Robust Decentralized Controller Design for Descriptor-type Systems with Distributed Time Delay

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Abstract: - Robust decentralized controller design is considered for linear time-invariant large-scale descriptor-type systems with distributed time delay which are composed of overlapping subsystems. A robustness bound, that accounts for the interactions among the subsystems and for modeling uncertainties both in the subsystem models and the interactions, is derived using overlapping decompositions and expansions. A robust decentralized controller design approach using this bound is then proposed. Once the robustness bound is derived, the proposed approach is decoupled for each subsystem and, for each subsystem, it is based on a local nominal model, which is also derived using overlapping decompositions and expansions. Satisfying a simple condition, involving the derived robustness bound, however, guarantees the robust stability of the overall actual closed-loop system.

Key-Words: - Robust controller design, Decentralized control, Time-delay systems, Descriptor-type systems, Distributed time delay, Large-scale systems, Overlapping decompositions

Received: April 12, 2023. Revised: February 7, 2024. Accepted: March 19, 2024. Published: April 18, 2024.

1 Introduction

Many practical systems, especially large-scale systems [1] may be subject to time delays [2]. Such systems, which are typically named as *time-delay systems* [3], can be described by delay-differential equations [4]. For some time-delay systems, such as telerobotic systems [5], however, delay-differential equations must be coupled with delay-algebraic equations to describe the dynamics of the system. Such systems are known as *descriptor-type time-delay* systems. Descriptor-type time-delay systems impose a challenge since their response may be discontinuous and even impulsive [6].

Time delays in a system may be pointwise or distributed [7]. Distributed time delay may appear in many application areas, such as neural networks [8], biology [9], traffic flow [10], logistics [11], and combustion control [12].

There are many examples of large-scale systems, such as interconnected power systems [13], freeway traffic regulation systems [14], intelligent vehicle-highway systems [15], and data-communication networks [16], which consist of subsystems with overlapping dynamics. To analyze and design controllers for such systems, the approach of *overlapping decompositions* was first introduced in [17]. Although, initially, this approach was introduced for linear time-invariant (LTI) delay-free systems, since then it has also been extended to time-delay systems with both pointwise and distributed time delays [18], [19], [20], [21].

In general, it is not possible to obtain an exact model of any practical system. Therefore, any designed controller for a practical system must be *robust* against modeling uncertainties [22]. To ensure such a robustness, a *robustness bound*, which accounts for modeling uncertainties in centralized descriptor-type time-delay systems were considered in [23]. In the case of large-scale systems, however, it may be necessary or practical to ignore the interactions among subsystems during controller design [24]. In this case, besides modeling uncertainties, the designed controllers must also be robust against the neglected interactions. Therefore, in the present work, we consider large-scale LTI descriptor-type distributed-time-delay systems consisting of subsystems which dynamically overlap and extend the robustness bound also to account for the neglected interactions between the subsystems and any modeling uncertainties in the interactions. We then propose a robust decentralized controller design approach using this bound. Once the robustness bound is derived, the proposed approach is decoupled for each subsystem and, for each subsystem, it is based on a local nominal model, which is also derived using overlapping decompositions and expansions. Satisfying a simple condition, involving the derived robustness bound, however, guarantees the robust stability of the overall actual closed-loop system.

Throughout the paper, $\mathbf{R}$ and $\mathbf{C}$ respectively denote the sets of real and complex numbers. For $s \in \mathbf{C}$, $\mathrm{Re}(s)$ denotes the real part of s. For positive integers k and l, $\mathbf{R}^k$ and $\mathbf{R}^{k\times l}$ denote the spaces of, respectively, k-dimensional real vectors and $k \times l$-dimensional real matrices. I_k denotes the $k \times k$-dimensional identity matrix and I denotes the identity matrix of appropriate dimensions. 0 may denote either the scalar zero, a zero matrix, or a matrix function which is identically zero. $\det(\cdot)$, $\mathrm{rank}(\cdot)$, $\bar{\sigma}(\cdot)$, and $\underline{\sigma}(\cdot)$ respectively denote determinant, the rank, the maximum singular value, and the minimum singular value of the indicated matrix. $\mathrm{bdiag}(\cdots)$ denotes a block diagonal matrix with the indicated matrices on the main diagonal. For a matrix function $M(\cdot) : [-\tau, 0] \to \mathbf{R}^{k\times l}$, $\|M\| := \int_{-\tau}^{0} \bar{\sigma}(M(\theta))\, d\theta$. Finally, j denotes the imaginary unit (i.e., $j := \sqrt{-1}$).

