-h}^{0} x^T(t+s)A_{\mathrm{d}}^T Px(t)ds + \int_{-h}^{0} x^T(t)PA_{\mathrm{d}}x(t+s)ds \quad (3)$$

Rearrange the two last terms on the right-hand side (3) as follows:

$$\int_{-h}^{0} (x^T(t+s)A_{\mathrm{d}}^T Px(t) + x^T(t)PA_{\mathrm{d}}x(t+s))ds = \int_{-h}^{0} x^T(t+s)dsA_{\mathrm{d}}^T Px(t) + x^T(t)PA_{\mathrm{d}}\int_{-h}^{0} x(t+s)ds$$

Thus,

$$\dot{V}_1\big(x(t)\big) = x^T(t)(A^TP + PA)x(t) + \int_{-h}^{0} x^T(t+s)dsA_{\mathrm{d}}^T Px(t) + x^T(t)PA_{\mathrm{d}}\int_{-h}^{0} x(t+s)ds \quad (4)$$

By following a similar given process, by applying the lemmas 1 and 2, the derivative of the second part of the functional V(x(t)) leads to:

$$V_2 = \mathrm{h}\int_{-h}^{0} x^T(t+s)Rx(t+s)ds$$

$$\dot{V}_2\big(x(t)\big) = x^T(t)Rx(t) - x^T(t-h)Rx(t-h) \quad (5)$$

Taking the time derivative of V(x(t)) by combining (4) and (5), we get:

$$\dot{V}(x_t) = \dot{V}_1 + \dot{V}_2$$

$$= x^T(t)(A^TP + PA)x(t) + \int_{-h}^{0} 2x^T(t+s)A_{\mathrm{d}}^T Px(t)ds + \int_{-h}^{0} x^T(t+s)(A^TR + RA)x(t+s)ds + x^T(t)Rx(t) - x^T(t-h)Rx(t-h)$$

Finally, the derivative of the proposed Lyapunov functional can be rewritten as shown below:

$$\dot{V}(x_t) = \begin{bmatrix} x(t) \\ x(t+s) \\ \int_{-h}^{0} x(t+s)ds \end{bmatrix}^T Q \begin{bmatrix} x(t) \\ x(t+s) \\ \int_{-h}^{0} x(t+s)ds \end{bmatrix} \quad (6)$$

for

$$\zeta^T(t) = \begin{bmatrix} x^T(t) & x^T(t+s) & \int_{-h}^{0} x^T(t+s)ds \end{bmatrix}$$

and

$$Q = \begin{bmatrix} A^TP + PA + R & 0 & PA_d \\ 0 & -R & 0 \\ {A_d}^TP & 0 & R \end{bmatrix} \quad (7)$$

$$\dot{V}(x_t) = \zeta^T(t)\, Q\zeta(t) \quad (8)$$

This shows that system (1) is asymptotically stable under the given condition (7). This completes the proof. ■

4 Numerical Results and Simulation

In this subsection, we provided an example to illustrate the effectiveness of our theorem.

Consider the considered autonomous system (1) with distributed delay as follows:

$$\dot{x}(t) = Ax(t) + A_d\int_{-h}^{0} x(t+s)ds$$

for all constant delays in the interval [0, 0.3]

