Well-posedness of the Optimal Control Problem Related to Degenerate Chemo-attraction Models

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Abstract: This paper delves into the mathematical analysis of optimal control for a nonlinear degenerate chemotaxis model with volume-filling effects. The control is applied in a bilinear form specifically within the chemical equation. We establish the well-posedness (existence and uniqueness) of the weak solution for the direct problem using the Faedo Galerkin method (for existence), and the duality method (for uniqueness). Additionally, we demonstrate the existence of minimizers and establish first-order necessary conditions for the adjoint problem. The main novelty of this work concerns the degeneracy of the diffusive term and the presence of control over the concentration in our nonlinear degenerate chemotaxis model. Furthermore, the state, consisting of cell density and chemical concentration, remains in a weak setting, which is uncommon in the literature for solving optimal control problems involving chemotaxis models.

Key-Words: Chemotaxis-model, degenerate parabolic problem, Existence, Optimal Control, Lagrange multipliers

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1 Introduction

Chemotaxis, the directed movement of organisms in response to chemical gradients, is crucial in various biological processes like embryogenesis, immunology, cancer growth and wound healing. It allows organisms to find nutrients, avoid predators or locate mates. For instance, cellular slime molds move towards higher chemical concentrations secreted by amoebae, while bacteria swim towards areas with more oxygen. In pancreatic cancer therapy, iodine acts as a chemoattractant to destroy cancer cells.

Mathematical models of chemotaxis are extensively studied to understand and predict these biological processes. Among these models, degenerate nonlinear chemotaxis systems have gained attention due to their irregularity and the difficult proof of the global existence and the uniqueness of weak solutions. Various works, particularly those by [1], have contributed to proving the global existence under certain assumptions, often employing methods like the duality method used in [2]. We are particularly interested in a control on this problem, where we manipulate chemical concentrations to influence cell behavior.

Bilinear control means that the control serves as a coefficient for a reaction term depends linearly on the state. The control in this system governs the injection or extraction of a chemical substance within a specified subdomain $\Omega_c \subset \Omega$. Here, Ω represents a bounded domain in $\mathbb{R}^2$ with a smooth boundary $\partial\Omega$ of class C^2. Specifically, in this paper, we are concerned with the study of an optimal control system arising in chemotaxis as follows: Let $(0,T)$ be a time interval, where $0 < T < +\infty$, and define $Q_T = (0,T) \times \Omega$, the control problem can be described by the following system in Q_T:

$$\begin{cases} \partial_t N - \nabla \cdot \big(a(N)\nabla N\big) + \nabla \cdot \big(\chi(N)\nabla C\big) = 0, \\ \partial_t C - \nabla \cdot (\nabla C) = \alpha N - \beta C + fC1_{\Omega_c}. \end{cases} \tag{1}$$

The homogeneous Neumann boundary conditions over $\Sigma_t = (0,T) \times \partial\Omega$ are

$$a(N)\nabla N \cdot \eta = 0, \nabla C \cdot \eta = 0, \tag{2}$$

where η is the exterior unit normal to $\partial\Omega$. The initial conditions on Ω are given by,

$$N(0,x) = N_0(x),\ C(0,x) = C_0(x). \tag{3}$$

In the model above, $f : Q_c := (0,T) \times \Omega_c \to \mathbb{R}$ is the control with $\Omega_c \subset \Omega \subset \mathrm{R}^d$ the control domain, where $d = 2$ or 3. The density of cells and the concentration of chemicals are denoted respectively by N and C. The function $a(N)$ represents a non-linear degenerate density-dependent diffusion coefficient. The function $\chi(N)$ determines the sensitivity of cells to be attracted or repulsed. Finally, the rates of production and degradation are denoted respectively by α and β.

Moreover, optimal theory provides well-designed strategies to determine an optimal control over chemoattractant concentration to achieve desired cell density and concentration, particularly in cancer treatment by example, decreasing cancer cell density while minimizing effects on healthy tissues This is achieved by minimization of a quadratic problem (defined in section 3) where the cost function measures the errors between the cell density, the concentration and the desired given states; the cost of the control is also minimized.
Many studies had significant attention to optimal control problems governed by interconnected partial differential equations. Notably, [3], discussed the optimal control of a fully parabolic attraction-repulsion chemotaxis model with a logistic source in a two-dimensional setting, while [4], conducted a thorough investigation into an optimal control problem for a haptotaxis model of solid tumor infiltration, exploring the use of multiple cancer treatments. Recently, [5], analyzed a non-degenerate parabolic chemo-repulsion model with nonlinear production in two-dimensional domains.
The novelty of our study lies in extending the concept of optimal control to our degenerate parabolic chemo-attractant problem, where the challenge stems from the strong degeneracy and non-linearity of the diffusive term. This degeneracy not only complicates the existence of weak solutions for the direct problem but also poses challenges in constructing the adjoint problem using Lagrange multipliers. In our work, we consider that the set of constraints is given by the concept of weak solutions with energy inequalities. This set of admissible constraints is an unusual and new concept in the literature for solving optimal control problems involving chemotaxis models.

The structure of our paper is organized as follows: In Section 2, we present the main results and establish the well-posedness (existence and uniqueness) of our nonlinear degenerate chemo-attractant problem using the Faedo-Galerkin method. Section 3 is dedicated to the optimal control problem. Here, we introduce our optimal control approach, define a functional useful for minimization, establish the existence of the control, and derive the adjoint-state problem. Finally, employing the technique of regular points, we demonstrate the existence of Lagrange multipliers, forming a solution to this adjoint problem.