2 Problem Statement

In this work, as mentioned in the introduction, we consider descriptor-type large-scale LTI distributed-time-delay systems, which are composed of overlapping subsystems. In practice, such subsystems may overlap in many different ways. For simplicity of presentation, however, in here we consider the case of only two subsystems which are interconnected through another dynamic subsystem, which form the overlapping part. Such a system can compactly be described as:

$$E\dot{x}(t) = A^0 x(t) + \int_{-\tau}^{0} \left(A(\theta)x(t+\theta) + B(\theta)u(t+\theta)\right) d\theta \quad (1)$$

$$y(t) = \int_{-\tau}^{0} C(\theta)x(t+\theta)d\theta \quad (2)$$

where $\tau > 0$ is the maximum time-delay in the system and t is the time variable. The state, the input, and the output vectors are decomposed as:

$$x = \begin{bmatrix} x_1 \\ x_c \\ x_2 \end{bmatrix},\ u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},\ y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad (3)$$

where $x_i \in \mathbf{R}^{n_i}$, $u_i \in \mathbf{R}^{p_i}$, and $y_i \in \mathbf{R}^{q_i}$ are, respectively, the state, the input, and the output vectors of the i^{th} subsystem ($i = 1, 2$), and $x_c \in \mathbf{R}^{n_c}$ is the state vector of the overlapping part. The matrices $E \in \mathbf{R}^{n\times n}$, $A^0 \in \mathbf{R}^{n\times n}$, and the matrix functions $A(\cdot) : [-\tau, 0] \to \mathbf{R}^{n\times n}$, $B(\cdot) : [-\tau, 0] \to \mathbf{R}^{n\times p}$, and $C(\cdot) : [-\tau, 0] \to \mathbf{R}^{q\times n}$ (where $n := n_1 + n_c + n_2$, $p := p_1 + p_2$, and $q := q_1 + q_2$) are also decomposed as:

$$E = \begin{bmatrix} E_1 & 0 & 0 \\ 0 & E_c & 0 \\ 0 & 0 & E_2 \end{bmatrix},\ A^0 = \begin{bmatrix} A_1^0 & 0 & 0 \\ 0 & A_c^0 & 0 \\ 0 & 0 & A_2^0 \end{bmatrix},$$

$$A(\cdot) = \begin{bmatrix} A_1(\cdot) & A_{1c}(\cdot) & 0 \\ A_{c1}(\cdot) & A_c(\cdot) & A_{c2}(\cdot) \\ 0 & A_{2c}(\cdot) & A_2(\cdot) \end{bmatrix},$$

$$B(\cdot) = \begin{bmatrix} B_1(\cdot) & 0 \\ 0 & 0 \\ 0 & B_2(\cdot) \end{bmatrix},$$

and

$$C(\cdot) = \begin{bmatrix} C_1(\cdot) & 0 & 0 \\ 0 & 0 & C_2(\cdot) \end{bmatrix},$$

where the partitionings are compatible with those in (3). All the submatrix functions shown above are assumed to be bounded, except that they are allowed to include Dirac-delta terms, $\delta(\theta + h)$, $h \in [0, \tau]$, except that $A(0)$ is assumed to be bounded (i.e., each submatrix function of $A(\theta)$ can include a term $\delta(\theta+h)$, $h \in (0, \tau]$, but not $\delta(\theta)$). Inclusion of these terms allow representation of pointwise time delays, together with distributed time delay. Furthermore, it is assumed that the submatrix functions of $A(\cdot)$ and $B(\cdot)$ are subject to uncertainties. Thus, representing any one of these submatrix functions by $M_\bullet(\cdot)$, we assume that $M_\bullet(\cdot) = M_\bullet^n(\cdot) + M_\bullet^u(\cdot)$, where $M_\bullet^n(\cdot)$ is the known nominal part and $M_\bullet^u(\cdot)$ is the unknown uncertain part. The uncertain parts, however, are assumed to be bounded as follows:

$$\begin{array}{c} \|A_1^u\| \le \alpha_1,\quad \|A_{1c}^u\| \le \alpha_{1c},\quad \|A_{c1}^u\| \le \alpha_{c1}, \\ \|A_c^u\| \le \alpha_c,\quad \|A_{c2}^u\| \le \alpha_{c2},\quad \|A_{2c}^u\| \le \alpha_{2c}, \\ \|A_2^u\| \le \alpha_2,\quad \|B_1^u\| \le \beta_1,\quad \|B_2^u\| \le \beta_2, \end{array} \quad (4)$$

where α's and β's are known non-negative bounds. Since the input-output uncertainties can in general be represented either at the input or at the output, here we assume that all such uncertainties are represented at the input, hence there are no uncertainties in $C(\cdot)$.

