# Fareeha Transform Performance In Solving Fractional Differential Telegraph Equations Combining Adomian Decomposition Method

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*Abstract:* Transformations have successfully outperformed a significant role in solving differential equations and have been applied in large-scale aspects of science. Fareeha transform has been illustrated effectively in data compression based on containing more information of the transform. In this paper, we expand the fractional Fareeha transform in the Caputo derivative sense combining the Adomian Decomposition Method to seek the solutions of fractional differential telegraph equations. The results of practical utilization have also been significantly shown successful in solving fractional telegraph differential equations.

*Key-Words:* Fareeha Transform, ADM method, Laplace-typed transform, Time-fractional telegraph equations.

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## 1 Introduction

Transformation is conveniently applied to perform solutions of differential equations with constant-coefficient equations, and usually combines other suited methods for the coefficients containing polynomial coefficients, for example, Laplace transform is a powerful technique applied in solving mathematics and physics equations, especially in mechanics, [1]. The application of Fourier and Laplace transformation is varied such as managing the transfer of electrical circuits, recognition systems, chemical components, and computer programs, [2]. In recent years, Fourier and Laplace transform has been used more in analyzing and controlling digital signal processing, [3]. The newfangled types of transforms produced based on Fourier and Laplace transforms to supply the needs of sciences have performed the reliable approaches to solve particular problems, for example, the Sumudu transform, [4]. Sumudu transformation has been used in combination with homotopy analysis to solve the delay Fractional differential Bagley-Torvik equation that does not need to calculate the fractional derivative or integration terms, easy and straightforward to get the exact solutions, [5]. The Sumudu transform created a new double integral Laplace-Sumudu transformation that has demonstrated efficiency in solving the nonhomogeneous linear and nonlinear partial differential equations, [6]. Another transform such as the ELzaki transform derived from the Fourier integral transform is a convenient mathematical tool for solving differential equations, [7]. Elzaki transform is effective enough to find the solution to the linear system expressed by an ordinary differential equation associated with initial or external disturbance condition, [8]. The association of ELzaki transforms and the homotopy perturbation method makes the Elzaki transform homotopy perturbation method (ETHPM) used in physics and engineering to solve linear and nonlinear differential equations, [9].

Aboodh transform illustrated the main properties to facilitate solving ordinary and partial differential equations, [10]. Aboodh transform connected the new iterative method to perform a new approximate analytical method to solve fractional differential equations in a biological population model, [11]. The new alpha-integral Laplace transform supported the general features of the Laplace transform for the classical sense definition and theory, [12]. HY integral transform, [13], is applied to solve Laguerre and Hermite differential equations by converting Lagoerre and Hermite differential equations to first-order differential equations. HY integral transform is useful in finding Newtons law of cooling, [14], and expanded for solving exponential growth and decay problems, [15].

Another variation, the Mohand transform, is created to efficiently apply in finding solutions of partial differential equations, [16]. The Mohand transform comes along with the Adomian decomposition method to make the new path to solve the nonlinear fractional evolution equations useful in mathematical models and engineering projects, [17]. Based on the mathematical Fourier integral transform, the Sawi transform was created providing the main properties to demonstrate the fundamental properties for solving differential equations, [18]. The establishment of the new transformation called the Kamal transform supported new applications for ordinary and partial differential equations, [19]. The duality relations make the Kamal transform visualize the full important sightseeing to other transformations, such as Laplace, LaplaceCarson, Aboodh, Sumudu, Elzaki, Mohand, and Sawi transforms, [20].

Attaining solutions of dynamical systems by using Laplace-Carson, fractional order associated with the terms in the differential equations were demonstrated to get the exact solution, [21]. Another performance of Laplace-Carson is to raise the development of the gradient flow of a viscoplastic medium, [22]. A Laplace-typed transform called G_Trasform established is applied to engineering problems, [23]. Anuj transforms illustrated new properties of linearity, scaling, translation, convolution, and application in solving ordinary differential equations, [21, 24]. The differential equations performance was significantly solved by using Laplace transform, [25], and ELzaki Transform, [26]. Another useful and effective application of the Anuj transform is to find the solution of Volterra integral equations, [27]. The general form of the transform can be seen in Table 1.

In this paper, the Fareeha transform will be extended for fractional differential derivative defined as the following, [28]:

$$\mathscr{G}_f(u(t)) = F(s^n) = \int_0^\infty u(t)e^{-s^n t}dt. \tag{1}$$

where $t>0, s^n>0, n=2k+1, k=0,1,2,\cdots$. The Fareeha transform has been built for the general form of some following specific cases shown in Table 2. The inverse Fareeha transform is defined as the following:

$$u(t) = \mathscr{G}_f^{-1}F(s^n). \tag{2}$$

Table 1: Demographic performance of recent transforms.

| Cases of transform $s^m\int_0^\infty u(t)e^{-s^n t}dt$ | Transform |
|---|---|
| $m=0, n=1$ | Laplace transform |
| $m=n=-1$ | Sumudu transform |
| $m=1, n=-1$ | Elzaki transform |
| $m=-1, n=1$ | Aboodh transform |
| $m=0, n=\alpha^{-1}$, $\alpha\in\mathbb{R}_0^+$ | $\alpha$-Integral Laplace-Transform |
| $m=1, n=2$ | HY integral transform |
| $m=2, n=1$ | Mohand transform |
| $m=-2, n=-1$ | Sawi transform |
| $m=0, n=-1$ | Kamal transform |
| $m=\alpha, n=-1$ | G_transform |
| $m=2, n=-1$ | Anuj transform |
| $m=1, n=1$ | LaplaceCarson transform |

In this paper, the Fareeha transform will be considered to solve the general fractional telegraph equation of the form

$$\mathscr{D}_t^\alpha u(x,t) + \mathscr{L}[u(x,t)] + \mathscr{N}[u(x,t)] = q(x,t). \tag{3}$$

subject to the initial and boundary conditions

$$u(x,0)=g(x), u_t(x,0)=h(x), 0<x<1, \quad t>0, 1<\alpha\le 2. \tag{4}$$

and $\mathscr{D}_t^\alpha u(x,t)$ represents Caputo derivative in terms of $t$ of order $\alpha$, $\mathscr{L}[u(x,t)], \mathscr{N}[u(x,t)]$ respectively perform the linear and nonlinear terms of function $u(x,t)$.

# 2 Basic Functions Using Fareeha Transform

**Definition 1** (Gamma function)**.** *(Shown in [29]) Given a value $z\in\mathbb{C}$, and $Re(z)>0$, the integration presented as following is valid*

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt. \tag{5}$$

We collect some particular results related to the gamma function:

$$\Gamma(z+1) = z\Gamma(z), \quad \Gamma(n+1) = n!. \tag{6}$$

**Definition 2** (Caputo's Fractional Derivatives)**.** *(See in [30], chapter 2) Given a continuous function $y=f(x)$, and an arbitrary value $m-1<\alpha\le m$, the Caputo's fractional derivative of order $\alpha$ is defined by:*

$$ {}_a^C\mathscr{D}_x^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)}\int_a^x (x-s)^{m-\alpha-1}f^{(m)}(s)ds. \tag{7}$$

In this section, we will expand the transform for the polynomial-coefficient functions and fractional derivatives that are performed in Table 2.