2 Setting of the Problem

First, let us introduce the notations that we will need later. We assume that the chemotactic sensitivity $\chi(N)$ vanishes when $N \geq N_m$ and that $\chi(0) = 0$. This condition, known as the volume-filling effect, has a straightforward biological interpretation. Upon normalization, we can assume that the threshold density is $N_m = 1$.
Moreover, the main data assumptions are:

$$a : [0,1] \longmapsto \mathbb{R}^+,\ a \in \mathcal{C}^1([0,1]), \tag{4}$$

where $a(0) = a(1) = 0, a(s) > 0$ for $0 < s < 1$,

$$\chi : [0,1] \longmapsto \mathbb{R},\ \chi \in \mathcal{C}^1([0,1]) \text{ and } \chi(0) = \chi(1) = 0 \tag{5}$$

and

$$f \in L^\infty(Q_T). \tag{6}$$

We will prove the existence and the uniqueness of a weak solution of (1)-(3) along with its continuous dependency on the control f.
This involves initially defining a weak solution for the system (1)-(3).

Definition 2.1. *Assume that* $0 \leq N_0 \leq 1$, $C_0 \geq 0$ *and* $C_0 \in L^\infty(\Omega)$. *A pair* (N,C) *is said to be a* **weak solution** *of* (1)-(3) *if*

$$\begin{aligned}
&0 \leq N(t,x) \leq 1,\ C(t,x) \geq 0 \textit{ a.e. in } Q_T,\\
&N \in C_w(0,T;L^2(\Omega)), \quad \partial_t N \in L^2(0,T;(H^1(\Omega))'),\\
&A(N) := \int_0^N a(r)dr \in L^2(0,T;H^1(\Omega)),\\
&C \in L^\infty(Q_T) \cap L^2(0,T;H^1(\Omega)) \cap C(0,T;L^2(\Omega)),\\
&\partial_t C \in L^2(0,T;(H^1(\Omega))')
\end{aligned}$$

and (N,C) satisfy

$$\begin{aligned}
\int_0^T &< \partial_t N, \psi_1 >_{(H^1)',H^1} dt\\
&+ \iint_{Q_T} a(N)\nabla N \cdot \nabla \psi_1 \, dxdt\\
&- \iint_{Q_T} \chi(N)\nabla C \cdot \nabla \psi_1 \, dxdt = 0,
\end{aligned} \tag{7}$$

$$\int_0^T < \partial_t C, \psi_2 >_{(H^1)',H^1} dt + \iint_{Q_T} \nabla C \cdot \nabla \psi_2 \, dxdt$$
$$= \iint_{Q_T} [\alpha N - \beta C + fC1_{\Omega_c}]\psi_2 \, dxdt \,, \tag{8}$$

for all ψ_1 and $\psi_2 \in L^2(0,T;H^1(\Omega))$, where $C_w(0,T;L^2(\Omega))$ denotes the space of continuous functions with values in $L^2(\Omega)$ endowed with the weak topology, and <.,.> denotes the duality pairing between $H^1(\Omega)$ and $(H^1(\Omega))'$.

The next theorems state the existence and the uniqueness of the weak solution.

Theorem 2.2. *Under the assumptions* (4) *to* (6)*, if $0 \le N_0 \le 1$, $C_0 \ge 0$ almost everywhere in Ω, and $C_0 \in L^\infty(\Omega)$, then System* (1)-(3) *possesses a global weak solution (N,C) as defined in Definition 2.1.*

Theorem 2.3. *The weak solution (N,C) of* (1)-(3) *is unique under the following assumption: there exists $K_0 \ge 0$ such that $\forall N_1, N_2 \in [0,1]$,*

$$(\chi(N_1)-\chi(N_2))^2 \le K_0(N_1-N_2)(A(N_1)-A(N_2)).$$

Outline of the proofs of the theorems 2.2 and 2.3: To prove Theorem 2.3, one can maintain the same necessary estimates of ([1], [2], [4], [5], [6], [7]) and choose a positive δ such that $\delta < \min(\frac{1}{K_0}, \frac{1}{c_\chi^2+\alpha^2})$.

For the proof of Theorem 2.2, we note that a major difficulty for the analysis of the first equation of (1) is the strong degeneracy of the diffusion term. To solve this problem, we modify the original diffusion term $a(N)$ by adding a small positive value ε to create $a_\varepsilon = a(N) + \varepsilon$. This adjustment allows us to consider the following non-degenerate system:

$$\begin{cases} \partial_t N_\varepsilon - \nabla \cdot (a_\varepsilon(N_\varepsilon)\nabla N_\varepsilon) + \nabla \cdot (\chi(N_\varepsilon)\nabla C_\varepsilon) = 0, \\ \partial_t C_\varepsilon - \Delta C_\varepsilon = \alpha N_\varepsilon - \beta C_\varepsilon + fC_\varepsilon 1_{\Omega_c}, \end{cases} \tag{9}$$

with the following conditions:

$$\begin{cases} a(N_\varepsilon)\nabla N_\varepsilon \cdot \eta = 0, \; \nabla C_\varepsilon \cdot \eta = 0 & \text{on } \Sigma_T \\ N_\varepsilon(0,x) = N_0(x), \; C_\varepsilon(0,x) = C_0(x) & \text{in } \Omega. \end{cases} \tag{10}$$

To establish the existence of a weak solution to the non-degenerate problem (9)-(10) in the sense of the definition 2.1, we use the Faedo-Galerkin method while the discrete maximum principle holds. Convergence is achieved using a priori estimates and compactness arguments.

The details of the Faedo-Galerkin method, the maximum principle and the convergence will be given in the next sections.