It is also assumed that $\operatorname{rank}(E_\bullet) = \bar{n}_\bullet \leq n_\bullet$, where $\bullet$ stands for 1, c, or 2, with $\bar{n}_\bullet < n_\bullet$, for at least one of 1, c, or 2. This assumption means that $\operatorname{rank}(E) < n$, which means that the system (1)–(2) is a descriptor-type system [6]. It is, however, asuumed that:

Assumption 1: If $\bar{n}_\bullet < n_\bullet$, then $\operatorname{rank}(\mathcal{L}_\bullet A^0_\bullet \mathcal{R}_\bullet) = n_\bullet - \bar{n}_\bullet$, where $\mathcal{L}_\bullet \in \mathbf{R}^{n_\bullet - \bar{n}_\bullet \times n_\bullet}$ and $\mathcal{R}_\bullet \in \mathbf{R}^{n_\bullet \times n_\bullet - \bar{n}_\bullet}$ are such that the rows of $\mathcal{L}_\bullet$ span the left null space of $E_\bullet$ and the columns of $\mathcal{R}_\bullet$ span the right null space of $E_\bullet$.

The above assumption implies that there exists a unique solution to (1), for any suitable initial condition, $x(t+\theta)$, $\theta \in [-\tau, 0]$ [4].

The *characteristic function* of the system (1) is given by

$$\psi(s) := \det(sE - \mathcal{A}(s)) \ , \tag{5}$$

where

$$\mathcal{A}(s) := A^0 + \int_{-\tau}^{0} A(\theta) e^{s\theta} d\theta \ . \tag{6}$$

The *modes* of the system (1) are the roots of $\psi(s) = 0$. It is known that the system (1) has infinitely many modes, in general [25]. However, under Assumption 1, it has only finitely many modes with real part greater than

$$\nu_f := \sup\{\operatorname{Re}(s) \mid \det(\bar{\mathcal{A}}(s)) = 0\} \ , \tag{7}$$

where $\bar{\mathcal{A}}(s) := \mathcal{L}\mathcal{A}(s)\mathcal{R}$, where $\mathcal{A}(s)$ is as given in (6),

$$\mathcal{L} := \operatorname{bdiag}(\mathcal{L}_1 \ , \ \mathcal{L}_c \ , \ \mathcal{L}_2) \tag{8}$$

and

$$\mathcal{R} := \operatorname{bdiag}(\mathcal{R}_1 \ , \ \mathcal{R}_c \ , \ \mathcal{R}_2) \ , \tag{9}$$

where $\mathcal{L}_\bullet$ and $\mathcal{R}_\bullet$ are as in Assumption 1, if $\bar{n}_\bullet < n_\bullet$, and they are missing in (8) and (9), otherwise [26].

It is known that (1)–(2) can not be stabilized by a finite-dimensional proper LTI controller unless $\nu_f < 0$ [27]. Thus, since our aim is to stabilize (1)–(2) for all uncertainties satisfying (4), we make the following assumption:

Assumption 2: $\nu_f < 0$ for any uncertainties satisfying (4).

This assumption implies that the system (1) has only finitely many unstable modes. A mode is called as an *unstable mode* if it has a non-negative real part.

Our aim is to design decentralized (possibly time-delay) LTI controllers:

$$U_i(s) = -K_i(s) Y_i(s) \ , \quad i = 1, 2 \ , \tag{10}$$

where $U_i(s)$ and $Y_i(s)$ are the Laplace transforms of, respectively, $u_i(t)$ and $y_i(t)$, and $K_i(s)$ is the transfer function matrix (TFM) of the i^{th} controller. These controllers are to be designed such that

- each local nominal closed-loop system is stable,
- a local performance criteria is satisfied for each local nominal closed-loop system, and
- the actual overall closed-loop system is robustly stable for all uncertainties that satisfy the bounds (4).

Furthermore, the design of each decentralized controller is to be based on a *local nominal model*, which is to be derived next.