**Proposition 1.** *Using the Fareeha transform, we apply some basic function derivatives, [28]:*

$$\mathscr{G}_f(u') = F(s^n) = s^n F(s^n) - u(0). \tag{8}$$

$$\mathscr{G}_f(u'') = F(s^n) = s^{2n} F(s^n) - s^n u(0) - u'(0). \tag{9}$$

$$\mathscr{G}_f(u''') = s^{3n} F(s^n) - s^{2n} u(0) - s^n u'(0) - u''(0). \tag{10}$$

$$\mathscr{G}_f(u^{(k)}(t)) = s^{kn}\mathscr{G}_f(u(t)) - \sum_{m=0}^{k-1} s^{(k-m-1)n} f^{(m)}(0). \tag{11}$$

**Proposition 2.** *Fareeha transform of the function $f(t) = t^\alpha$:*

$$\mathscr{G}_f(t^\alpha) = \frac{\Gamma(\alpha+1)}{s^{(\alpha+1)n}}. \tag{12}$$

*Proof.*
*Using Fareeha's definition, [28], Equation (1), and Gamma function (5), we have*

$$\begin{aligned}\mathscr{G}_f(t^\alpha) &= \int_0^\infty t^\alpha e^{-s^n t} dt, \quad (\textit{substituting} \quad u = s^n t) \\ &= \frac{1}{s^{(\alpha+1)n}} \int_0^\infty u^\alpha e^{-u} du = \frac{\Gamma(\alpha+1)}{s^{(\alpha+1)n}}. \end{aligned} \tag{13}$$

**Proposition 3.** *Now, we will extend the Fareeha transform for the Caputo derivative which is a main point applied to this work:*

$$\mathscr{G}_f\left[{}_0^C\mathscr{D}_t^\alpha u(t)\right] = s^{n\alpha} F(s^n) - \sum_{k=0}^{m-1} s^{n(\alpha-k-1)} u^{(k)}(0). \tag{14}$$

Proof.
Using convolution property of $f(t)$ and $g(t)$ in Table 2,

$$f * g = \int_0^t f(u)\, g(t-u)\, du. \tag{15}$$

From definition 2 of the Caputo derivative, we have

$$\begin{aligned}{}_0^C\mathscr{D}_t^\alpha u(t) &= \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} x^{(m)}(\tau)\, d\tau \\ &= \frac{1}{\Gamma(m-\alpha)} t^{m-\alpha-1} * x^{(m)}(t). \end{aligned} \tag{16}$$

Using Fareeha definition, [28], Equation (11), we have

$$\begin{aligned}\mathscr{G}_f\left[{}_0^C\mathscr{D}_t^\alpha u(t)\right] &= \frac{1}{\Gamma(m-\alpha)} \mathscr{G}_f\left(t^{m-\alpha-1}\right) \times \mathscr{G}_f\left(x^{(m)}(t)\right) \\ &= \frac{1}{\Gamma(m-\alpha)} \frac{\Gamma(m-\alpha)}{s^{(m-\alpha)n}} \\ &\times \left[s^{nm}F(s^n) - \sum_{k=0}^{m-1} s^{n(m-k-1)} u^{(k)}(0)\right] \\ &= \frac{1}{s^{(m-\alpha)n}} \\ &\times \left[s^{nm}F(s^n) - \sum_{k=0}^{m-1} s^{n(m-k-1)} u^{(k)}(0)\right] \\ &= s^{n\alpha}F(s^n) - \sum_{k=0}^{m-1} s^{n(\alpha-k-1)} u^{(k)}(0). \end{aligned} \tag{17}$$

Table 2: The Fareeha transform of basic functions.

| Functions $u(t)$ | $F(s^n)(s) = \int_0^\infty u(t) e^{-s^n t} dt$ |
|---|---|
| $1$ | $s^{-n}$ |
| $t$ | $s^{-2n}$ |
| $e^{at}$ | $\frac{1}{s^n - a}$ |
| $t^\alpha$ | $\frac{\Gamma(\alpha+1)}{s^{(\alpha+1)n}}$ |
| $\sin at$ | $\frac{a}{s^{2n}+a^2}$ |
| $t \sin at$ | $\frac{2as^n}{(s^{2n}+a^2)^2}$ |
| $\cos at$ | $\frac{s^n}{s^{2n}+a^2}$ |
| $t \cos at$ | $\frac{s^{2n}-a^2}{(s^{2n}+a^2)^2}$ |
| $\sinh at$ | $\frac{a}{s^{2n}-a^2}$ |
| $\cosh at$ | $\frac{s^n}{s^{2n}-a^2}$ |
| $t \sinh at$ | $\frac{2as^n}{s^{2n}-a^2}$ |
| $t \cosh at$ | $\frac{s^{2n}+a^2}{s^{2n}-a^2}$ |
| $e^{at} \sin \omega t$ | $\frac{\omega}{(s^n-a^2)^2+\omega^2}$ |
| $e^{at} \cos \omega t$ | $\frac{s^n-a}{(s^n-a^2)^2+\omega^2}$ |
| $e^{at} \sinh \omega t$ | $\frac{\omega}{(s^n-a^2)^2-\omega^2}$ |
| $e^{at} \cosh \omega t$ | $\frac{s^n-a}{(s^n-a^2)^2-\omega^2}$ |
| $\sin t \cos t$ | $\frac{1}{s^{2n}+4}$ |
| $u(t) * v(t)$ | $U(s^n) \times V(s^n)$ (convolution of $u(t)$ and $v(t)$, |

# 3 Methodology

In this section, we will combine the transform of the fractional derivative using Proposition 3 and the Adomian decomposition method for solving the factional differential telegraph equations. First, we remind the Adomian decomposition method deployed in this work

### 3.1 Adomian decomposition method

The Adomian decomposition method is an effective procedure for finding the analytic solutions introduced to solve linear and nonlinear differential equations, [31]. The Adomain decomposition method was summarized to determine the exact solutions of Bratu-type equations, [32], and has been used in most system theories to simplify the disjoint state spaces. The differential equations, [33], have been developed enormously and expanded in some models of real life such as the circuit model, Spring-mass system, diffusion system, and population model, [25]. In this paper, the newfangled Fareeha transform, [28], combines the Adomian decomposition method to perform the effectiveness of the transform for solving fractional telegraph equations. The Adomain decomposition method, [34], expresses a function $u(x)$ in the form of

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{18}$$

where the term $u_n(x)$ be iteratively defined. Especially, the nonlinear term $N(u)$ can be decomposition into an infinite series of polynomials in the form of

$$N(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, \cdots, u_n), \tag{19}$$

where $A_n$ be the so-called Adomian polynomial terms of $u_0, u_1, ..., u_n$ defined by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[F(\sum_{i=0}^{n}\lambda^i u_i)]_{\lambda=0}, n = 0, 1, 2, \cdots \tag{20}$$

where the terms are determined as the following

$$\begin{aligned}
&A_0 = F(u_0),\\
&A_1 = u_1 F'(u_0),\\
&A_2 = u_2 F'(u_0) + \frac{1}{2}u_1^2 F''(u_0),\\
&A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3}u_1^3 F'''(u_0,\\
&A_4 = u_4 F\prime(u_0) + (u_1 u_3 + \frac{1}{2}u_2^2)F''(u_0)\\
&\quad + \frac{1}{2}u_1^2 u_2 F'''(u_0) + \frac{1}{24}u_1^4 F^{(4)} u_0,\\
&\cdots
\end{aligned}$$

### 3.2 Using Fareeha transform for the fractional telegraph equations

In this section, we deploy the Fareeha transform for general fractional telegraph equations as the following:

$$\mathscr{D}_t^{\alpha} u(x,t) + \mathscr{L}[u(x,t)] + \mathscr{N}[u(x,t)] = q(x,t), \tag{21}$$

satisfy the condition

$$u(x,0) = g(x), u_t(x,0) = h(x), 0 < \alpha \leq 2. \tag{22}$$

**Step 1:** Taking Fareeha transform for the equation (21), and using property (14) including condition (22):

$$\mathscr{G}_f[\mathscr{D}_t^{\alpha} u(x,t) + \mathscr{L}[u(x,t)] + \mathscr{N}[u(x,t)]] = \mathscr{G}_f[q(x,t)]. \tag{23}$$

**Step 2:** Simplify the equation in the form:

$$\begin{aligned}
u(x,t) &= \mathscr{G}_f^{-1}\left\{g(x)s^{-n} + h(x)s^{-2n} + \frac{1}{s^{n\alpha}}\mathscr{G}_f[q(x,t)]\right\}\\
&+ \mathscr{G}_f^{-1}\left\{\frac{1}{s^{n\alpha}}\mathscr{G}_f\{-\mathscr{L}[u(x,t)] - \mathscr{N}[u(x,t)]\}\right\}.
\end{aligned} \tag{24}$$

**Step 3:** Setting $u(x,t) = \sum_{k=0}^{\infty} u_k(x,t), N(u) = -\mathscr{L}[u(x,t) - \mathscr{N}[u(x,t)]$ then calculate the terms:

$$\begin{aligned}
u_0 &= \mathscr{G}_f^{-1}\left\{g(x)s^{-n} + h(x)s^{-2n} + \frac{1}{s^{n\alpha}}\mathscr{G}_f[q(x,t)]\right\}\\
u_{k+1} &= \mathscr{G}_f^{-1}\left\{\frac{1}{s^{n\alpha}}\mathscr{G}_f(A_k)\right\}, k = 0, 1, 2, \cdots
\end{aligned}$$

where $A_k$ is the Adomian terms given by the Equation (20).