2.1 The Faedo-Galerkin Solution

In this subsection, we will give the proof of the existence of a weak solution of the non-degenerate problem (9)-(10) by the Faedo Galerkin method. We consider an appropriate spectral problem introduced in [8], [9], where the eigenfunctions $e_l(x)$ form an orthogonal basis in $H^1(\Omega)$ and an orthonormal basis in $L^2(\Omega)$.

Our goal is to find finite-dimensional approximations to the solutions of system (9)-(10) in the form of sequences $\{N_{n,\varepsilon}\}_{n>1}$ and $\{C_{n,\varepsilon}\}_{n>1}$, defined for $s \ge 0$ and $x \in \bar{\Omega}$ as follows:

$$N_{n,\varepsilon}(t,x) = \sum_{l=1}^{n} q_{n,l}(t)e_l(x)$$
$$C_{n,\varepsilon}(t,x) = \sum_{l=1}^{n} d_{n,l}(t)e_l(x).$$

It's important to note that this solution satisfies the necessary boundary conditions.

Next, we need to determine the coefficients $\{q_{n,l}(t)\}_{l=1}^n$ and $\{d_{n,l}(t)\}_{l=1}^n$ such that for $l = 1, \dots, n$, the following equations hold:

$$< \partial_t N_{n,\varepsilon}, \, e_l > + \int_\Omega a_\varepsilon(N_{n,\varepsilon})\nabla N_{n,\varepsilon} \cdot \nabla e_l \, dx$$
$$- \int_\Omega \chi(N_{n,\varepsilon})\nabla C_{n,\varepsilon} \cdot \nabla e_l \, dx = 0, \tag{11}$$

$$< \partial_t C_{n,\varepsilon} \, e_l > + \int_\Omega \nabla C_{n,\varepsilon} \cdot \nabla e_l \, dx$$
$$= \int_\Omega (\alpha N_{n,\varepsilon} - \beta C_{n,\varepsilon})e_l \, dx + \int_\Omega fC_{n,\varepsilon}1_{\Omega_c}e_l \, dx, \tag{12}$$

along with the initial conditions:

$$\begin{cases} N_{n,\varepsilon}(0,x) = N_0, \\ C_{n,\varepsilon}(0,x) = C_0. \end{cases}$$

Next, we use the orthonormality of the basis and hence the above equations can be rewritten in the following form:

$$\begin{aligned} q'_{n,l}(t) &= - \int_\Omega a_\varepsilon(N_{n,\varepsilon})\nabla N_{n,\varepsilon} \cdot \nabla e_l \, dx \\ &+ \int_\Omega \chi(N_{n,\varepsilon})\nabla C_{n,\varepsilon} \cdot \nabla e_l \, dx \\ &=: E_1^l(t, \{q_{n,l}\}_{l=1}^n, \{d_{n,l}\}_{l=1}^n) \end{aligned} \tag{13}$$

$$\begin{aligned} d'_{n,l}(t) = &-\int_\Omega \nabla C_{n,\varepsilon}\cdot\nabla e_l\,dx \\ &+\int_\Omega (\alpha N_{n,\varepsilon}-\beta C_{n,\varepsilon})e_l\,dx \\ &+\int_\Omega f C_{n,\varepsilon} 1_{\Omega_c} e_l\,dx \\ =:&\, E_2^l(t,\{q_{n,l}\}_{l=1}^n,\{d_{n,l}\}_{l=1}^n). \end{aligned} \tag{14}$$

Let $t' \in (0,T)$ and set $X=[0,t']$. Choose $R>0$ large enough so that the ball $B_R \subset \mathbb{R}^N$ contains $\{q_{n,l}(0)\}$ and $\{d_{n,l}(0)\}$, and set $Y=\bar{B}_R$. The components $E_i^l, i=1,2$, can be bounded on $X\times Y$: $|E_i^l| \le K(\varepsilon,R,n)$, $i=1,2$ where $K(\varepsilon,R,n)>0$ depends only on ε,R,n. Due to the concept of Carathéodory functions and the standard ODE theory, we can conclude the existence of absolutely continuous functions $\{q_{n,l}\}_{l=1}^n$ and $\{d_{n,l}\}_{l=1}^n$ for almost everywhere $s\in[0,t')$; $t'>0$. This proves that the sequences $(N_{n,\varepsilon},C_{n,\varepsilon})$ are well defined and the approximate solutions of (9)-(10) exist locally on $[0,t')$. Furthermore, the local solution constructed above can be extended to the entire time interval $[0,T)$ (see, [10]).

Next, we define $\phi_{i,n}(t,x)=\sum_{l=1}^n b_{i,n,l}(s)e_l(x)$ for $i=1,2$, where the coefficients $\{b_{i,n,l}\}$, for $i=1,2$, are absolutely continuous functions. Then, the approximate solution satisfies the weak formulation:

$$\begin{aligned} &< \partial_t N_{n,\varepsilon},\ \phi_{1,n} > + \int_\Omega a_\varepsilon(N)\nabla N_{n,\varepsilon}\cdot\nabla\phi_{1,n}\,dx \\ &+\int_\Omega \chi(N_{n,\varepsilon})\nabla C_{n,\varepsilon}\cdot\nabla\phi_{1,n}\,dx = 0, \end{aligned} \tag{15}$$

$$\begin{aligned} &< \partial_t C_{n,\varepsilon}\ \phi_{2,n} > + \int_\Omega \nabla C_{n,\varepsilon}\cdot\nabla\phi_{2,n}\,dx \\ &= \int_\Omega (\alpha N_{n,\varepsilon}-\beta C_{n,\varepsilon})\phi_{2,n}\,dx \\ &\quad + \int_\Omega f C_{n,\varepsilon} 1_{\Omega_c}\phi_{2,n}\,dx. \end{aligned} \tag{16}$$

2.2 Maximum Principle

In this section, we prove that the approximate solution of the non-degenerate problem (9) -(10) satisfies the following maximum principle.

