I/O SNR and Noise Covariances Norm Ratio Relation in Kalman Filter

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Abstract: - Kalman filters are used with great success to solve filtering problems in many fields of science and engineering. The ignorance of state noise covariance or the measurement noise covariance often creates difficulties in the practical application of Kalman filters. In this paper, the relation between the Input/Output signal-to-noise ratio (I/O SNR) and the noise covariance norm ratio for the discrete-time steady-state Kalman filter is established. The state or measurement noise covariance can be tuned via the I/O SNR. This result can be applied in time-varying systems and in steady-state systems, without the a priori knowledge of the state or measurement noise covariance.

Key-Words: - Kalman filter, Discrete-time, Steady-state, Riccati equation, Lyapunov equation, State Noise Covariance, Measurement Noise Covariance, Signal-to-Noise Ratio.

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1 Introduction

State estimation uses measurements to estimate/predict the system states. A popular algorithm for this purpose is the Kalman filter [1], which has been successfully used in various fields: object detection and tracking [2], robotic applications [3], electric load estimation [4], stock price prediction [5], weather forecasts [6], satellite orbit determination [7], power generation prediction [8], cases prediction of Covid-19 [9], multi-observation fusion applications related to timescale [10], DC-Drives and sensors applications [11], estimation with unlimited sensing measurements [12], applications where the measurement noise is correlated with the state noise [13], multi-target localization [14].

The discrete-time Kalman filter is associated with discrete-time state space systems, which describe the relation between the $n \times 1$ state vector $x(k)$ and the $m \times 1$ measurement vector $z(k)$, at time k. In the time-invariant case, all the Kalman filter parameters are constant real matrices: F is the transition matrix, H is the output matrix, Q is the state noise covariance matrix and R is the measurement noise covariance matrix.

Kalman filter computes the mean $x(k/k)$ and covariance $P(k/k)$ of estimation, as well as the mean $x(k+1/k)$ and covariance $P(k+1/k)$ of prediction and the Kalman filter gain $K(k)$.

Time-invariant Kalman filter takes the form of steady-state Kalman filter, when well-defined conditions [15] are satisfied. In the steady-state case, the estimation error covariance matrix, the prediction error covariance matrix, and the Kalman filter gain remain constant. The steady-state prediction error is the unique solution P_p of the associated Riccati equation:

$$P_p = Q + F \cdot P_p \cdot F^T - F \cdot P_p \cdot H^T \cdot \left(H \cdot P_p \cdot H^T + R\right)^{-1} \cdot H \cdot P_p \cdot F^T \quad (1)$$

Note that the existence of the inverse in the Riccati equation is guaranteed when the measurement noise covariance matrix R is positive definite (which means that no measurement is exact).

Because of the importance of the Riccati equation, there is considerable literature on its solution, [15], [16], [17], [18], [19], [20].

In the infinite measurement noise case, the Riccati equation takes the form of the Lyapunov equation:

$$P_L = Q + F \cdot P_L \cdot F^T \tag{2}$$

Due to the importance of the Lyapunov equation, there exists considerable literature on its solution, [16], [20].

Using the matrix inversion lemma, the Riccati equation is written as:

$$P_p = Q + F \cdot \left(P_p^{-1} + H^T \cdot R^{-1} \cdot H\right)^{-1} \cdot F^T \tag{3}$$

The nonsingularity of Q and R (which then are positive definite matrices) ensures the nonsingularity of P_p.

The steady-state estimation error covariance matrix is:

$$P_e = \left(P_p^{-1} + H^T \cdot R^{-1} \cdot H\right)^{-1} \tag{4}$$

Note that the steady-state prediction error covariance matrix P_p and the steady-state estimation error covariance matrix P_e are real square symmetric positive definite matrices.

The steady-state Kalman filter gain K is:

$$K = P_p \cdot H^T \cdot \left(H \cdot P_p \cdot H^T + R\right)^{-1} \tag{5}$$

The steady-state Kalman filter produces the state estimation using the previous state estimation and the actual measurement:

$$x(k+1/k+1) = [I - K \cdot H] \cdot F \cdot x(k/k) + K \cdot z(k) \tag{6}$$

2 Noise Covariance Matrices

Kalman filter assumes the knowledge of all Kalman filter parameters, i.e. the matrices F, H, Q, R are known. In the case where the noise covariances are unknown, the identification of the noise covariances of the Kalman filter is discussed in [21]. Kalman filter statistics (Q, R) tuning is discussed in [22]. In fact, R can be estimated by computing the covariance of measurements, but Q cannot be easily estimated, due to the fact that a) the state is not measured directly and b) the state noise covariance functions as a “waste basket” for unknown modeling errors. As explained in [23], if we choose a too small Q, then the Kalman filter will converge too slowly, while if we choose a too large Q, then P_p will become large, and the filter becomes over-sensitive, [23]. The idea proposed in [23] is to make Q so large that it just about matches the effects of the measurement noise covariance R:

$$H \cdot P_p \cdot H^T = c \cdot R \tag{7}$$

where c is a scalar positive tuning factor.