**Step 4:** The analytic solution is depicted in the form that could approach to the exact solution:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \tag{25}$$

## 4 Application

The Fareeha transform seems well suited to solutions of ordinary differential equations when they are linearly combined with constant coefficients, and when initial conditions create the solutions. We will apply the Fareeha transform for the fractional differential telegraph equations using the Adomian decomposition method. We consider the first example as the following

*Example* 4.1. Consider the fractional hyperbolic telegraph equation as follows, [35]:

$$\mathscr{D}_t^{\alpha} u(x,t) = u(x,t) - 2u_t(x,t) - u_{xx}(x,t), \tag{26}$$

$$u(x,0) = e^x, u_t(x,0) = -2e^x, 0 < \alpha \leq 2. \tag{27}$$

Taking Fareeha transform on both sides from Equation (26), using equation (14) including condition (27)

$$\mathcal{G}_f\left[\mathcal{D}_t^{\alpha}u(x,t)\right]=\mathcal{G}_f\left[u(x,t)-2u_t(x,t)-u_{xx}(x,t)\right]$$
$$s^{n\alpha}F(x,s^n)-s^{n(\alpha-1)}e^x+2e^xs^{n(\alpha-2)} =\mathcal{G}_f\left[u(x,t)-2u_t(x,t)-u_{xx}(x,t)\right].$$
$$F(x,s^n)=\frac{e^x(s^{n(\alpha-1)}-2s^{n(\alpha-2)})}{s^{n\alpha}} +\frac{1}{s^{n\alpha}}\mathcal{G}_f\left[u(x,t)-2u_t(x,t)-u_{xx}(x,t)\right]. \tag{28}$$

Simplify the equations (28) and taking converse Laplace on both sides, we can get

$$\begin{aligned}u(x,t)&=\mathcal{G}_f^{-1}\left[e^x(s^{-n}-2s^{-2n})\right.\\&\left.+\frac{1}{s^{n\alpha}}\mathcal{G}_f\left[u(x,t)-2u_t(x,t)-u_{xx}(x,t)\right]\right]\\&=\mathcal{G}_f^{-1}\left[e^x(s^{-n})-2s^{-2n})\right]\\&+\mathcal{G}_f^{-1}\left\{\frac{1}{s^{n\alpha}}\mathcal{G}_f\left[u(x,t)-2u_t(x,t)-u_{xx}(x,t)\right]\right\}.\end{aligned} \tag{29}$$

Taking $F[u]=u-2u_t-u_{xx}$, and setting the terms of ADM $F[u]=\sum_{k=0}^{\infty}A_k$, where

$$A_k=\frac{1}{k!}\frac{d^k}{d\lambda^k}\left[F\left(\sum_{i=0}^{k}\lambda^i u_i\right)\right]_{\lambda=0};k=0,1,2,3,\cdots \tag{30}$$

Assume the solution $u(x,t)=\sum_{i=0}^{\infty}u_k(x,t)$, so

$$\sum_{i=0}^{\infty}u_k(x,t)=\mathcal{G}_f^{-1}\left[e^x(s^{-n})-2s^{-2n})\right] +\mathcal{G}_f^{-1}\left\{\frac{1}{s^{n\alpha}}\mathcal{G}_f\left[A_k\right]\right\},k=0,1,2,3,\cdots \tag{31}$$

Construct the recursion form of the Adomian decomposition method (29), we calculate the terms as follows:

$$u_0=\mathcal{G}_f^{-1}\left[e^x(s^{-n}-2s^{-2n})\right]=e^x(1-2t) \tag{32}$$

$$u_{k+1}=\mathcal{G}_f^{-1}\left\{\frac{1}{s^{n\alpha}}\mathcal{G}_f\left[A_k\right]\right\},k=0,1,2,3,\cdots \tag{33}$$

Using (32), we can find the term $A_0$ as follows:

$$\begin{aligned}A_0&=\frac{1}{0!}\frac{d^0}{d\lambda^0}\left[F\left(\sum_{i=0}^{0}\lambda^0 u_0\right)\right]_{\lambda=0}=u_0-2u_{0t}-u_{0xx}\\&=e^x(1-2t)-2(-2e^x)-e^x(1-2t)=4e^x.\end{aligned} \tag{34}$$

From equation (33), we evaluate the terms of solution $u_1$ using (34) as follows

$$\begin{aligned}u_1&=\mathcal{G}_f^{-1}\left[\frac{1}{s^{n\alpha}}\mathcal{G}_f(A_0)\right]=\mathcal{G}_f^{-1}\left[\frac{1}{s^{n\alpha}}\mathcal{G}_f(4e^x)\right]\\&=4e^x\frac{t^{\alpha}}{\Gamma(\alpha+1)}.\end{aligned} \tag{35}$$

Now we calculate the term $A_1$ using equation (35)

$$\begin{aligned}A_1&=\frac{1}{1!}\frac{d^1}{d\lambda^1}\left[F\left(\sum_{i=0}^{1}\lambda^1 u_i\right)\right]_{\lambda=0}=F(u_1)\\&=u_1-2u_{1t}-u_{1xx}=-8e^x\frac{\alpha t^{\alpha-1}}{\Gamma(\alpha+1)}.\end{aligned} \tag{36}$$

From equation (36), $A_1$, we can find the value of $u_2$

$$\begin{aligned}u_2&=\mathcal{G}_f^{-1}\left[\frac{1}{s^n}\mathcal{G}_f(A_1)\right]=\mathcal{G}_f^{-1}\left[\frac{1}{s^{n\alpha}}\mathcal{G}_f\left(-8e^x\frac{\alpha t^{\alpha-1}}{\Gamma(\alpha+1)}\right)\right]\\&=-8e^x\mathcal{G}_f^{-1}\left[\frac{1}{s^{n\alpha}}\mathcal{G}_f\left(\frac{\alpha t^{\alpha-1}}{\Gamma(\alpha+1)}\right)\right]=-8e^x\frac{t^{2\alpha-1}}{\Gamma(2\alpha-1)}.\end{aligned} \tag{37}$$

Using equation (37), $u_2$, we can calculate $A_2$:

$$\begin{aligned}A_2&=\frac{1}{2!}\frac{d^2}{d\lambda^2}\left[F\left(\sum_{i=0}^{2}\lambda^2 u_{ixx}\right)\right]_{\lambda=0}\\&=\frac{1}{2!}\frac{d^2}{d\lambda^2}\left[\lambda^0u_0+\lambda u_1+\lambda^2u_2\right]_{\lambda=0}\\&=\frac{1}{2}2F[u_2]=u_2-2u_{2t}-u_{2xx}=16e^x\frac{t^{2\alpha-2}}{\Gamma(2\alpha-1)}.\end{aligned} \tag{38}$$

Using equation (38), $A_2$, we can calculate $u_3$:

$$\begin{aligned}u_3&=\mathcal{G}_f^{-1}\left[\frac{1}{s^{n\alpha}}\mathcal{G}_f(A_2)\right]=\frac{16e^x}{\Gamma(2\alpha-1)}\mathcal{G}_f^{-1}\left[\frac{1}{s^{n\alpha}}\mathcal{G}_f\left(t^{2\alpha-2}\right)\right]\\&=\frac{16e^x}{\Gamma(2\alpha-1)}\mathcal{G}_f^{-1}\left[\frac{1}{s^{n\alpha}}\left(\frac{\Gamma(2\alpha-1)}{s^{(2\alpha-1)n}}\right)\right]=\frac{16e^x}{\Gamma(3\alpha-1)}t^{3\alpha-2}.\end{aligned} \tag{39}$$

The solution collected using (25) is performed as follows

$$\begin{aligned}u(x,t)&=e^x(1-2t)+4e^x\frac{t^{\alpha}}{\Gamma(\alpha+1)}-8e^x\frac{t^{2\alpha-1}}{\Gamma(2\alpha)}\\&+\frac{16e^x}{\Gamma(3\alpha-1)}t^{3\alpha-2}+\cdots\\&=e^x\left[1-2t+4\frac{t^{\alpha}}{\Gamma(\alpha+1)}-8\frac{t^{2\alpha-1}}{\Gamma(2\alpha)}\right.\\&\left.+\frac{16}{\Gamma(3\alpha-1)}t^{3\alpha-2}+\cdots\right],\end{aligned} \tag{40}$$

when $\alpha = 2$, the solution (40) becomes as the following

$$u(x,t) = e^x \left[1 - 2t + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} - \cdots\right] = e^{x-2t}. \tag{41}$$

and their graphs have been depicted in Fig. 1, and Fig. 2, respectively (Appendix). The comparison of the solutions by different values of $\alpha$ is shown in Table 3 (Appendix).