3 Local Nominal Models

In order to obtain a local nominal model, we *expand* [21] the above overlappingly decomposed system by using the transformation

$$T = \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & I_{n_c} & 0 \\ 0 & I_{n_c} & 0 \\ 0 & 0 & I_{n_2} \end{bmatrix} \ . \tag{11}$$

The *expanded system* is described as:

$$\hat{E}\dot{\hat{x}}(t) = \hat{A}^0 \hat{x}(t) + \int_{-\tau}^{0} \Big(\hat{A}(\theta)\hat{x}(t+\theta) + \hat{B}(\theta)\hat{u}(t+\theta) \Big) d\theta \tag{12}$$

$$\hat{y}(t) = \int_{-\tau}^{0} \hat{C}(\theta)\hat{x}(t+\theta) d\theta \tag{13}$$

where

$$\hat{E} := \operatorname{bdiag}(\hat{E}_1, \hat{E}_2) \ ,$$

where

$$\hat{E}_1 := \operatorname{bdiag}(E_1, E_c) \ , \ \hat{E}_2 := \operatorname{bdiag}(E_c, E_2) \ ,$$

$$\hat{A}^0 := \operatorname{bdiag}(\hat{A}^0_1, \hat{A}^0_2) \ ,$$

where

$$\hat{A}^0_1 := \operatorname{bdiag}\left(A^0_1, A^0_c\right) \ , \ \hat{A}^0_2 := \operatorname{bdiag}\left(A^0_c, A^0_2\right) \ ,$$

$$\hat{A}(\theta) := \begin{bmatrix} \hat{A}_1(\theta) & \hat{A}_{12}(\theta) \\ \hat{A}_{21}(\theta) & \hat{A}_2(\theta) \end{bmatrix} ,$$

where

$$\hat{A}_1(\theta) := \begin{bmatrix} A_1(\theta) & A_{1c}(\theta) \\ A_{c1}(\theta) & A_c(\theta) \end{bmatrix} ,$$

$$\hat{A}_{12}(\theta) := \begin{bmatrix} 0 & 0 \\ 0 & A_{c2}(\theta) \end{bmatrix} ,$$

$$\hat{A}_{21}(\theta) := \begin{bmatrix} A_{c1}(\theta) & 0 \\ 0 & 0 \end{bmatrix} ,$$

and

$$\hat{A}_2(\theta) := \begin{bmatrix} A_c(\theta) & A_{c2}(\theta) \\ A_{2c}(\theta) & A_2(\theta) \end{bmatrix} ,$$

$$\hat{B}(\theta) := \text{bdiag}\left(\hat{B}_1(\theta), \hat{B}_2(\theta)\right) ,$$

where

$$\hat{B}_1(\theta) := \begin{bmatrix} B_1(\theta) \\ 0 \end{bmatrix} , \ \hat{B}_2(\theta) := \begin{bmatrix} 0 \\ B_2(\theta) \end{bmatrix} ,$$

and

$$\hat{C}(\theta) := \text{bdiag}\left(\hat{C}_1(\theta), \hat{C}_2(\theta)\right) ,$$

where,

$$\hat{C}_1(\theta) := \begin{bmatrix} C_1(\theta) & 0 \end{bmatrix} ,$$

and

$$\hat{C}_2(\theta) := \begin{bmatrix} 0 & C_2(\theta) \end{bmatrix} .$$

Note that relations $\hat{E}T = TE$, $\hat{A}^0 T = TA^0$, $\hat{A}(\theta)T = TA(\theta)$, $\hat{B}(\theta) = TB(\theta)$, and $\hat{C}(\theta)T = C(\theta)$ are satisfied. This implies that the *expanded system* (12)–(13) is an *extension* of the *original system* (1)–(2) [21]. Thus, the expanded system *includes* the original system, and hence the two systems have the same input-output map [20].

We note that, because of the block diagonal structure of $\hat{E}$ and of $\hat{A}^0$, Assumption 1 implies that a unique solution to (12), for any suitable initial condition, $\hat{x}(t+\theta)$, $\theta \in [-\tau, 0]$, is guaranteed. Here, we also make the following assumption:

Assumption 3:

$$\sup\left\{\text{Re}(s) \mid \det\left(\bar{\hat{\mathcal{A}}}(s)\right) = 0\right\} < 0 , \qquad (14)$$

for any uncertainties satisfying (4), where $\bar{\hat{\mathcal{A}}}(s) := \hat{\mathcal{L}}\hat{\mathcal{A}}(s)\hat{\mathcal{R}}$, where

$$\hat{\mathcal{A}}(s) := \hat{A}^0 + \int_{-\tau}^{0} \hat{A}(\theta)e^{s\theta}d\theta , \qquad (15)$$

$$\hat{\mathcal{L}} := \text{bdiag}(\mathcal{L}_1 \ , \ \mathcal{L}_c \ , \ \mathcal{L}_c \ , \ \mathcal{L}_2) , \qquad (16)$$

and

$$\hat{\mathcal{R}} := \text{bdiag}(\mathcal{R}_1 \ , \ \mathcal{R}_c \ , \ \mathcal{R}_c \ , \ \mathcal{R}_2) , \qquad (17)$$

where $\mathcal{L}_\bullet$ and $\mathcal{R}_\bullet$ are as in Assumption 1, if $\bar{n}_\bullet < n_\bullet$, and they are missing in (16) and (17), otherwise. We note that, since the expanded system includes the original system, Assumption 3, in particular, implies Assumption 2. Thus, under this assumption, both the expanded system and the original system have only finitely many unstable modes.

We note that the expanded system (12)–(13) is composed of two disjoint subsystems with only *weak* interactions (through $\hat{A}_{12}(\cdot)$ and $\hat{A}_{21}(\cdot)$). Thus, a *local model* can be obtained by ignoring these interactions as:

$$\hat{E}_i\dot{\hat{x}}_i(t) = \hat{A}_i^0\hat{x}_i(t) + \int_{-\tau}^{0} \Big(\hat{A}_i(\theta)\hat{x}_i(t+\theta) + \hat{B}_i(\theta)\hat{u}_i(t+\theta)\Big) d\theta \qquad (18)$$

$$\hat{y}_i(t) = \int_{-\tau}^{0} \hat{C}_i(\theta)\hat{x}_i(t+\theta)d\theta \qquad (19)$$

for $i = 1, 2$. A *local nominal model*, for $i = 1, 2$, can then be obtained by taking the nominal part of (18)–(19):

$$\hat{E}_i\dot{\hat{x}}_i(t) = \hat{A}_i^0\hat{x}_i(t) + \int_{-\tau}^{0} \Big(\hat{A}_i^n(\theta)\hat{x}_i(t+\theta) + \hat{B}_i^n(\theta)\hat{u}_i(t+\theta)\Big) d\theta \qquad (20)$$

$$\hat{y}_i(t) = \int_{-\tau}^{0} \hat{C}_i(\theta)\hat{x}_i(t+\theta)d\theta \qquad (21)$$

where

$$\hat{A}_1^n(\theta) := \begin{bmatrix} A_1^n(\theta) & A_{1c}^n(\theta) \\ A_{c1}^n(\theta) & A_c^n(\theta) \end{bmatrix} ,$$

$$\hat{A}_2^n(\theta) := \begin{bmatrix} A_c^n(\theta) & A_{c2}^n(\theta) \\ A_{2c}^n(\theta) & A_2^n(\theta) \end{bmatrix} ,$$

$$\hat{B}_1^n(\theta) := \begin{bmatrix} B_1^n(\theta) \\ 0 \end{bmatrix} ,$$

and

$$\hat{B}_2^n(\theta) := \begin{bmatrix} 0 \\ B_2^n(\theta) \end{bmatrix} .$$

Our final two assumptions are as follows:

Assumption 4: For $i = 1, 2$,

$$\sup\left\{\text{Re}(s) \mid \det\left(\bar{\hat{\mathcal{A}}}_i^n(s)\right) = 0\right\} < 0 , \qquad (22)$$

where $\bar{\hat{\mathcal{A}}}_i^n(s) := \hat{\mathcal{L}}_i \hat{\mathcal{A}}_i^n(s) \hat{\mathcal{R}}_i$, where

$$\hat{\mathcal{A}}_i^n(s) := \hat{A}_i^0 + \int_{-\tau}^{0} \hat{A}_i^n(\theta) e^{s\theta} d\theta \ , \tag{23}$$

$$\hat{\mathcal{L}}_1 := \mathrm{bdiag}(\mathcal{L}_1, \mathcal{L}_c) \ , \ \hat{\mathcal{L}}_2 := \mathrm{bdiag}(\mathcal{L}_c, \mathcal{L}_2) \ , \tag{24}$$

and

$$\hat{\mathcal{R}}_1 := \mathrm{bdiag}(\mathcal{R}_1, \mathcal{R}_c), \hat{\mathcal{R}}_2 := \mathrm{bdiag}(\mathcal{R}_c, \mathcal{R}_2), \tag{25}$$

where $\mathcal{L}_\bullet$ and $\mathcal{R}_\bullet$ are as in Assumption 3.