Lemma 2.4. *Assume that* $0 \le N_0 \le 1$ *and* $C_0 \ge 0$ *, then the approximate solution* $(N_{n,\varepsilon},C_{n,\varepsilon})$ *satisfy*

$$0 \le N_{n,\varepsilon} \le 1 \text{ and } C_{n,\varepsilon} \ge 0 \text{ for a.e } (t,x) \in Q_T.$$

Sketch of proof: To begin with, we denote by $N^- = \max(-N,0)$ and $N^+ = \max(N,0)$. To establish the non-negativity of $N_{n,\varepsilon}$, we multiply (15) by $-N^-_{n,\varepsilon}$ and then introduce the continuous and Lipschitz extension $\tilde{\chi}$ of χ on $\mathbb{R}$ such that

$$\tilde{\chi}(s) = \begin{cases} 0 & \text{if } s<0 \\ \chi(s) & \text{if } 0\le s\le 1 \\ 0 & \text{if } s>1, \end{cases}$$

to obtain $\chi(N_{n,\varepsilon})=0$. This leads us to

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |N^-_{n,\varepsilon}|^2\,dx \le 0.$$

Finally, utilizing the fact that N_0 is non-negative, we deduce that $N^-_{n,\varepsilon}=0$ for almost every $(t,x)\in Q_T$. Therefore $N_{n,\varepsilon}\ge 0$. Similarly to demonstrate that $N_{n,\varepsilon}\le 1$, we follow the same steps by taking $(N_{n,\varepsilon}-1)^+$ as a test function.

Similarly, to demonstrate $C_{n,\varepsilon}\ge 0$, when f is negative, we proceed with the same approach. However, when f is positive, for technical reasons, we extend the function $F(C,f) := fC1_\Omega$ to ensure measurability on Ω_T and continuity with respect to C. This is achieved by defining

$$\tilde{F}(C,f) = \begin{cases} F(C,f) & \text{if } C \geqslant 0, \\ 0 & \text{else.} \end{cases}$$

Therefore, it follows form (16) that $C_{n,\varepsilon}$ satisfy

$$\begin{aligned} &< \partial_t C_{n,\varepsilon}\ \phi_{2,n} > + \int_\Omega \nabla C_{n,\varepsilon}\cdot\nabla\phi_{2,n}\,dx \\ &= \int_\Omega \alpha N_{n,\varepsilon}\phi_{2,n}\,dx + \int_\Omega \tilde{F}(C^+_{n,\varepsilon},f)\phi_{2,n}\,dx. \end{aligned} \tag{17}$$

To complete the proof, we proceed similarly as before, taking $-C^-_{n,\varepsilon}$ as a test function, and utilizing the non-negativity of C_0. This completes the proof of the lemma 2.4. ☐

2.3 Convergence (Passing to the Limit with n)

Finally, to ensure the existence of a weak solution for the non-degenerate system (9)-(10), we still need to pass to the limit with n. Thus, we take $N_{n,\varepsilon}$ and $C_{n,\varepsilon}$ as a test function respectively in (15) and (16). Then by integrating over $(0,T)$ and employing Gronwall's inequalities, we establish that

$$N_{n,\varepsilon} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$$

and

$$C_{n,\varepsilon} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)).$$

Additionally, we can easily demonstrate that $\| (\partial_t N_{n,\varepsilon},\partial_t C_{n,\varepsilon}) \|_{L^2(0,T,H^{-1}(\Omega))} \le A$, where the constant A is independent of n.

Therefore, as $n\to+\infty$, we observe the following weak convergences:

- $(N_{n,\varepsilon}, C_{n,\varepsilon}) \rightharpoonup (N_\varepsilon, N_\varepsilon)$ weakly* in $L^\infty(0,T;L^2(\Omega))$,
- $(N_{n,\varepsilon}, C_{n,\varepsilon}) \rightharpoonup (N_\varepsilon, C_\varepsilon)$ weakly in $L^2(0,T;H^1(\Omega))$,
- $(\partial_t N_{n,\varepsilon}, \partial_t C_{n,\varepsilon}) \rightharpoonup (\partial_t N_\varepsilon, \partial_t C_\varepsilon)$ weakly in $L^2(0,T;H^{-1}(\Omega))$.

From these, we can deduce easily the weak convergence of $N_{n,\varepsilon}$ and $C_{n,\varepsilon}$ to N_ε and C_ε respectively, satisfying the definition of a weak solution of the non-degenerate system.

Next, we can prove that the sequence of approximate solutions $(N_\varepsilon, C_\varepsilon)$ converges to a weak solution (N, C) of (1)-(3) in the sense of definition 2.1, when ε tends to zero by following the same guidelines of [1].

In the absence of the control term f, numerical simulations were done in [7], [11], [12], using the finite element method, and in [6], using the finite volume method.
In our paper, we focus only on the theoretical study of the optimal model.

3 Optimal Control Problem

This section focuses on establishing the existence of optimal control. For that, we introduce the Lagrangian functions and construct the adjoint problem. Additionally, we will prove the existence of a weak solution to the adjoint problem.
Let us consider

$$f \in \mathcal{F} = \{L^\infty(Q_T), \text{ such that } \forall (t,x) \in Q_T, \\ f_{\min} \le f(t,x) \le f_{\max}\}.$$

We also have $0 \le N_0 \le 1$ and $C_0 \in L^\infty(\Omega)$ with $C_0 \ge 0$ in Ω. The function f describes the bilinear control acting on the equation of the concentration.

