General Integral Transform Performance for Space-Time Fractional Telegraph Equations

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Abstract: The development of technology has supported effective tools in industrial machines and set up the remarkable phase that serves well-being such as kinetic energy, kinetic movement, and nuclear energy. Applied mathematics has also contributed valuable procedures in various fields of these sciences, especially the creation of transformation. With practical relevance, a new general integral (NGI) transform has also shown a crucial role in the same pragmatic methods. In this paper, the NGI transform using the combination of Padé approximation including continued fraction expansions (CFE) has been used to attain approximate solutions of space-time fractional telegraph equations by directly getting the inverse transform.

Key-Words: New general integral transform; Padé approximation; Continued fraction expansions (CFE)

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1 Introduction

Transformation is an increasingly astounding variation created and applied in solving partial differential equations, [1]. In recent years, transform has been shown not only in different fields of mathematics but also in other majors such as physics, chemistry, and technology, [2]. The following decades produced a lot of new transformations, for example, the Shehu transformation was also created and applied to find the approximate solutions, [3]; the Sumudu transformation is used to solve delay fractional Bagley-Torvik equations shown combining efficient method, [4]; the Elzaki transformation is created from Laplace transform and applied in finding solutions of partial differential equations, [5], [6]; the natural transform (N-transform) was invented by, [7], and was performed in solving unstable fluid flow problems; the Aboodh transforms, [8], and new $\alpha-$Integral Laplace transform, [9], derived from Laplace transform are useful tools combining the Homotopy method for finding the exact solution of various differential equations. Following this stream, Pourreza transforms, [10], and Mohand transforms, [11], set a new expansion to solve higher differential equations with constant coefficients. Sawi transforms, [12], and Kamal transforms, [13], are new structures of Fourier integration establishing new methods to find the zeros of partial differential equations. G_transform was introduced, [14], [15], as a state-of-the-art method to evaluate the improper integral. However, it also has some limitations as shown in, [16]. Almost all transforms were introduced in L^1 or L^2 space. H-transform followed by the sequences of relatives was written in, [17], and has been generalized with hypergeometric and Bessels-typed kernels related to fractional differential equations. H-Transform, a whole transformation motivated by H-functions kernels in class Lebesgue functions, has set up an expansion on the Space $L_{\nu,r}$ for $1 \leq r \leq \infty$ on the upper half-space with the types of Bessel Integral Transforms and Hypergeometric Integral Transforms. The role of the H-function, a form banked on Mellin-Barnes integrals, has been illustrated in, [18], and applied in partial differential equations with rational coefficients.

In this paper, a new general integral (NGI) has been performed to find solutions of fractional differential equations as shown in equation (1).

$$\frac{\partial^{k\alpha}}{\partial t^{\alpha}} f(x,t) = \frac{\partial^{\beta}}{\partial t^{\beta}} f(x,t) + \frac{\partial}{\partial t} f(x,t) + f(x,t),$$
$$t \geq 0, 0 < \alpha \leq 2. \qquad (1)$$

satisfy the initial conditions $f(x,0) = h(x), f_t(x,0) = 0$, and $\frac{\partial^{\alpha}}{\partial t^{\alpha}}$ is fractional Caputo derivative with order α, $\frac{\partial^{\beta}}{\partial t^{\beta}}$ represents the integer or fractional Caputo derivative of the function.
The new general integral (NGI) transform, [19], is defined as the following:

$$\mathbb{T}[f(t)] = F(s) = p(s)\int_0^{\infty} e^{-q(s)t} f(t)dt. \qquad (2)$$

The function $f(t)$ is defined in the set

$$A = \{f(t) : \exists N, n_1, n_2 > 0, |f(t)| < Ne^{\frac{|t|}{n_i}},$$
$$\text{if} \quad t \in (-1)^i \times [0,\infty)]\}. \qquad (3)$$

The inverse new general integral transform of the function $f(t)$ is defined by

$$\mathbb{T}^{-1}F(s) = f(t), \quad for \quad t \geq 0. \qquad (4)$$

Another form is

$$u(t) = \mathbb{T}^{-1}[F(s)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{q(s)t}}{p(s)} F(s)ds. \qquad (5)$$

where s is the new general integral transform variable, and c is a real constant and the integral has got along the $s = c$ in the complex plane $s = x + iy$.

2 Fundamental Definitions

This section will perform some useful definitions applied in fractional differential equations and this paper.

Definition 1. *Gamma function, [20]:*

$$\Gamma(p) = \int_0^{\infty} e^{-x} x^{p-1} dx. \qquad (6)$$

Definition 2. *Beta function, [20]:*

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx \qquad (7)$$

where $\mathrm{Re}(p), \mathrm{Re}(q) > 0$.