*Example* 4.2. Consider a non-linear time-fractional hyperbolic telegraph equation, [35]:

$$\mathscr{D}_t^{\alpha} u(x,t) = u_{xx}(x,t) + u_t(x,t) - u^2(x,t) + xu(x,t).u_x(x,t), \tag{42}$$

$$u(x,0) = x; u_t(x,0) = x, 0 < \alpha \leq 2. \tag{43}$$

Let $F[u] = u_{xx}(x,t) + u_t(x,t) - u^2(x,t) + xu(x,t).u_x(x,t)$, and taking Fareeha transform on both sides of equation (42), using equation (14) consisting of condition (43)

$$\mathscr{G}_f[\mathscr{D}_t^{\alpha} u(x,t)] = \mathscr{G}_f[N(u)] \tag{44}$$

$$s^{n\alpha} F(x,s^n) - s^{n(\alpha-1)} x - s^{n(\alpha-2)} x = \mathscr{G}_f[N(u)]. \tag{45}$$

Simplify equation (45) and collect the terms $F(s^n)$, we have

$$F(x,s^n) = s^{-n\alpha}[x(s^{n(\alpha-1)} - s^{n(\alpha-2)})] + s^{-n\alpha}\mathscr{G}_f[N(u)]. \tag{46}$$

Deriving the solution by dividing $s^{n\alpha}$ on both sides of equation (46) by inverse Fareeha transform

$$u(x,t) = \mathscr{G}_f^{-1}\left\{s^{-n\alpha}[x(s^{n(\alpha-1)} - s^{n(\alpha-2)})]\right\} + \mathscr{G}_f^{-1}\left\{\frac{1}{s^{n\alpha}}\mathscr{G}_f[N(u)]\right\}. \tag{47}$$

Adopting $N(u) = \sum_{k=0}^{\infty} A_k$, $u(x,t) = \sum_{k=0}^{\infty} u_k(x,t)$, where

$$A_k = \frac{1}{k!}\frac{d^k}{d\lambda^k}\left[F\left(\sum_{i=0}^{k} \lambda^i u_i\right)\right]_{\lambda=0} ; n = 0,1,2,\cdots \tag{48}$$

the equation (47) could be rewritten as follows

$$\sum_{k=0}^{\infty} u_k(x,t) = \mathscr{G}_f^{-1}\left\{s^{-n\alpha}[x(s^{n(\alpha-1)} - s^{n(\alpha-2)})]\right\} + \mathscr{G}_f^{-1}\left\{\frac{1}{s^{n\alpha}}\mathscr{G}_f[N(u)]\right\}, \tag{49}$$

and we calculate the iterative form of ADM using equation (47) as the following

$$u_0 = \mathscr{G}_f^{-1}\left[s^{-n\alpha}[x(s^{n(\alpha-1)} - s^{n(\alpha-2)})]\right] = x(1+t). \tag{50}$$

$$u_{k+1} = \mathscr{G}_f^{-1}\left\{\frac{1}{s^{n\alpha}}\mathscr{G}_f[A_k]\right\}, k = 0,1,2,\cdots \tag{51}$$

We can calculate the terms $A_0$ using equation (50) as follows

$$\begin{aligned} A_0 &= \frac{1}{0!}\frac{d^0}{d\lambda^0}\left[F\left(\sum_{i=0}^{0} \lambda^0 u_0\right)\right]_{\lambda=0} \\ &= u_{0xx} + u_{0t} - u_0^2 + xu_0.u_{0x} \\ &= x - [x(1+t)]^2 + x^2(1+t)(1+t) = x. \end{aligned} \tag{52}$$

Considering the terms $u_k$ from equation (51), and equation (52) as follows:

$$u_1 = \mathscr{G}_f^{-1}\left[\frac{1}{s^{n\alpha}}\mathscr{G}_f(A_0)\right] = \mathscr{G}_f^{-1}\left[\frac{1}{s^{n\alpha}}\mathscr{G}_f(x)\right] = x\frac{t^{\alpha}}{\Gamma(\alpha+1)}. \tag{53}$$

Using equation (52), $u_1$, we can calculate $A_1$

$$\begin{aligned} A_1 &= \frac{1}{1!}\frac{d^1}{d\lambda^1}\left[F\left(\sum_{i=0}^{1} \lambda^1 u_{ixx}\right)\right]_{\lambda=0} = \frac{d}{d\lambda}\left[\lambda^0 u_0 + \lambda u_1\right]_{\lambda=0} \\ &= u_{1xx} + u_{1t} - u_1^2 + xu_1.u_{1x}. \\ &= x\frac{\alpha t^{\alpha-1}}{\Gamma(\alpha+1)} - x^2\frac{t^{2\alpha}}{(\Gamma(\alpha+1))^2} + x^2\frac{t^{\alpha}}{\Gamma(\alpha+1)}\frac{t^{\alpha}}{\Gamma(\alpha+1)} \\ &= x\frac{\alpha t^{\alpha-1}}{\Gamma(\alpha+1)}. \end{aligned} \tag{54}$$

Using equation (54), $A_1$, we can calculate $u_2$

$$\begin{aligned} u_2 &= \mathscr{G}_f^{-1}\left[\frac{1}{s^n}\mathscr{G}_f(A_1)\right] = \mathscr{G}_f^{-1}\left[\frac{1}{s^{n\alpha}}\mathscr{G}_f\left(x\frac{\alpha t^{\alpha-1}}{\Gamma(\alpha+1)}\right)\right] \\ &= \frac{x\alpha}{\Gamma(\alpha+1)}\mathscr{G}_f^{-1}\left[\frac{1}{s^{n\alpha}}\left(\frac{\Gamma(\alpha)}{s^{\alpha n}}\right)\right] = x\frac{t^{2\alpha-1}}{\Gamma(2\alpha-1)}. \end{aligned} \tag{55}$$

Using equation (55), $u_2$, we calculate $A_2$:

$$\begin{aligned} A_2 &= \frac{1}{2!}\frac{d^2}{d\lambda^2}\left[F\left(\sum_{i=0}^{2} \lambda^2 u_{ixx}\right)\right]_{\lambda=0} \\ &= \frac{1}{2!}\frac{d^2}{d\lambda^2}\left[\lambda^0 u_0 + \lambda u_1 + \lambda^2 u_2\right]_{\lambda=0} \\ &= u_{2xx} + u_{2t} - u_2^2 + xu_2.u_{2x} = x\frac{t^{2\alpha-2}}{\Gamma(2\alpha-1)}. \end{aligned} \tag{56}$$

Now, we can calculate $u_3$ using equation (56)

$$\begin{aligned} u_3 &= \mathscr{G}_f^{-1}\left[\frac{1}{s^{n\alpha}}\mathscr{G}_f(A_2)\right] \\ &= \frac{x}{\Gamma(2\alpha-1)}\mathscr{G}_f^{-1}\left[\frac{1}{s^{n\alpha}}\mathscr{G}_f\left(t^{2\alpha-2}\right)\right] \\ &= x\frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha+1)}\frac{(2\alpha-1)}{\Gamma(2\alpha)}\mathscr{G}_f^{-1}\left[\frac{1}{s^{n\alpha}}\left(\frac{\Gamma(2\alpha-1)}{s^{(2\alpha-1)n}}\right)\right] \\ &= \frac{xt^{3\alpha-2}}{\Gamma(3\alpha-1)}. \end{aligned} \tag{57}$$

Solution collected using equation (25) is performed as follows

$$\begin{aligned} u(x,t) &= u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+\cdots \\ &= x(1+t)+\frac{xt^{\alpha}}{\Gamma(\alpha+1)}+\frac{xt^{2\alpha-1}}{\Gamma(2\alpha)} \\ &+\frac{xt^{3\alpha-2}}{\Gamma(3\alpha-1)}+\cdots \end{aligned} \tag{58}$$

When $\alpha = 2$, the solution from equation (58) becomes

$$u(x,t) = x\left[1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots\right] = xe^t. \tag{59}$$

and their graphs has been respectively portrayed in Fig. 3, and Fig. 4 (Appendix). The comparison of different values of $\alpha$ is shown in Table 4 (Appendix).