where $A = \begin{bmatrix} -0.5 & -1 \\ 0 & -0.5 \end{bmatrix}$,

and $A_d = \begin{bmatrix} 0.5 & 0 \\ 1 & -0.5 \end{bmatrix}$

and a delay equals to 0.1s

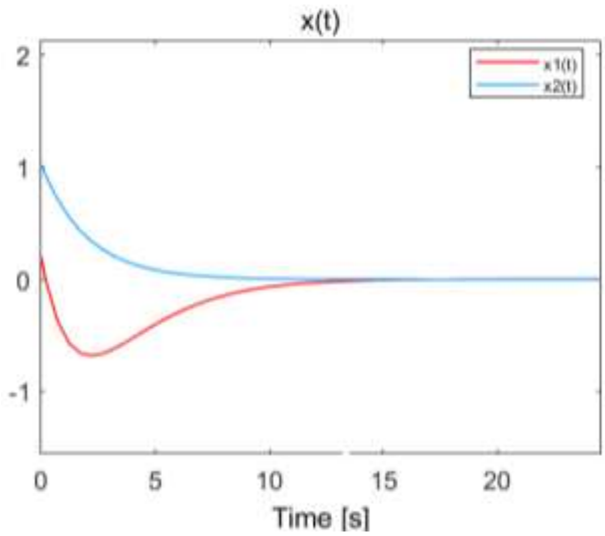

Fig. 1: System dynamics for the first given A and A_d

Take another example $A = \begin{bmatrix} 0 & -1 \\ 1 & -0.5 \end{bmatrix}$,
and $A_d = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.05 \end{bmatrix}$

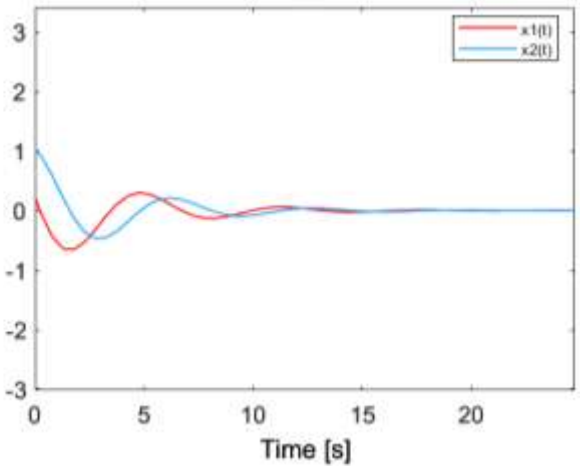

Fig. 2: System dynamics for the second A and A_d

A third example of $A = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$,
and $A_d = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}$

for all constant delays in the interval [0, 0.3], and for the three values of A and A_d different, respecting the stability condition obtained, the dynamics of $x_1(t)$ and $x_2(t)$ converge towards zero, which means that the considered system is stable under the derived stability condition.

These three examples prove the effectiveness of our analysis.

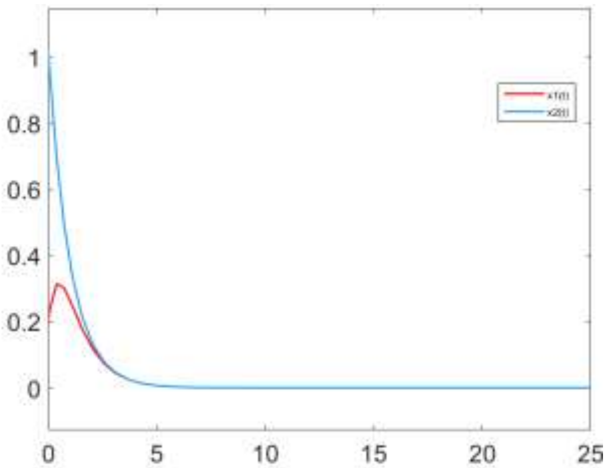

Fig. 3: System dynamics for the third A and A_d

5 Conclusion

This work presents a simplified methodological approach for the stability analysis of distributed delay systems. By leveraging Leibniz integrals, this method significantly simplifies the calculations while preserving the rigor of classical approaches based on Lyapunov-Krasovskii functionals. In this study, a Lyapunov functional defined by (2a-b) is proposed to evaluate the stability of the system with bounded distributed delay (1). The results obtained, summarized by condition (7), and validated through the simulation of three given examples (Figure 1, Figure 2 and Figure 3), demonstrate that any system of this type is asymptotically stable. The result provides condition (7), which ensures that any system of this form is asymptotically stable. This result demonstrates that this simplified method is particularly well-suited for linear systems with bounded distributed delays and could prove valuable for analyzing other types of more complex delay systems for future work.

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