Definition 3. *Mittag-Leffler Function, [20]:*

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \qquad \mathrm{Re}(\alpha) > 0. \qquad (8)$$

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0, \beta \in \mathbb{C}. \qquad (9)$$

$$E(t,\alpha,a) = \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(k+\alpha+1)} = t^{\alpha} E_{1,\alpha+1}(at). \qquad (10)$$

We have the following special values

$$E_{1,1}(z) = e^z; \quad E_{1,2}(z) = \frac{e^z - 1}{z}. \qquad (11)$$

$$E_{2,1}(z^2) = \cosh(z); \quad E_{2,2}(z^2) = \frac{\sinh z}{z}. \qquad (12)$$

Definition 4. *Hypergeometric functions, [20]:*

$${}_pF_q(a_1,..,a_p,b_1,..,b_q;z) = \frac{\Gamma(b_1)...\Gamma(b_q)}{\Gamma(a_1)...\Gamma(a_q)} \times \sum_{k=0}^{\infty} \frac{\Gamma(a_1+k)...\Gamma(a_p+k)z^k}{\Gamma(b_1+k)...\Gamma(a_q)}. \qquad (13)$$

In particular, $\mathrm{Re}\, c > \mathrm{Re}(a+b)$ and c is not a nonpositive integer

$${}_2F_1(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \qquad (14)$$

$${}_1F_1(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(c+k)} \frac{z^k}{k!}. \qquad (15)$$

Definition 5. *The incomplete gamma function, [20]:*

The incomplete gamma function $\gamma^*(v,z)$ is defined

$$\gamma^*(v,z) = e^{-z} \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(v+k+1)}. \qquad (16)$$

If $\mathrm{Re}\, z > 0$, then $\gamma^*(v,z)$ has the integral presentation

$$\gamma^*(v,z) = \frac{1}{\Gamma(v) z^v} \int_0^z t^{v-1} e^{-t} dt. \qquad (17)$$

The special forms of the confluent hypergeometric functions:

$$\gamma^*(v,t) = \frac{e^{-t}}{\Gamma(v+1)} \, {}_1F_1(1, v+1; t). \qquad (18)$$

An alternative form

$$\gamma^*(v,t) = \frac{1}{\Gamma(v+1)} \, {}_1F_1(1, v+1; -t). \qquad (19)$$

when $t = 0$ then

$$\gamma^*(v,0) = \frac{1}{\Gamma(v+1)}. \tag{20}$$

Considering p is a non-negative integer, we can readily deduce in the form

$$\gamma^*(p,at) = e^{-at}\sum_{k=p}^{\infty}\frac{(at)^{k-p}}{k!}. \tag{21}$$

and

$$\gamma^*(-p,at) = (at)^p.$$

In particular, we have

$$\gamma^*(1,at) = \frac{1-e^{-at}}{at};\ \gamma^*(0,at) = 1;\ \gamma^*(-1,at) = at. \tag{22}$$

Now we define the function

$$E_t(v,a) = z^v e^{az}\gamma^*(v,z). \tag{23}$$

The properties of the $E_t(v,a)$ may be determined by

$$E_t(v,a) = t^v\sum_{k=0}^{\infty}\frac{(at)^k}{\Gamma(v+k+1)}. \tag{24}$$

We have the following special values

$$E_t(0,a) = e^{at};\ E_0(v,a) = 0; E_t(-1,a) = aE_t(0,a);$$
$$E_t(1,a) = \frac{E_t(0,a)-1}{a}; E_t(v,0) = \frac{t^v}{\Gamma(v+1)}. \tag{25}$$

3 Transformation of Basic Functions

Some fundamental functions using new general integral transformation, [19], are shown in Table 1, and some transcendental functions are summarized in the form of propositions derived from, [20], as the following:

Proposition 1. *The inverse NGT derived from, [21], of special functions as follows*

$$\mathbb{T}^{-1}\left[\frac{p(s)q(s)^{\alpha-\beta}}{q(s)^{\alpha}-\lambda}\right] = t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha}),$$
$$\text{for } |\lambda| < |q(s)^{\alpha}|. \tag{26}$$