*Example* 4.3. Taking into account 2D time-fractional telegraph equation given as follows, [35]:

$$\begin{aligned} \mathscr{D}_t^{2\alpha}u(x,y,t)+3\mathscr{D}_t^{\alpha}u(x,y,t)+2u(x,y,t) &= u_{xx}(x,y,t) \\ &+u_{yy}(x,y,t), \end{aligned} \tag{60}$$

satisfy the initial conditions

$$u(x,y,0) = e^{x+y}; u_t(x,y,0) = -3e^{x+y}, 0<\alpha\le 1. \tag{61}$$

Let $F[u] = -3\mathscr{D}_t^{\alpha}u-2u+u_{xx}+u_{yy}$, and apply the Fareeha transform on both sides of equation (60) including (61)

$$\begin{aligned} &\mathscr{G}_f\left[\mathscr{D}_t^{2\alpha}u(x,y,t)\right] = \mathscr{G}_f[N(u)]. \\ &s^{2n\alpha}F(x,y,s^n)-s^{n(2\alpha-1)}u(x,y,0)-s^{n(2\alpha-2)}u_t(x,y,0) \\ &= \mathscr{G}_f[N(u)]. \end{aligned} \tag{62}$$

Taking inverse Fareeha transform from equation (62), we have

$$\begin{aligned} u(x,y,t) &= \mathscr{G}_f^{-1}\left\{s^{n(2\alpha-1)}u(x,y,0)-s^{n(2\alpha-2)}u_t(x,y,0)\right\} \\ &+\mathscr{G}_f^{-1}\left\{\frac{1}{s^{2n\alpha}}\mathscr{G}_f[N(u)]\right\}. \end{aligned} \tag{63}$$

By setting $N(u) = \sum_{k=0}^{\infty}A_k$, and $u(x,y,t) = \sum_{k=0}^{\infty}u_k(x,y,t)$, where

$$A_k = \frac{1}{k!}\frac{d^k}{d\lambda^k}\left[F\left(\sum_{i=0}^{k}\lambda^i u_i\right)\right]_{\lambda=0}, k=0,1,2,3,\cdots \tag{64}$$

the equation (65) is rewritten as follows

$$\begin{aligned} \sum_{k=0}^{\infty}u_k(x,y,t) &= \mathscr{G}_f^{-1}\left\{s^{n(2\alpha-1)}u(x,y,0)-s^{n(2\alpha-2)}u_t(x,y,0)\right\} \\ &+\mathscr{G}_f^{-1}\left\{\frac{1}{s^{2n\alpha}}\mathscr{G}_f[N(u)]\right\}. \end{aligned} \tag{65}$$

Using equation (65), and setting reiterated terms as the following

$$\begin{aligned} u_0 &= \mathscr{G}_f^{-1}\left[s^{n(2\alpha-1)}u(x,y,0)-s^{n(2\alpha-2)}u_t(x,y,0)\right]. \\ &= u(x,y,0)+tu_t(x,y,0) = e^{x+y}(1-3t) \end{aligned} \tag{66}$$

$$\begin{aligned} u_{r+1} &= \mathscr{G}_f^{-1}\left\{s^{-2n\alpha}\mathscr{G}_f\left[-3D_t^{\alpha}u-2u+u_{xx}+u_{yy}\right]\right\}, \\ r &= 1,2,\cdots \end{aligned} \tag{67}$$

We calculate the terms of ADM, $A_0$

$$\begin{aligned} A_0 &= \frac{1}{0!}\frac{d^0}{d\lambda^0}\left[F\left(\sum_{i=0}^{0}\lambda^0 u_0\right)\right]_{\lambda=0} \\ &= -3\mathscr{D}_t^{\alpha}u_0-2u_0+u_{0xx}+u_{0yy} = -3\mathscr{D}_t^{\alpha}u_0. \end{aligned} \tag{68}$$

Using an equation (68), we calculate the term $u_1$

$$\begin{aligned} u_1 &= \mathscr{G}_f^{-1}\left[\frac{1}{s^{2n\alpha}}\mathscr{G}_f(A_0)\right] = \mathscr{G}_f^{-1}\left[\frac{1}{s^{2n\alpha}}\mathscr{G}_f\left(-3\mathscr{D}_t^{\alpha}u_0\right)\right] \\ &= -3\mathscr{G}_f^{-1}\left[\frac{1}{s^{2n\alpha}}s^{n\alpha}\mathscr{G}_f[u_0]-s^{n(\alpha-1)}u_0(x,y,0)\right] \\ &= -3\mathscr{G}_f^{-1}\left[\frac{1}{s^{2n\alpha}}e^{x+y}\left[s^{n\alpha}\left(\frac{1}{s^n}-\frac{3}{s^{2n}}\right)-s^{n(\alpha-1)}\right]\right] \\ &= -3\mathscr{G}_f^{-1}\left[\frac{1}{s^{2n\alpha}}-\frac{3}{s^{(2-\alpha)n}}e^{x+y}\right] \\ &= -3\mathscr{G}_f^{-1}\left[\frac{1}{s^{2n\alpha}}\mathscr{G}_f\left(D_t^{\alpha}u_0\right)\right] \\ &= -3\mathscr{G}_f^{-1}\left\{s^{-2n\alpha}\left(-\frac{3}{s^{(2-\alpha)n}}e^{x+y}\right)\right\} \\ &= 9e^{x+y}\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}. \end{aligned} \tag{69}$$

Using equation (69), we can calculate the term $A_1$

$$\begin{aligned} A_1 &= \frac{1}{1!}\frac{d^1}{d\lambda^1}\left[F\left(\sum_{i=0}^{1}\lambda^1 u_{ixx}\right)\right]_{\lambda=0} \\ &= \frac{d}{d\lambda}\left[\lambda^0 u_0+\lambda u_1\right]_{\lambda=0} \\ &= -3\mathscr{D}_t^{\alpha}u_1-2u_1+u_{1xx}+u_{1yy}. \end{aligned} \tag{70}$$

Using equation (69), and equation (70), we have

$$\begin{aligned} u_2 &= \mathscr{G}_f^{-1}\left[\frac{1}{s^{2n\alpha}}\mathscr{G}_f(A_1)\right] \\ &= \mathscr{G}_f^{-1}\left[\frac{1}{s^{2n\alpha}}\mathscr{G}_f\left(-3\mathscr{D}_t^{\alpha}u_1 - 2u_1 + u_{1xx} + u_{1yy}\right)\right] \\ &= -3\mathscr{G}_f^{-1}\left[\frac{1}{s^{2n\alpha}}\left(s^{n\alpha}\mathscr{G}_f[u_1] - s^{n(\alpha-1)}u_1(x,y,0)\right)\right] \\ &= -3\mathscr{G}_f^{-1}\left[\frac{1}{s^{2n\alpha}}s^{n\alpha}\mathscr{G}_f\left(9e^{x+y}\frac{t^{\alpha+1}}{\Gamma(\alpha)}\right)\right] \\ &= -\frac{27e^{x+y}t^{2\alpha+1}}{\Gamma(2\alpha+2)}. \end{aligned} \tag{71}$$