Then, an acceptable choice for P_p is:

$$P_p = c \cdot H^+ \cdot R \cdot (H^+)^T \tag{8}$$

where M^+ denotes the Moore-Penrose pseudoinverse of M.

Note that it is required that the output matrix H is full rank [23]; if not, then it the can be replaced by a proper output matrix by using the observability matrix [23].

Thus, the desired Q is derived:

$$Q = P_p - \frac{1}{1+c} \cdot F \cdot P_p \cdot F^T \tag{9}$$

The noise covariances norm ratio is the ratio defined by the state noise covariance norm divided by the measurement noise covariance norm:

$$\lambda = \frac{\|Q\|_F}{\|R\|_F} \tag{10}$$

where the subscript F indicates the Frobenius norm.

In [23] it is depicted that if the ratio λ is known, then the tuning factor c is derived:

$$c = \lambda \cdot \|H\|^2 \tag{11}$$

where $\|H\|$ is the largest singular value of H [23].

Then, obviously, the desired Q can be derived via the ratio λ:

$$Q = P_p - \frac{1}{1+\lambda \cdot \|H\|^2} \cdot F \cdot P_p \cdot F^T \tag{12}$$

where

$$P_p = \lambda \cdot \|H\|^2 \cdot H^+ \cdot R \cdot (H^+)^T \tag{13}$$

3 SNR Definitions

The input signal-to-noise ratio (input SNR) and output signal-to-noise ratio (output SNR) are defined in [16]:

The input signal-to-noise ratio (input SNR) is:

$$H \cdot P_L \cdot H^T = r_{in} \cdot R \tag{14}$$

The output signal-to-noise ratio (output SNR) is:

$$H \cdot P_L \cdot H^T = r_{out} \cdot H \cdot P_e \cdot H^T \tag{15}$$

where the following signal-to-noise ratio improvement property holds:

$$r_{out} \geq r_{in} \quad (16)$$

The I/O SNR is the ratio defined by the output SNR divided by the input SNR:

$$r = \frac{r_{in}}{r_{out}} \quad (17)$$

Obviously,

$$r \leq 1 \quad (18)$$

4 Relation between SNR and Noise Covariances

In the following, the relation between the I/O SNR r and the noise covariances norm ratio λ is established.

From (14) and (15) we get:

$$r_{out} \cdot H \cdot P_e \cdot H^T = r_{in} \cdot R$$

From (4) and (7) we get:

$$H \cdot P_e \cdot H^T = H \cdot \left(P_p^{-1} + H^T \cdot R^{-1} \cdot H\right)^{-1} \cdot H^T = \frac{c}{1+c} \cdot R$$

Then, we have:

$$r_{out} \cdot \frac{c}{1+c} \cdot R = r_{in} \cdot R$$

and using (17) we get:

$$r = \frac{c}{1+c} \quad (19)$$

or

$$c = \frac{r}{1-r} \quad (20)$$

Finally, using (11) we get the relation between r and λ:

$$r = \frac{\lambda \cdot \|H\|^2}{1+\lambda \cdot \|H\|^2} \quad (21)$$

or

$$\lambda = \frac{r}{1-r} \cdot \frac{1}{\|H\|^2} \quad (22)$$

5 Noise Covariances Estimation

5.1 State Noise Covariance Estimation

When the state noise covariance Q is unknown, we are able to estimate it via the I/O SNR. In fact, we rewrite (12) and (13) using (22). Then, the desired Q can be derived via the ratio r:

$$Q = P_p - (1-r) \cdot F \cdot P_p \cdot F^T \quad (23)$$

where

$$P_p = \frac{r}{1-r} \cdot H^+ \cdot R \cdot (H^+)^T \quad (24)$$

5.2 Measurement Noise Covariance Estimation

When the state noise covariance R is unknown, we are able to estimate it via the I/O SNR. In fact, we rewrite (23) as:

$$P_p = Q + \left(\sqrt{1-r} \cdot F\right) \cdot P_p \cdot \left(\sqrt{1-r} \cdot F\right)^T \quad (25)$$

The solution of this Lyapunov equation depends on the known parameters F, Q, r. It is worth to note that a proper selection of r is prerequisite for the existence of the unique solution of this Lyapunov equation.

Then, the desired R can be derived via the ratio r by rewriting (7) as:

$$R = \frac{1-r}{r} \cdot H \cdot P_p \cdot H^T \quad (26)$$

6 Application in Steady-State Kalman Filter

In both the above cases where the state or the measurement noise covariance is unknown, the steady-state Kalman filter gain (5) becomes [23]:

$$K = P_p \cdot H^T \cdot \left(H \cdot P_p \cdot H^T + R\right)^{-1} = H^+ \cdot H \cdot P_p \cdot H^T \cdot \left(H \cdot P_p \cdot H^T + R\right)^{-1}$$

Then, using (26) we get:

$$K = r \cdot H^+ \quad (27)$$

Hence, the steady-state Kalman filter becomes:

Steady-State Kalman Filter

$$x(k+1/k+1) = (1-r) \cdot F \cdot x(k/k) + r \cdot H^+ \cdot z(k)$$

It is obvious that the steady-state Kalman filter parameters depend on r.