We start first by proving the existence of a solution for the following optimal control problem:

$$\begin{cases} \text{Find } (N,C,f) \in \mathcal{W} \times \mathcal{X} \times \mathcal{F} \text{ minimizing} \\ \text{the functional} \\ J(N,C,f) := \dfrac{\beta_1}{2}\displaystyle\int_0^T \|N(t)-N_d(t)\|^2_{L^2(\Omega)}\,dt \\ \qquad + \dfrac{\beta_2}{2}\displaystyle\int_0^T \|C(t)-C_d(t)\|^2_{L^2(\Omega)}\,dt \\ \qquad + \dfrac{\gamma_f}{2}\displaystyle\int_0^T \|f(t)\|^2_{L^2(\Omega_c)}\,dt, \\ \text{subject to } (N,C) \text{ be a weak solution} \\ \text{of the PDE system (1)-(3). in the sense of} \\ \text{Definition (2.1) and } f \in \mathcal{F}, \end{cases} \tag{18}$$

where

$$\mathcal{W} = \{N \in C_w(0,T;L^2(\Omega)) \\ \text{and } \partial_t N \in L^2(0,T;(H^1(\Omega))')\}, \tag{19}$$

$$\mathcal{X} = \{C \in L^\infty(Q_T) \cap L^2(0,T;H^1(\Omega)) \\ \cap C(0,T;L^2(\Omega)) \text{ and } \partial_t C \in L^2(0,T;(H^1(\Omega))')\}. \tag{20}$$

Here $(N_d, C_d) \in L^2(Q_T)^2$ represent the target states, and the non-negative numbers β_2, β_2 and γ_f measure the cost of the states and control, respectively.
We define the set of admissible solutions of (18) as

$$\mathcal{S}_{ad} = \{s = (N,C,f) \in \mathcal{W} \times \mathcal{X} \times \mathcal{F}, \text{ where} \\ s \text{ is a weak solution of } (1)-(3) \text{ in } Q_T\}.$$

3.1 Existence of the Control

In this section, we prove the existence of a solution to the control problem of (18).

Theorem 3.1. *Assume that* $0 \le N_0 \le 1$*,* $C_0 \in L^\infty(\Omega), N_d \in L^2(Q_T)$*,* $C_d \in L^2(Q_T)$ *and* $f \in \mathcal{F}$*. Then, there exists at least one solution of the optimal control problem* (18).

Proof. Theorem (2.2) ensures that $\mathcal{S}_{ad}$ is non-empty. With the non-negativity of the cost functional, J has a greatest lower bound, leading to a minimizing sequence $\{s_n\}_{n\in\mathbb{N}} \subset \mathcal{S}_{ad}$ with $\lim\limits_{n\to+\infty} J(s_n) = \inf\limits_{s\in\mathcal{S}_{ad}} J(s)$.
By the definition of J, $(f_n)_n$ is bounded in $L^2(Q_T)$, indicating the convergence of f_n weakly to f^*. As $\mathcal{F}$ is closed, $f^* \in \mathcal{F}$. Let (N_n, C_n) be weak solutions to (1)-(3) with f_n. Now, as in section 2, we can deduce the convergence of N_n and C_n to a weak solution (N^*, C^*) of (1)-(3). This implies that $s^* = (N^*, C^*, f^*)$ is a weak solution of the system where $s^* \in \mathcal{S}_{ad}$.
Finally, since J is weakly lower semi-continuous, then we conclude that

$$J(N^*, C^*, f^*) \leqslant \liminf_{n\to\infty} J(N_n, C_n, f_n) \\ \leqslant \inf_{s\in\mathcal{S}_{ad}} J(s) \leqslant J(N^*, C^*, f^*).$$

This establishes the existence of an optimal control solution (18). □

3.2 Optimality Condition and Dual Problem

We will now study the first-order optimality conditions necessary for a local optimal solution (N, C, f) of problem (18).
First, we consider the following (generic) optimization problem:

$$\min_{s\in\mathbb{M}} J(s) \text{ subject to } G(s) = 0, \tag{21}$$

where $J : \mathrm{X} \to \mathbb{R}$ is a functional, $G : \mathrm{X} \to \mathrm{Y}$ is an operator, X and Y are Banach spaces, and M is a nonempty closed and convex subset of X.

Definition 3.2. *-([[13], chapter 6] (Lagrange multiplier) Let $\bar{s} \in \mathcal{S}$ be a local optimal solution for problem* (21). *Suppose that J and G are Fréchet differentiable in $\bar{s}$. Then, any $\xi \in \mathrm{Y}'$ is called a Lagrange multiplier for* (21) *at the point $\bar{s}$ if for all $r \in \mathcal{C}(\bar{s})$, we have*

$$J'(\bar{s})[r] - \langle \xi, G'(\bar{s})[r] \rangle \geq 0,$$

where $\mathcal{C}(\bar{s}) = \{\theta(s - \bar{s}) : s \in \mathbb{M}, \theta \geq 0\}$ is the conical hull of $\bar{s}$ in $\mathbb{M}$.