Using equation (71), we have

$$\begin{aligned} A_2 &= \frac{1}{2!}\frac{d^2}{d\lambda^2}\left[F\left(\sum_{i=0}^{2}\lambda^2 u_{ixx}\right)\right]_{\lambda=0} \\ &= \frac{1}{2!}\frac{d^2}{d\lambda^2}\left[\lambda^0 u_0 + \lambda u_1 + \lambda^2 u_2\right]_{\lambda=0} \\ &= -3\mathscr{D}_t^{\alpha}u_2 - 2u_2 + u_{2xx} + u_{2yy}. \end{aligned} \tag{72}$$

$$\begin{aligned} u_3 &= \mathscr{G}_f^{-1}\left[\frac{1}{s^{n\alpha}}\mathscr{G}_f(A_2)\right] \\ &= \mathscr{G}_f^{-1}\left[\frac{1}{s^{n\alpha}}\mathscr{G}_f\left(-3\mathscr{D}_t^{\alpha}u_2 - 2u_2 + u_{2xx} + u_{2yy}\right)\right] \\ &= \mathscr{G}_f^{-1}\left[\frac{1}{s^{n\alpha}}\mathscr{G}_f\left(-3\mathscr{D}_t^{\alpha}u_2\right)\right] = \mathscr{G}_f^{-1}\left[\frac{1}{s^{n\alpha}}\mathscr{G}_f\left(-3\mathscr{D}_t^{\alpha}u_2\right)\right] \\ &= -3\mathscr{G}_f^{-1}\left[\frac{1}{s^{n\alpha}}\left(s^{\alpha n}\mathscr{G}_f[u_2] - s^{\alpha-1}u_2(x,y,0)\right)\right] \\ &= -3\mathscr{G}_f^{-1}\left[\frac{1}{s^{n\alpha}} - \frac{27e^{x+y}}{s^{(\alpha+2)n}}\right] = \frac{81e^{x+y}t^{3\alpha+1}}{\Gamma(3\alpha+2)}. \end{aligned} \tag{73}$$

So the solution is collected as the following

$$u(x,y,t) = e^{x+y}\left[1 - 3t + \frac{9t^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{27t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{81t^{3\alpha+1}}{\Gamma(3\alpha+2)}\cdots\right]. \tag{74}$$

When $\alpha = 1$, the solution from the equation (74) shown as follows

$$u(x,y,t) = e^{x+y}\left[1 - 3t + \frac{(3t)^2}{2!} - \frac{(3t)^3}{3!} + \cdots\right] = e^{x+y-3t}. \tag{75}$$

and the solutions have been respectively shown in Fig. 7, and Fig. 8 (Appendix). The comparison of the solutions of arbitrary values of $\alpha$ is shown in Table 5 (Appendix).

*Example* 4.4. To perform the effectiveness of the Fareeha transform, we take into account the space-differential telegraph equation as follows, [36]:

$$\mathscr{D}_x^{\alpha}u(x,t) = u_{tt}(x,t) + u_t(x,t) + u(x,t), \tag{76}$$

$$u(0,t) = e^{-t}, u_x(0,t) = e^{-t}, 0 < \alpha \le 2, t \ge 0. \tag{77}$$

Considering $F[u] = u_{tt} + u_t + u$, taking Fareeha transform on both sides equation (76), using equation (14) including equation (77):

$$\mathscr{G}_f\left(\mathscr{D}_x^{\alpha}u(x,t)\right) = \mathscr{G}_f[N(u)]. \tag{78}$$

$$s^{n\alpha}F(x,s^n) - s^{n(\alpha-1)}e^{-t} - s^{n(\alpha-2)}e^{-t} = \mathscr{G}_f[N(u)].$$

$$F(x,s^n) = \frac{[s^{n(\alpha-1)} - s^{n(\alpha-2)}]e^{-t}}{s^{n\alpha}} + \frac{1}{s^{n\alpha}}\mathscr{G}_f[(N(u)]. \tag{79}$$

Taking inverse Fareeha transform of equation (79) by $s^{n\alpha}$, and we have

$$\begin{aligned} u(x,t) &= \mathscr{G}_f^{-1}\left\{\frac{[s^{n(\alpha-1)} - s^{n(\alpha-2)}]e^{-t}}{s^{n\alpha}} + \frac{1}{s^{n\alpha}}\mathscr{G}_f[N(u)]\right\} \\ &= \mathscr{G}_f^{-1}\left\{\frac{[s^{n(\alpha-1)} - s^{n(\alpha-2)}]e^{-t}}{s^{n\alpha}}\right\}. \\ &\quad + \mathscr{G}_f^{-1}\left\{\frac{1}{s^{n\alpha}}\mathscr{G}_f[N(u)]\right\} \end{aligned} \tag{80}$$

Setting $N(u) = \sum_{k=0}^{\infty}A_k$, $u(x,t) = \sum_{k=0}^{\infty}u_k(x,t)$, where

$$A_k = \frac{1}{n!}\frac{d^k}{d\lambda^k}\left[F\left(\sum_{i=0}^{k}\lambda^i u_i\right)\right]_{\lambda=0} ; k = 0,1,2,3,\cdots \tag{81}$$

and the equation (80) is rewritten as follows

$$\begin{aligned} \sum_{k=0}^{\infty}u_k(x,t) &= \mathscr{G}_f^{-1}\left\{\frac{[s^{n(\alpha-1)} - s^{n(\alpha-2)}]e^{-t}}{s^{n\alpha}}\right\} \\ &\quad + \mathscr{G}_f^{-1}\left\{\frac{1}{s^{n\alpha}}\mathscr{G}_f[N(u)]\right\}. \end{aligned} \tag{82}$$

Now, we will establish repeated terms of ADM using (82)

$$\begin{aligned} u_0 &= \mathscr{G}_f^{-1}\left[\frac{(s^{n(\alpha-1)} - s^{n(\alpha-2)})e^{-t}}{s^{n\alpha}}\right] \\ &= e^{-t} + xe^{-t} = e^{-t}(1+x). \end{aligned} \tag{83}$$

$$u_{k+1} = \mathscr{G}_f^{-1}\left\{\frac{1}{s^{n\alpha}}\mathscr{G}_f(u_{tt} + u_t + u)\right\} = \mathscr{G}_f^{-1}\left\{\frac{1}{s^{n\alpha}}\mathscr{G}_f(A_n)\right\}. \tag{84}$$

Using equation (83), we evaluate $A_0$ as follows

$$A_0 = \frac{1}{0!}\frac{d^0}{d\lambda^0}\left[F\left(\sum_{i=0}^{0}\lambda^0 u_0\right)\right]_{\lambda=0} = u_{0tt} + u_{0t} + u_0$$
$$= (1+x)e^{-t} + (1+x)(-e^{-t}) + (1+x)e^{-t} = (1+x)e^{-t}. \tag{85}$$

Based on equation (85), and equation (84), we can calculate the terms of the solution as follows:

$$u_1 = \mathscr{G}_f^{-1}\left[\frac{1}{s^{n\alpha}}\mathscr{G}_f(A_0)\right] = \mathscr{G}_f^{-1}\left[\frac{1}{s^{n\alpha}}\mathscr{G}_f\left((1+x)e^{-t}\right)\right]$$
$$= e^{-t}\mathscr{G}_f^{-1}\left[\frac{1}{s^{n\alpha}}(\frac{1}{s^n}+\frac{1}{s^{2n}})\right] = e^{-t}\left[\frac{x^\alpha}{\Gamma(\alpha+1)}+\frac{x^{\alpha+1}}{\Gamma(\alpha)}\right]. \tag{86}$$

The following steps are to calculate the rest of ADM and solution, for example, using equation (86), we can calculate $A_1$

$$A_1 = \frac{1}{1!}\frac{d^1}{d\lambda^1}\left[F\left(\sum_{i=0}^{1}\lambda^1 u_{ixx}\right)\right]_{\lambda=0} = \frac{d}{d\lambda}\left[\lambda^0 u_0 + \lambda u_1\right]_{\lambda=0}$$
$$= u_{1tt} + u_{1t} + u_1 = e^{-t}\left[\frac{x^\alpha}{\Gamma(\alpha+1)}+\frac{x^{\alpha+1}}{\Gamma(\alpha)}\right]. \tag{87}$$

Using equation (87), we calculate $u_2$

$$u_2 = \mathscr{G}_f^{-1}\left[\frac{1}{s^n}\mathscr{G}_f(A_1)\right]$$
$$= \mathscr{G}_f^{-1}\left[\frac{1}{s^{n\alpha}}\mathscr{G}_f\left(e^{-t}\left[\frac{x^\alpha}{\Gamma(\alpha+1)}+\frac{x^{\alpha+1}}{(\alpha+1)!}\right]\right)\right]$$
$$= e^{-t}\mathscr{G}_f^{-1}\left[\frac{1}{s^{n\alpha}}\left(\frac{1}{s^{(\alpha+1)n}}+\frac{1}{s^{(\alpha+2)n}}\right)\right]$$
$$= e^{-t}\left[\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right]. \tag{88}$$