The resulting steady-state Kalman filter is suboptimal [23], but it can be implemented without the a priori knowledge of the state or measurement noise covariance.

Note that in the time-invariant case, all the Kalman filter parameters constant. In the steady-state case, the Kalman filter gain in (27) is constant as well. The estimation error covariance and the prediction error covariance are also constant. The state estimation is derived by the steady-state Kalman filter equation. The state prediction is:

$$x(k+1/k) = F \cdot x(k/k) \quad (28)$$

7 Application in Time-Varying Kalman Filter

Consider the time-varying case, where all the Kalman filter parameters are time-varying, i.e. $F(k+1,k), H(k), Q(k), R(k)$.

Then we use the time-varying noise covariances norm ratio $\lambda(k)$:

$$\lambda(k) = \frac{\|Q(k)\|_F}{\|R(k)\|_F} \quad (29)$$

Following the ideas in [23], we define the time-varying scalar positive tuning factor $c(k)$:

$$H(k) \cdot P(k/k-1) \cdot H^T(k) = c(k) \cdot R(k) \quad (30)$$

with

$$c(k) = \lambda(k) \cdot \|H(k)\|^2 \quad (31)$$

Then, the Kalman filter gain becomes:

$$K(k) = \frac{c(k)}{1+c(k)} \cdot H^+ \quad (32)$$

since

$$K(k) = P(k/k-1) \cdot H^T(k) \cdot \left(H(k) \cdot P(k/k-1) \cdot H^T(k) + R(k)\right)^{-1}$$

$$H(k) \cdot K(k) = H(k) \cdot P(k/k-1) \cdot H^T(k) \cdot \left(H(k) \cdot P(k/k-1) \cdot H^T(k) + R(k)\right)^{-1}$$

$$H(k) \cdot K(k) = c(k) \cdot R(k) \cdot (c(k) \cdot R(k) + R(k))^{-1}$$

$$H(k) \cdot K(k) = \frac{c(k)}{1+c(k)}$$

The estimation and the estimation error covariance matrix are:

$$x(k/k) = [I - K(k) \cdot H(k)] \cdot x(k/k-1) + K(k) \cdot z(k) \quad (33)$$

$$P(k/k) = [I - K(k) \cdot H(k)] \cdot P(k/k-1) \quad (34)$$

The prediction and the prediction error covariance matrix are:

$$x(k+1/k) = F \cdot x(k/k) \quad (35)$$

$$P(k+1/k) = Q(k) + F(k+1,k) \cdot P(k/k) + F^T(k+1,k) \quad (36)$$

or

$$P(k+1/k) = c(k) \cdot H^+(k) \cdot R(k) \cdot \left(H^+(k)\right)^T \quad (37)$$

Finally, we define the time-varying factor $r(k)$:

$$r(k) = \frac{c(k)}{1+c(k)} \quad (38)$$

Of course

$$c(k) = \frac{r(k)}{1-r(k)} \quad (39)$$

Then, from (32) and (39) we get:

$$r(k) = \frac{\lambda(k) \cdot \|H(k)\|^2}{1+\lambda(k) \cdot \|H(k)\|^2} \quad (40)$$

Thus, we are able to use the time-varying noise covariances norm ratio $\lambda(k)$ in order to derive the time-varying Kalman filter:

Time-Varying Kalman Filter

$$K(k) = r(k) \cdot H^+(k)$$
$$x(k/k) = \left(1 - r(k)\right) \cdot x(k/k-1) + K(k) \cdot z(k)$$
$$P(k/k) = \left(1 - r(k)\right) \cdot P(k/k-1)$$
$$x(k+1/k) = F(k+1,k) \cdot x(k/k)$$
$$P(k+1/k) = \frac{r(k)}{1-r(k)} \cdot H^+(k) \cdot R(k) \cdot \left(H^+(k)\right)^T$$

8 Conclusions

Kalman filters are successfully used for solving filtering problems in many different areas of science and engineering. Especially in the field of electrical engineering and electric controls, Kalman filters are an integral part of many states of the art of electric controls.

However, the practical implementation of Kalman filters often presents difficulties due to the ignorance of noise covariances. In this paper, the relation between the I/O SNR and the noise covariances norm ratio for the discrete-time steady-state Kalman filter has been determined and it is shown that when the state or measurement noise covariance is unknown, it can be tuned via the I/O SNR.

This result can be applied in time-varying systems and in steady-state systems, without the a priori knowledge of the state or measurement noise covariance. The impact of this result on Kalman filtering, combined with AI techniques to estimate the noise covariances, can be to derive reliable estimates, in the absence of noise covariances.

Future work includes solving real-world electrical engineering and electronic problems using the proposed approach and investigating the extension of the proposed method in nonlinear prediction and estimation applications.

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