Definition 3.3. *Let $\bar{s} \in \mathcal{S}$ be a local optimal solution for problem* (21). *we say that $\bar{s}$ is a regular point if*

$$G'(\bar{s})[\mathcal{C}(\bar{s})] = \mathrm{Y}.$$

Theorem 3.4. *([14], Theorem 3.1) Let $\bar{s} \in \mathcal{S}$ be a local optimal solution for problem* (21). *Suppose that J is Fréchet differentiable in $\bar{s}$, and G is continuous Fréchet-differentiable in $\bar{s}$. If $\bar{s}$ is a regular point, then the set of Lagrange multipliers for* (21) *at $\bar{s}$ is nonempty.*

Now, we will reformulate the optimal control problem (18) in the abstract setting (21).
We define the following Banach spaces:

$$\begin{aligned} X &:= \mathcal{W} \times \mathcal{X} \times L^2(Q_c), \\ Y &:= L^2(Q_T) \times L^2\left(0, T; (H^1(\Omega))'\right) \times L^2(\Omega) \\ &\quad \times L^2(\Omega). \end{aligned}$$

The operator $G = (G_1, G_2, G_3, G_4) : X \to Y$, where

$$\begin{aligned} &G_1 : X \to L^2(Q_T), \\ &G_2 : X \to L^2(0, T; (H^1(\Omega))'), \\ &G_3 : X \to L^2(\Omega), \\ &G_4 : X \to L^2(\Omega) \end{aligned}$$

are defined at each point $s = (N, C, f) \in X$ by

$$\begin{aligned} < G_1(s), \varphi_1 > =& < \partial_t N, \varphi_1 >_{L^2(H^1), L^2((H^1)')} \\ &+ (a(N)\nabla N, \nabla\varphi_1)_{L^2(Q_T)} \\ &- (\chi(N)\nabla C \cdot \nabla\varphi_1)_{L^2(Q_T)}, \end{aligned}$$

$$\begin{aligned} < G_2(s), \varphi_2 > =& < \partial_t C, \varphi_2 >_{L^2(H^1), L^2((H^1)')} \\ &+ (\nabla C, \nabla\varphi_2)_{L^2(Q_T)} \\ &- (\alpha N - \beta C + f C 1_{\Omega_c}, \varphi_2)_{L^2(Q_T)}, \end{aligned}$$

and G_3 and G_4 satisfy respectively the constraints on the initial conditions of $N(t, x)$ and $C(t, x)$ as follows:

$$G_3(s) = N(0) - N_0 \quad \text{and} \quad G_4(s) = C(0) - C_0.$$

Now, we take $\mathbb{M} = \mathcal{W} \times \mathcal{X} \times \mathcal{F}$ (a closed convex subset of X). Thus, our optimal control problem (18) is reformulated as follows:

$$\min_{s \in \mathbb{M}} J(s) \text{ subject to } G(s) = 0. \tag{22}$$

Regarding the differentiability of the functional J and the constraint operator G, we use a direct calculation similar to [15], [16], to obtain the following results.

Lemma 3.5. *The functional $J : X \to \mathbb{R}$ is Fréchet differentiable and its derivative at $s = (N, C, f) \in X$ in the direction $r = (U, V, F) \in X$ is given by*

$$\begin{aligned} J'(s)[r] =& \beta_1 \int_0^T \int_\Omega (N - N_d)\, U \; dxdt \\ &+ \beta_2 \int_0^T \int_\Omega (C - C_d)\, V \; dxdt \\ &+ \gamma_f \int_0^T \int_{\Omega_c} f F \; dxdt. \end{aligned}$$

Lemma 3.6. *The operator $G : X \to Y$ is continuous Fréchet differentiable and its derivative at $s = (N, C, f) \in X$ in the direction $r = (U, V, F) \in X$ is the linear operator*

$$G'(\bar{s})[r] = \left(G_1'(\bar{s})[r], G_2'(\bar{s})[r], G_3'(\bar{s})[r], G_4'(s)[r]\right)$$

defined by

$$\begin{aligned} < G_1'(s)[r], \varphi_1 >=& < \partial_t U, \varphi_1 > + (a(N)\nabla U, \nabla\varphi_1) \\ &+ (a'(N)U\nabla N, \nabla\varphi_1) - (\chi(N)\nabla V, \nabla\varphi_1) \\ &- (\chi'(N)U\nabla C, \nabla\varphi_1), \end{aligned}$$

$$\begin{aligned} < G_2'(s)[r], \varphi_2 >=& < \partial_t V, \varphi_2 > + (\nabla V, \nabla\varphi_2) \\ &- (\alpha U, \varphi_2) + (\beta V, \varphi_2) \\ &- (fV + FC, \varphi_2), \end{aligned}$$

and

$$G_3'(s)[r] = U(0), \quad G_4'(s)[r] = V(0).$$

Next, we aim to prove the existence of Lagrange multipliers.

3.2.1 Existence of Lagrange Multipliers

In this section, we prove the existence of Lagrange Multipliers under the assumptions:

$$\nabla N \in L^\infty(Q_T), \tag{23}$$

$$a(N) \text{ vanishes at a finite number of points in } Q_T. \tag{24}$$

Furthermore, the existence is guaranteed if a local optimal solution of problem (22) is a regular point of operator G. Therefore, we first need to prove the following Lemma:

Lemma 3.7. *Let $a'(N) \leqslant ca(N)$ and $\chi'(N) \leq Ka(N)$ for a positive constant c and K, and suppose that* (23) *and* (24) *hold. If $s = (N, C, f) \in \mathcal{S}_{ad} = \{s = (N, C, f) \in \mathbb{M} : G(s) = 0\}$, then s is a regular point.*

Proof. For a given $(N, C, f) \in \mathcal{S}_{ad}$, let $(g_N, g_C, U_0, V_0) \in Y$. As 0 is in $\mathcal{C}(\bar{f})$, we only need to prove the existence of (U, V) that solves the following problem:
$\forall \varphi_1, \varphi_2 \in L^2(H^1)$

$$\begin{cases} < \partial_t U, \varphi_1 > +(a(N)\nabla U, \nabla\varphi_1) \\ \quad + (a'(N)U\nabla N, \nabla\varphi_1) - (\chi(N)\nabla V, \nabla\varphi_1) \\ \quad - (\chi'(N)U\nabla C, \nabla\varphi_1) =< g_N, \varphi_1 >, \\ < \partial_t V, \varphi_2 > +(\nabla V, \nabla\varphi_2) - (\alpha U, \varphi_2) \\ \quad + (\beta V, \varphi_2) - (fV, \varphi_2) =< g_C, \varphi_2 >, \\ U(0) = U_0, \quad V(0) = V_0. \end{cases} \tag{25}$$