Assumption 5: $\hat{m} = m_1 + m_2$, where $\hat{m}$ denotes the number of unstable modes of the expanded system (12)–(13) (which is finite by Assumption 3) and m_i ($i = 1, 2$) denotes the number of unstable modes of the i^{th} local nominal model (20)–(21) (which is also finite by Assumption 4).

4 A Robustness Bound

In this section, we derive a robustness bound, to account for the

- uncertainties in the subsystem models,
- neglected interactions between the subsystem models, and
- uncertainties in the interactions between the subsystem models.

For this, from (20)–(21), we obtain the TFM of the i^{th} local nominal model ($i = 1, 2$) as:

$$\Gamma_i^n(s) = \hat{\mathcal{C}}_i(s) \left(s\hat{E}_i - \hat{\mathcal{A}}_i^n(s) \right)^{-1} \hat{\mathcal{B}}_i^n(s) \tag{26}$$

where $\hat{\mathcal{C}}_i(s) := \int_{-\tau}^{0} \hat{C}_i(\theta) e^{s\theta} d\theta$, $\hat{\mathcal{A}}_i^n(s)$ is defined in (23), and $\hat{\mathcal{B}}_i^n(s) := \int_{-\tau}^{0} \hat{B}_i^n(\theta) e^{s\theta} d\theta$. The TFM for the *overall design model* is then given as

$$\Gamma^n(s) = \mathrm{bdiag}\left(\Gamma_1^n(s), \Gamma_2^n(s)\right) \ . \tag{27}$$

On the other hand, since the expanded system (12)–(13) includes the original system (1)–(2), the TFM, $\Gamma(s)$, of the actual original system is equal to the TFM, $\hat{\Gamma}(s)$, of the expanded system:

$$\Gamma(s) = \hat{\Gamma}(s) = \hat{\mathcal{C}}(s) \left(s\hat{E} - \hat{\mathcal{A}}(s) \right)^{-1} \hat{\mathcal{B}}(s) \tag{28}$$

where $\hat{\mathcal{C}}(s) := \int_{-\tau}^{0} \hat{C}(\theta) e^{s\theta} d\theta$, $\hat{\mathcal{A}}(s)$ is defined in (15), and $\hat{\mathcal{B}}(s) := \int_{-\tau}^{0} \hat{B}(\theta) e^{s\theta} d\theta$.

Now, let $\Delta(s)$ be such that:

$$\Gamma(s) = \Gamma^n(s) \left(I + \Delta(s) \right) \ . \tag{29}$$

Then, we can use a frequency-dependent upper bound on the norm of $\Delta(j\omega)$ as a robustness bound. Such a bound can be obtained as follows:

Lemma 1: Suppose that $\delta_d(\omega) > 0$, $\forall \omega \in \mathbf{R}$. Then

$$\bar{\sigma}(\Delta(j\omega)) \le \delta(\omega) := \frac{\delta_n(\omega)}{\delta_d(\omega)} \ , \quad \forall \omega \in \mathbf{R} \ , \tag{30}$$

where

$$\delta_d(\omega) := \underline{\sigma}\left(H(j\omega)\right) - \max\left\{\delta_d^1(\omega), \delta_d^2(\omega)\right\} - \max\left\{\delta_d^{1,2}(\omega), \delta_d^{2,1}(\omega)\right\} \ , \tag{31}$$

where

$$H(s) := \begin{bmatrix} \hat{\mathcal{B}}_1^n(s) & H_{12}(s) \\ H_{21}(s) & \hat{\mathcal{B}}_2^n(s) \end{bmatrix} \ , \tag{32}$$

where

$$H_{12}(s) := \begin{bmatrix} 0 & 0 \\ 0 & -\int_{-\tau}^{0} A_{c2}^n(\theta) e^{s\theta} d\theta \end{bmatrix} G_2(s)$$

and

$$H_{21}(s) := \begin{bmatrix} -\int_{-\tau}^{0} A_{c1}^n(\theta) e^{s\theta} d\theta & 0 \\ 0 & 0 \end{bmatrix} G_1(s)$$

where, for $i = 1, 2$,

$$G_i(s) := \left(s\hat{E}_i - \hat{\mathcal{A}}_i^n(s) \right)^{-1} \hat{\mathcal{B}}_i^n(s) \ ,$$