From equation (88), we can calculate $A_2$

$$A_2 = \frac{1}{2!}\frac{d^2}{d\lambda^2}\left[F\left(\sum_{i=0}^{2}\lambda^2 u_{ixx}\right)\right]_{\lambda=0}$$
$$= \frac{1}{2!}\frac{d^2}{d\lambda^2}\left[\lambda^0 u_0 + \lambda u_1 + \lambda^2 u_2\right]_{\lambda=0}$$
$$= u_{2tt} + u_{2t} + u_2 = e^{-t}\left[\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right]. \tag{89}$$

Using equation (89), we can find the term $u_3$

$$u_3 = \mathscr{G}_f^{-1}\left[\frac{1}{s^{n\alpha}}\mathscr{G}_f(A_2)\right]$$
$$= e^{-t}\mathscr{G}_f^{-1}\left[\frac{1}{s^{n\alpha}}\mathscr{G}_f\left(e^{-t}\left[\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right]\right)\right]$$
$$= e^{-t}\mathscr{G}_f^{-1}\left[\frac{1}{s^{n\alpha}}\left(\frac{1}{s^{(2\alpha+1)n}}+\frac{1}{s^{(2\alpha+2)n}}\right)\right]$$
$$= e^{-t}\left[\frac{1}{s^{(3\alpha+1)n}}+\frac{1}{s^{(3\alpha+2)n}}\right]$$
$$= e^{-t}\left[\frac{x^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{x^{3\alpha+1}}{\Gamma(3\alpha+2)}\right]. \tag{90}$$

Then the solution was derived using (25) as follows

$$u(x,t) = e^{-t}\Bigg[1 + x + \frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{x^{\alpha+1}}{\Gamma(2\alpha+2)} + \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{x^{3\alpha+1}}{\Gamma(2\alpha+2)} + \cdots\Bigg]. \tag{91}$$

When $\alpha = 2$ the solution gained from equation (91) as the following

$$u(x,t) = e^{x-t}. \tag{92}$$

and the solutions have been respectively depicted using the Matlab application in Fig. 5 and Fig. 6 (Appendix). The comparison of the solutions by different values of $\alpha$ is shown in Table 6 (Appendix).

# 5 Conclusion

In this paper, some extensive properties for the Fareeha transform have been created, especially the fractional derivative in the Caputo sense. Fareeha transform has also performed successfully by solving time-fractional telegraph equations. To show the effectiveness of the transform, a space-fractional telegraph equation was conducted briefly to get the analytic solution. Using series expansion, the close forms have been derived that approach exact solutions played an essential role in physics [37], and their graphs have been performed to show the necessary role of the methods. Like other Laplace-typed transforms, the inverse Fareeha transform is derived from the direct Fareeha transform summarized in Table 2. The transform is conducted by taking Fareeha transform for fractional terms and other linear and nonlinear terms are included in Adomian decomposition method terms. In general, the Fareeha transform has effectively contributed to finding the solution to fractional telegraph equations.

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**Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)**
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Sanoe Koonprasert: Conceptualization, data curation, formal analysis, investigation, methodology, project administration, resources, supervision, validation, visualization, writing-original draft, writing-review and editing.
Phayung Meesad: Conceptualization, data curation, formal analysis, investigation, resources, supervision, validation, visualization, writing-original draft, and editing.

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# APPENDIX

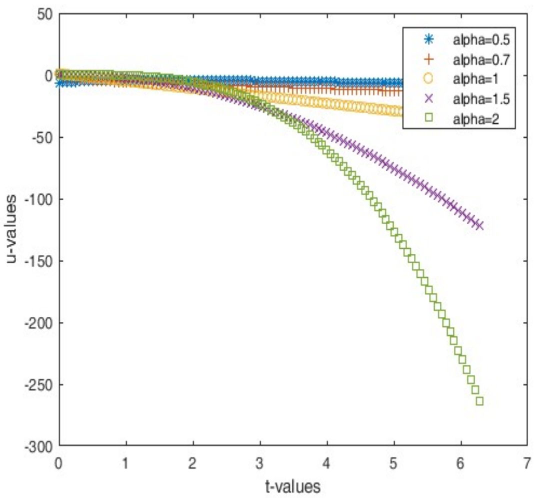


Figure 1: 2D solution depict example 4.1.

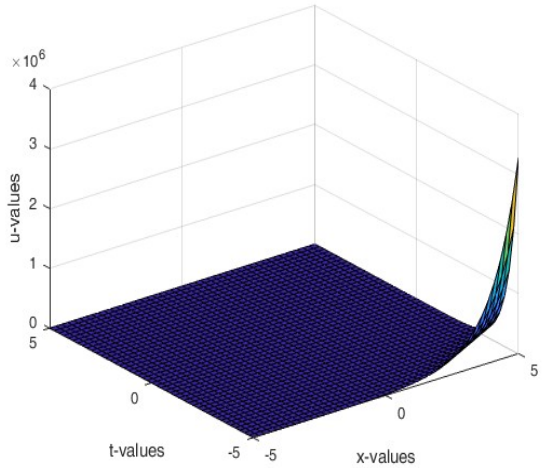


Figure 2: 3D solution example 4.1, $\alpha = 2$.

Table 3: Solutions of example 4.1 for x=0.5.

| $t$ | $\alpha = 1.1$ | $\alpha = 1.2$ | $\alpha = 1.5$ | $\alpha = 1.7$ | $\alpha = 2$ |
|---|---|---|---|---|---|
| 0.0 | 1.6487 | 1.6487 | 1.6487 | 1.6487 | 1.6487 |
| 0.1 | 2.1974 | 1.7374 | 1.4350 | 1.3896 | 1.3499 |
| 0.2 | 3.1172 | 2.1464 | 1.3112 | 1.1994 | 1.1052 |
| 0.3 | 4.2396 | 2.7909 | 1.2724 | 1.0576 | 0.9048 |
| 0.4 | 5.5136 | 3.6384 | 1.3328 | 0.9644 | 0.7408 |
| 0.5 | 6.9118 | 4.6687 | 1.5085 | 0.9274 | 0.6065 |
| 0.6 | 8.4166 | 5.8679 | 1.8152 | 0.9581 | 0.4966 |
| 0.7 | 10.0155 | 7.2254 | 2.2686 | 1.0707 | 0.4066 |
| 0.8 | 11.6991 | 8.7328 | 2.8836 | 1.2814 | 0.3329 |
| 0.9 | 13.4600 | 10.3831 | 3.6745 | 1.6077 | 0.2725 |
| 1.0 | 15.2923 | 12.1706 | 4.6550 | 2.0685 | 0.2231 |

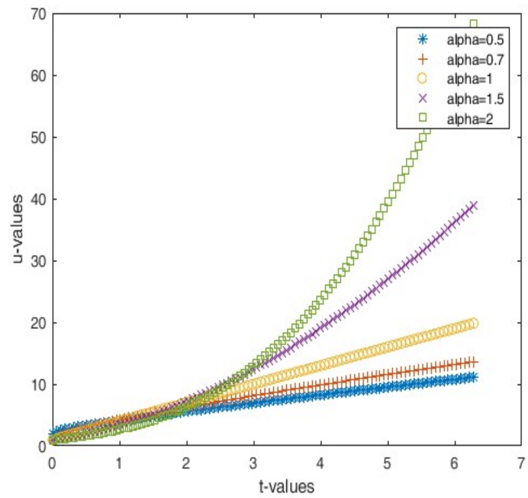


Figure 3: 2D solution depict example 4.2.

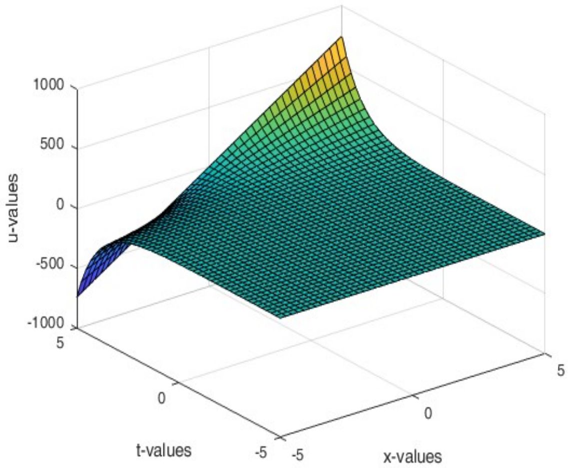


Figure 4: 3D solution example 4.2, $\alpha = 2$.