First, we note that despite the linearity of the problem, the degeneracy due to the term $a(N)$ adds a layer of difficulty to the analysis. Therefore, as discussed in Section 2 of this article, we replace $a(N)$ with $a(N) + \varepsilon$ and define the following non-degenerate problem:

$$\begin{aligned} &< \partial_t U_\varepsilon, \varphi_1 > +(a_\varepsilon(N)\nabla U_\varepsilon, \nabla\varphi_1) \\ &\quad + (a'(N)U_\varepsilon\nabla N, \nabla\varphi_1) - (\chi(N)\nabla V_\varepsilon, \nabla\varphi_1) \\ &\quad - (\chi'(N)U_\varepsilon\nabla C, \nabla\varphi_1) =< g_N, \varphi_1 >, \end{aligned} \tag{26}$$

$$\begin{aligned} &< \partial_t V_\varepsilon, \varphi_2 > +(\nabla V_\varepsilon, \nabla\varphi_2) - (\alpha U_\varepsilon, \varphi_2) \\ &\quad + (\beta V_\varepsilon, \varphi_2) - (fV_\varepsilon, \varphi_2) =< g_C, \varphi_2 >, \end{aligned} \tag{27}$$

$$U_\varepsilon(0) = U_0, \quad V_\varepsilon(0) = V(0). \tag{28}$$

We can establish the existence of a solution to the non-degenerate problem by following the same steps as in Section 2, employing the Faedo-Galerkin method. Now, we still need to tend ε to zero.
Taking V_ε as a test function in (27), multiplying (26) by δ and then taking U_ε as a test function in (26), we apply Young's inequality to each equation, using the assumptions (23)-(24), and $a'(N) \leqslant ca(N)$ and $\chi(N) \leqslant ka(N)$, for positive constants c and K. After that, we sum these two equations together and choose $\delta \geq 0$ such that
$1 - 2\delta K^2\|a(N)\|_{L^\infty} \geq 0$.
Finally by applying Gronwall's inequality, we conclude that $(U_\varepsilon, V_\varepsilon)$ satisfies:

$$U_\varepsilon \in L^\infty\left(0, T; L^2(\Omega)\right), \tag{29}$$

$$V_\varepsilon \in L^2(0, T; H^1(\Omega)), \tag{30}$$

$$\sqrt{\varepsilon}\nabla U_\varepsilon \in L^2\left(0, T; L^2(\Omega)\right), \tag{31}$$

$$\sqrt{a(N)}\nabla U_\varepsilon \in L^2\left(0, T; L^2(\Omega)\right), \tag{32}$$

and are bounded in these spaces independently of ε.
Moreover, one can show that

$$\|(\partial_t U_\varepsilon, \partial_t V_\varepsilon)\|_{L^2(0,T,H^{-1}(\Omega))} \leq A,$$

where the constant A is independent of ε.
Thus, there exist a solution (U, V) and a subsequence of $(U_\varepsilon, V_\varepsilon)$ still denotes as the sequence such that, as ε goes to 0,

$$U_\varepsilon \rightharpoonup U \text{ weakly* in } L^\infty\left(0, T; L^2(\Omega)\right), \tag{33}$$

$$V_\varepsilon \rightharpoonup V \text{ weakly in } L^2(0, T; H^1(\Omega)), \tag{34}$$

$$\varepsilon\nabla U_\varepsilon \rightharpoonup 0 \text{ weakly in } L^2(Q_T), \tag{35}$$

$$\sqrt{a(N)}\nabla U_\varepsilon \rightharpoonup \xi \text{ weakly in } L^2(Q_T), \tag{36}$$

$$\partial_t U_\varepsilon \rightharpoonup \partial_t U_\varepsilon \text{ weakly in } L^2\left(0, T; (H^1(\Omega))'\right), \tag{37}$$

$$\partial_t V_\varepsilon \rightharpoonup \partial_t V_\varepsilon \text{ weakly in } L^2\left(0, T; (H^1(\Omega))'\right). \tag{38}$$

We still have to show that $\xi = \sqrt{a(N)}\nabla N$ a.e. on Q_T. To do this, we use the assumption (24) and follow the guidelines presented in [17].
With the above convergence, we can easily show that (U, V) is a solution to the problem (25). □

Now, using Lemma 3.7 and Definition 3.3, we can conclude the following result about the existence of Lagrange multipliers.

Theorem 3.8. *Let $s = (N, C, f) \in \mathcal{S}_{ad}$ be a local optimal solution for the control problem* (22) *and suppose that there exist positive constants c and K such that $a'(N) \leqslant ca(N)$ and $\chi(N) \leqslant Ka(N)$. If the assumptions* (23) *and* (24) *hold, then all for all $(U, V, F) \in W \times X \times \mathcal{C}(f)$, there exists a Lagrange multiplier $(\lambda, \eta, \phi, \psi) \in L^2(Q_T) \times L^2(0, T, H^1(\Omega)) \times L^2(Q_T) \times (H^1(\Omega))'$ such that*