and

$$\delta_d^1(\omega) := \hat{\alpha}_1 g_1(\omega) \ , \ \ \delta_d^{1,2}(\omega) := \alpha_{c2} g_2(\omega) \ ,$$
$$\delta_d^2(\omega) := \hat{\alpha}_2 g_2(\omega) \ , \ \ \delta_d^{2,1}(\omega) := \alpha_{c1} g_1(\omega) \ ,$$

where, for $i = 1, 2$, $g_i(\omega) := \bar{\sigma}\left(G_i(j\omega)\right)$ and $\hat{\alpha}_i := \max(\alpha_i, \alpha_c) + \max(\alpha_{ci}, \alpha_{ic})$, and

$$\delta_n(\omega) := \max\left\{\delta_n^1(\omega), \delta_n^2(\omega)\right\} + \max\left\{\delta_n^{1,2}(\omega), \delta_n^{2,1}(\omega)\right\} \ , \tag{33}$$

where

$$\delta_n^1(\omega) := \beta_1 + \hat{\alpha}_1 g_1(\omega) \ , \ \ \delta_n^2(\omega) := \beta_2 + \hat{\alpha}_2 g_2(\omega) \ ,$$
$$\delta_n^{1,2}(\omega) := \bar{\sigma}(H_{12}(j\omega)) + \alpha_{c2} g_2(\omega) \ ,$$

and

$$\delta_n^{2,1}(\omega) := \bar{\sigma}(H_{21}(j\omega)) + \alpha_{c1} g_1(\omega) \ .$$

Proof: $\Delta(s)$ in (29) can be chosen to satisfy

$$\left(s\hat{E} - \hat{\mathcal{A}}(s) \right)^{-1} \hat{\mathcal{B}}(s) = \left(s\hat{E} - \hat{\mathcal{A}}^n(s) \right)^{-1} \hat{\mathcal{B}}^n(s) \left(I + \Delta(s) \right) \ , \tag{34}$$

where $\hat{\mathcal{A}}^n(s) := \text{bdiag}\left(\hat{\mathcal{A}}_1^n(s), \hat{\mathcal{A}}_2^n(s)\right)$ and $\hat{\mathcal{B}}^n(s) := \text{bdiag}\left(\hat{\mathcal{B}}_1^n(s), \hat{\mathcal{B}}_2^n(s)\right)$. By premultiplying both sides of (34) by $(s\hat{E} - \hat{\mathcal{A}}(s))$ and rearranging terms, we obtain $N(s) = D(s)\Delta(s)$, where

$$D(s) := \left(s\hat{E} - \hat{\mathcal{A}}(s)\right) G(s)$$

and

$$N(s) := \hat{\mathcal{B}}(s) - D(s) \ ,$$

where $G(s) := \text{bdiag}(G_1(s), G_2(s))$. The result then follows by noting that $\bar{\sigma}(N(j\omega)) \leq \delta_n(\omega)$ and $\underline{\sigma}(D(j\omega)) \geq \delta_d(\omega)$. □

Remark: The bound (30) is not defined if $\delta_d(\omega) \leq 0$, i.e., if $\underline{\sigma}(H(j\omega)) \leq \max\{\delta_d^1(\omega), \delta_d^2(\omega)\} + \max\{\delta_d^{1,2}(\omega), \delta_d^{2,1}(\omega)\}$, for some $\omega \in \mathbf{R}$. However, note that the matrix $H(j\omega)$ contains the nominal input terms $\hat{\mathcal{B}}_i^n(j\omega)$ on the diagonal blocks (which typically have large norm for any ω) and interaction terms (which typically have smaller norm) on the off-diagonal blocks. Furthermore, the terms $\delta_d^{\bullet}(\omega)$ contain bounds on the uncertainties (which are typically small). Therefore, typically, $\underline{\sigma}(H(j\omega)) > \max\{\delta_d^1(\omega), \delta_d^2(\omega)\} + \max\{\delta_d^{1,2}(\omega), \delta_d^{2,1}(\omega)\}$, and hence $\delta_d(\omega) > 0$.

5 Controller Design

We now consider the problem of designing decentralized controllers (10) so that the following requirements are satisfied:

(i) each local nominal closed-loop system, (20)–(21) under the local controller $K_i(s)$, is stable,
(ii) a local performance criteria (if any) is satisfied for each local nominal closed-loop system, and
(iii) the actual overall closed-loop system, (1)–(2) under the overall decentralized controller (10) is robustly stable for all uncertainties that satisfy (4).