Table 4: Solutions of example 4.2 for x=0.5.

| $t$ | $\alpha = 1.1$ | $\alpha = 1.2$ | $\alpha = 1.5$ | $\alpha = 1.7$ | $\alpha = 2$ |
|---|---|---|---|---|---|
| 0.0 | 0.5000 | 0.5000 | 0.5000 | 0.5000 | 0.5000 |
| 0.1 | 0.5808 | 0.5788 | 0.5779 | 0.5751 | 0.5526 |
| 0.2 | 0.6685 | 0.6616 | 0.6567 | 0.6521 | 0.6107 |
| 0.3 | 0.7597 | 0.7470 | 0.7350 | 0.7285 | 0.6749 |
| 0.4 | 0.8535 | 0.8348 | 0.8125 | 0.8038 | 0.7459 |
| 0.5 | 0.9494 | 0.9246 | 0.8896 | 0.8778 | 0.8244 |
| 0.6 | 1.0472 | 1.0163 | 0.9664 | 0.9504 | 0.9111 |
| 0.7 | 1.1465 | 1.1099 | 1.0432 | 1.0217 | 1.0069 |
| 0.8 | 1.2472 | 1.2052 | 1.1204 | 1.0918 | 1.1128 |
| 0.9 | 1.3493 | 1.3023 | 1.1981 | 1.1610 | 1.2298 |
| 1.0 | 1.4525 | 1.4010 | 1.2766 | 1.2294 | 1.3591 |

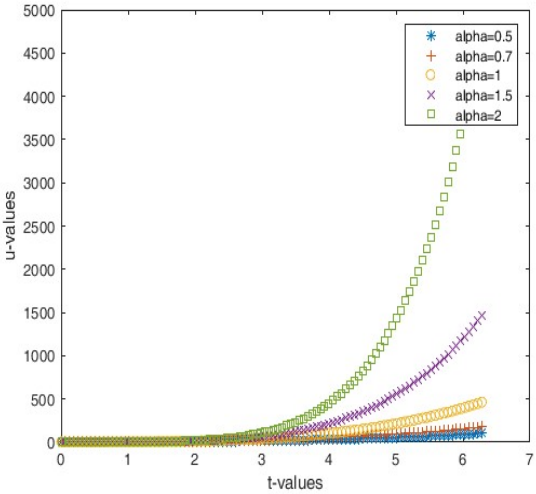


Figure 5: 2D solution depict example 4.3.

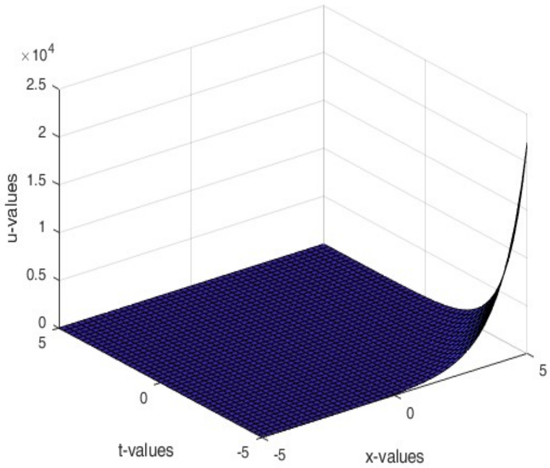


Figure 6: 3D solution example 4.3, $\alpha = 2$.

Table 5: Solutions of example 4.3 for x=0.5,y=0.5.

| $t$ | $\alpha = 1.1$ | $\alpha = 1.2$ | $\alpha = 1.5$ | $\alpha = 1.7$ | $\alpha = 2$ |
|---|---|---|---|---|---|
| 0.0 | 0.2500 | 0.2500 | 0.2500 | 0.2500 | 0.2500 |
| 0.1 | 0.1826 | 0.1805 | 0.1771 | 0.1760 | 0.2763 |
| 0.2 | 0.1303 | 0.1235 | 0.1107 | 0.1063 | 0.3054 |
| 0.3 | 0.0912 | 0.0774 | 0.0514 | 0.0416 | 0.3375 |
| 0.4 | 0.0634 | 0.0399 | -0.0033 | -0.0199 | 0.3730 |
| 0.5 | 0.0461 | 0.0081 | -0.0582 | -0.0829 | 0.4122 |
| 0.6 | 0.0396 | -0.0201 | -0.1198 | -0.1551 | 0.4555 |
| 0.7 | 0.0459 | -0.0465 | -0.1965 | -0.2476 | 0.5034 |
| 0.8 | 0.0684 | -0.0715 | -0.2983 | -0.3747 | 0.5564 |
| 0.9 | 0.1125 | -0.0944 | -0.4363 | -0.5542 | 0.6149 |
| 1.0 | 0.1854 | -0.1130 | -0.6228 | -0.8074 | 0.6796 |

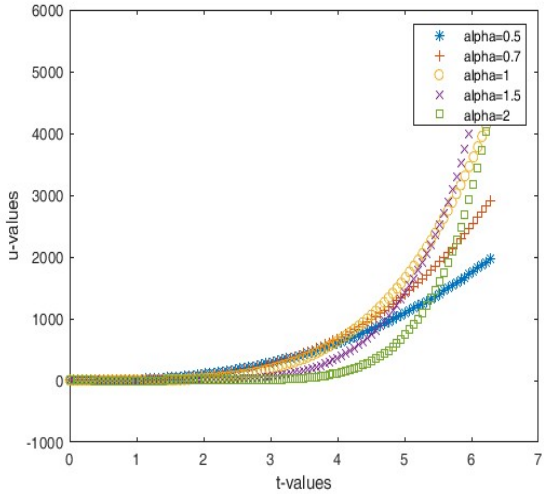


Figure 7: 2D solution depict example 4.4.

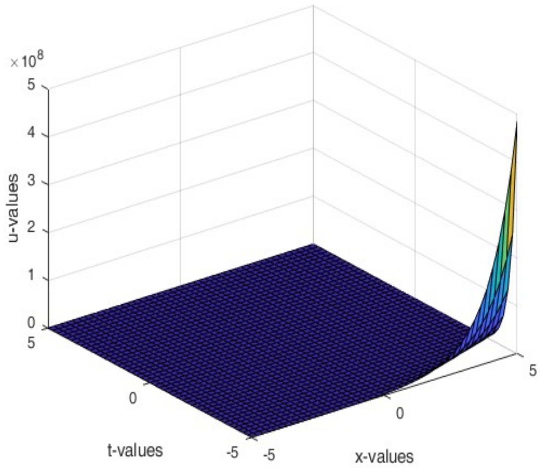


Figure 8: 3D solution example 4.4, $\alpha = 1$.

Table 6: Solutions of example 4.4 for t=0.5.

| $x$ | $\alpha = 1.1$ | $\alpha = 1.2$ | $\alpha = 1.5$ | $\alpha = 1.7$ | $\alpha = 2$ |
|---|---|---|---|---|---|
| 0.0 | 0.6065 | 0.6065 | 0.6065 | 0.6065 | 0.6065 |
| 0.1 | 0.7098 | 0.6986 | 0.6796 | 0.6737 | 0.6703 |
| 0.2 | 0.8129 | 0.7932 | 0.7573 | 0.7449 | 0.7408 |
| 0.3 | 0.9125 | 0.8845 | 0.8327 | 0.8144 | 0.8187 |
| 0.4 | 1.0087 | 0.9715 | 0.9026 | 0.8785 | 0.9048 |
| 0.5 | 1.1022 | 1.0540 | 0.9649 | 0.9344 | 1.0000 |
| 0.6 | 1.1938 | 1.1323 | 1.0181 | 0.9795 | 1.1052 |
| 0.7 | 1.2848 | 1.2072 | 1.0611 | 1.0118 | 1.2214 |
| 0.8 | 1.3765 | 1.2795 | 1.0934 | 1.0297 | 1.3499 |
| 0.9 | 1.4703 | 1.3505 | 1.1147 | 1.0317 | 1.4918 |
| 1.0 | 1.5679 | 1.4214 | 1.1250 | 1.0166 | 1.6487 |