$$
\begin{aligned}
&\beta_1 \int_0^T \int_\Omega (N - N_d)\, U \, dxdt \\
&+ \beta_2 \int_0^T \int_\Omega (C - C_d)\, V \, dxdt \\
&+ \gamma_f \int_0^T \int_{\Omega_c} fF \, dxdt - \int_0^T \langle \partial_t U, \lambda \rangle \, dt \\
&+ \int_0^T \int_\Omega (-a(N)\nabla U + \chi(N)\nabla V) \cdot \nabla\lambda \, dxdt \\
&+ \int_0^T \int_\Omega (-a'(N)U\nabla N + \chi'(N)U\nabla C) \cdot \nabla\lambda \, dxdt \\
&- \int_0^T < \partial_t C, \eta > \, dt - \int_0^T \int_\Omega \nabla C \cdot \nabla\eta \, dxdt \\
&+ \int_0^T \int_\Omega (\alpha U - \beta V + fV 1_{\Omega_c})\eta \, dxdt \\
&+ \int_0^T \int_\Omega FC1_{\Omega_c}\eta \, dxdt - \int_\Omega U(0)\phi \, dx \\
&- \int_\Omega V(0)\psi \, dx \geq 0.
\end{aligned}
$$

Remark 3.9. *From the previous theorem 3.8, we obtain an optimality system, for which we can consider the following spaces:*

$$
\mathcal{W}_{u_0} := \{u \in \mathcal{W} : u(0) = 0\} \quad \textit{and} \\
\mathcal{X}_{v_0} := \{v \in \mathcal{X} : v(0) = 0\}.
$$

Corollary 3.10. *Let $s = (N, C, f)$ represent a local optimal solution for the optimal control problem* (22) *and suppose that there exist positive constants c and K such that $a'(N) \leqslant ca(N)$ and $\chi(N) \leqslant Ka(N)$. If the assumptions* (23) *and* (24) *hold, then any Lagrange multiplier $(\lambda, \eta) \in L^2(Q_T) \times L^2(0, T; H^1(\Omega))$ provided by Theorem 3.8 , satisfies the following system :*
$\forall\, U \in \mathcal{W}_{u_0}$ and $\forall\, V \in \mathcal{X}_{v_0}$,

$$
\begin{aligned}
&\int_0^T \langle \partial_t U, \lambda \rangle \, dt + \int_0^T \int_\Omega a(N)\nabla U \cdot \nabla\lambda \, dxdt \\
&+ \int_0^T \int_\Omega (a'(N)U\nabla N - \chi'(N)U\nabla C) \cdot \nabla\lambda \, dxdt \\
&- \int_0^T \int_\Omega \alpha U\eta \, dxdt \qquad (39) \\
&= \beta_1 \int_0^T \int_{\Omega_d} (N - N_d)\, U \, dxdt,
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^T \langle \partial_t C, \lambda \rangle \, dt + \int_0^T \int_\Omega \nabla C \cdot \nabla\lambda \, dxdt \\
&- \int_0^T \int_\Omega \chi(N)\nabla V \cdot \nabla\lambda \, dxdt + \int_0^T \int_\Omega \beta U\eta \, dxdt \\
&- \int_0^T \int_{\Omega_C} fV\eta \, dxdt. \qquad (40) \\
&= \beta_2 \int_0^T \int_\Omega (C - C_d)\, V \, dxdt,
\end{aligned}
$$

and the optimality condition: $\forall \bar{f} \in \mathcal{F}$,

$$
\int_0^T \int_{\Omega_\varepsilon} (\gamma_f f + C\eta)\,(\bar{f} - f) dxdt \geq 0. \qquad (41)
$$

Outline of proof: To get the equations (39) and (40) satisfied by λ and η, we set $(V, F) = (0, 0)$, and since $\mathcal{W}_{u_0}$ is a vector space, we obtain the first equation (39). Similarly, by setting $(U, F) = (0, 0)$ and considering that $\mathcal{X}_{V_0}$ is a vector space, we derive the second equation (40).
Now, to obtain the optimality condition, we take $(U, V) = (0, 0)$ and choose $F = \bar{f} - f \in \mathcal{C}(\bar{f})$ for all $\tilde{f} \in \mathcal{F}$.

Remark 3.11. *A pair (λ, η) satisfying first equation* (39) *and second equation*(40) *corresponds to the concept of a weak solution of the problem*

$$
\begin{cases}
-\partial_t\lambda - \nabla\cdot(a(N)\nabla\lambda) + a'(N)\nabla N \cdot \nabla\lambda \\
\quad - \chi'(N)\nabla C \cdot \nabla\lambda - \alpha\eta = \beta_1 (N - N_d), \textit{in } Q_T \\
-\partial_t\eta - \Delta\eta + \beta\eta + \nabla\cdot(\chi(N)\nabla\lambda) - f\eta 1_{\Omega_c}, \\
\quad = \beta_2 (C - C_d), \textit{ in } Q_T \\
\lambda(T) = 0, \quad \eta(T) = 0 \quad \textit{in } \Omega, \\
a(N)\nabla\lambda \cdot \vec{n} = 0, (\nabla\eta - \chi(N)\nabla\lambda) \cdot \vec{n} = 0, \\
\qquad \textit{on } \Sigma_T.
\end{cases}
$$

4 Conclusion

In this paper, we addressed an optimal control problem for the concentration in a non-linear parabolic system with a two-sidedly degenerate equation and volume-filling effects equation. We provided a rigorous analysis of the mathematical model, proposing an optimal control strategy to reduce cancer cell density while minimizing the impact on healthy cells in the body, and this model can also help to reduce pattern formation. We demonstrated that the direct control problem is well-posed. Additionally, we established the characterization of the global optimal solution. By imposing regularity conditions and utilizing the technique of Fréchet derivatives, we give a rigorous mathematical justification of the existence of the adjoint problem and demonstrate the existence of Lagrange multipliers within the admissible constraint set as weak solutions.

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