Requirements (i) and (ii) are local requirements which can be satisfied independently for each channel. To satisfy requirement (iii), we propose to design the i^{th} controller ($i = 1, 2$), $K_i(s)$, to satisfy

$$\bar{\sigma}\left(T_i^n(j\omega)\right) < \frac{1}{\delta(\omega)} \ , \quad \forall \omega \in \mathbf{R} \ , \qquad (35)$$

where $T_i^n(s) := \Gamma_i^n(s) K_i(s) \left[I + \Gamma_i^n(s) K_i(s)\right]^{-1}$ is the complementary sensitivity matrix for the i^{th} local nominal closed-loop system and $\delta(\omega)$ is given by (30). The following theorem shows that when requirement (i) is satisfied for each local nominal closed-loop system, satisfying (35) guarantees robust stability of the actual overall closed-loop system.

Theorem 1: Suppose that Assumptions 1, 3, 4, and 5 hold. Also suppose that each local nominal closed-loop system is stable and that (35) is satisfied. Then, the actual overall closed-loop system is robustly stable for all uncertainties that satisfy (4).

Proof: Assumption 1 is needed for the validity of the model (1)–(2). This also implies the validity of the models (12)–(13), (18)–(19), and (20)–(21). The block diagonal structure (27) of the overall design model, together with Assumption 4, implies that there are no modes of the overall design model approaching the imaginary axis from left and that the number of unstable modes of the overall design model is finite. Assumption 3, on the other hand, implies that there are no modes of the expanded open-loop system approaching the imaginary axis from left and that the number of unstable modes of the expanded open-loop system is finite. The complementary sensitivity matrix for the overall closed-loop design model is

$$T^n(s) = \Gamma^n(s) K(s) \left[I + \Gamma^n(s) K(s)\right]^{-1} = \text{bdiag}\left(T_1^n(s), T_2^n(s)\right) \qquad (36)$$

where $K(s) := \text{bdiag}(K_1(s), K_2(s))$ is the TFM for the overall controller. Due to the block diagonal structure of the overall closed-loop design model, stability of the local nominal closed-loop systems implies the stability of the overall closed-loop design model. The block diagonal structure of the overall design model, on the other hand, implies that the number of unstable modes of the overall design model is equal to the sum of the number of the unstable modes of the local nominal models. Then, by Assumption 5, the number of the unstable modes of the expanded system (12)–(13) is equal to the number of unstable modes of the overall design model. Furthermore, by (36),

$$\bar{\sigma}\left(T^n(j\omega)\right) = \max_{i \in \{1,2\}} \left\{\bar{\sigma}\left(T_i^n(j\omega)\right)\right\} \ .$$

Thus, when (35) is satisfied, by (30), we have

$$\bar{\sigma}\left(T^n(j\omega)\right) < \frac{1}{\bar{\sigma}\left(\Delta(j\omega)\right)} \ , \quad \forall \omega \in \mathbf{R} \ . \qquad (37)$$

This, however, implies that the actual expanded closed-loop system is robustly stable [22]. However, since the actual expanded system includes the actual original system, this implies the robust stability of the actual original closed-loop system [20]. □

Note that, once the robustness bound (30) is obtained, (35) is in fact a local requirement. This means that each local controller $K_i(s)$ can be designed independently of others based on the local nominal model (20)–(21). For this, any centralized controller design approach developed for descriptor-type LTI time-delay systems (e.g., [28], [29]) can be used.

6 Conclusions

Robust decentralized controller design has been considered for large-scale LTI descriptor-type systems with distributed time delay which are composed of overlapping subsystems. Using overlapping decompositions and expansions, a frequency dependent robustness bound, (30), has been derived. A robust decentralized controller design approach using this bound has then been proposed. Once the robustness bound is derived, the proposed approach is decoupled for each subsystem and, for each subsystem, it is based on a local nominal model. Satisfying a simple condition, (35), however, guarantees the robust stability of the overall actual closed-loop system. Since the derived bound is frequency dependent, the approach is, in general, less conservative than an approach in which a constant bound is used. Furthermore, it also allows frequency shaping [30].

Although, for simplicity of presentation, we have considered only the case of two overlapping subsystems, the proposed approach can be extended to such cases as N subsystems with a common overlapping part or a string of N overlapping subsystems.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The author contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding

This work has been supported by the Scientific Research Projects Commission of Eskişehir Technical University.

Conflicts of Interest

The author has no conflicts of interest to declare that are relevant to the content of this article.

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