Proposition 2. *The inverse NGT for specific case is derived from, [20]:*

$$\mathbb{T}^{-1}\left[\frac{p(s)}{q(s)\left(q(s)^{\alpha}-a\right)^2}\right]$$
$$= \sum_{j=1}^{q}\sum_{k=1}^{q}a^{j+k-2}\Big\{tE_t\left[(j+k)\alpha-1,a^q\right]$$
$$-\left[(j+k)\alpha-1\right]E_t\left[(j+k)\alpha,a^q\right]\Big\}, \alpha = \frac{1}{q}, q\in Z_+. \tag{27}$$

Proposition 3. *The inverse NGT for the general case is derived from, [20]:*

$$\mathbb{T}^{-1}\left[\frac{p(s)}{q^u(s)[q(s)^{\alpha}-a]^n}\right]$$
$$= \sum_{j_1=1}^{q}\cdots\sum_{j_n=1}^{q}\frac{a^{J-n}}{(n-1)!\Gamma(u-n+J\alpha)}$$
$$\times\sum_{k=0}^{n-1}(-1)^k C_k^{n-1}\Gamma(u-n+J\alpha+k)$$
$$\times t^{n-1-k}E_t(u-n+J\alpha+k,a^q). \tag{28}$$

where $J = \sum_{i=1}^{n} j_i$; $\alpha = \frac{1}{q}, q\in Z_+, C_k^n = \frac{n!}{k!(n-k)!}$.

Table 1: Special definition of the newfangled Laplace-typed NGLT transform.

Cases	Transform
1	$\frac{p(s)}{q(s)}$
t	$\frac{p(s)}{q(s)^2}$
t^{α}	$\frac{\Gamma(\alpha+1)p(s)}{q(s)^{\alpha+1}}$
$\sin t$	$\frac{p(s)}{q(s)^2+1}$
$\sin at$	$\frac{ap(s)}{q(s)^2+a^2}$
$\cos t$	$\frac{q(s)p(s)}{q(s)^2+1}$
$\cos at$	$\frac{aq(s)p(s)}{q(s)^2+a^2}$
e^t	$\frac{p(s)}{q(s)-1}$
$tf(t-1)$	$\frac{e^{-q(s)}(q(s)+1)p(s)}{q(s)^2}$
te^{at}	$\frac{p(s)}{(p-a)^2}$
$f'(t)$	$q(s)F(s)-p(s)f(0)$
$f^{(\alpha)}(t)$	$q^{\alpha}(s)F(s)-p(s)\sum_{k=0}^{n-1}q^{\alpha-(k+1)}(s)f^{(k)}(0)$

Theorem 3.1. Caputo's Fractional Derivatives, [21], [22]: Considering a continuous function $y = f(t)$, and an arbitrary order $n-1 < \alpha \leq n$, the Caputo's fractional derivative of order α is given by:

$${}_a^C\mathbb{D}_x^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x (x-s)^{n-\alpha-1}f^{(n)}(s)ds. \tag{29}$$

Theorem 3.2. Residue theorem (Cauchy's residue theorem), [23], [24], [25]. Let C be a simple closed contour, described positively. If a function f is analytic, then

$$f(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{-q(s)t}}{p(s)}F(x,s)ds$$
$$= \sum_{j=1}^{m}Res[f(x,z_j),z_j] \text{ in u.h.p.} \tag{30}$$

where z_j is a finite number of isolated singularities in the upper half-plane (u.h.p) in the complex plane $z = x+iy$, and $f(x,z)$ is the function under the integration, and if $f(z)$ has a pole of order N at $z = z_0, N \geq m$, we use the particular expression as follows:

$$Res[f(z),z_0] = \lim_{z\to z_0} \frac{1}{(N-1)!}\frac{d^{N-1}}{dz^{N-1}}[(z-z_0)^N f(z)] \text{ in u.h.p.} \tag{31}$$

Theorem 3.3. CFE (Continued Fraction Expansions) approximation method, [26], [27]:
Continued Fraction Expansions is one of the prestigious theories applied to find the approximate expansion, especially in fractional order derivatives. The main idea is to express the infinitive expansion of truncated form as the following:

$$\begin{aligned}(1+x)^\alpha &= 1 + \cfrac{\alpha x}{1+\cfrac{(1-\alpha)x}{2+\cfrac{(1+\alpha)x}{3+\cfrac{(2-\alpha)x}{2+\cfrac{(2+\alpha)x}{5+\cfrac{(3-\alpha)x}{2+\cfrac{(3+\alpha)x}{7+\cdots}}}}}}} \\ &= b_0 + \cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\cfrac{a_3}{b_3+\cfrac{(a_4}{b_4+\cfrac{a_5}{b_5+\cfrac{(a_6}{b_6+\cfrac{a_7}{b_7+\cdots}}}}}}}\end{aligned} \tag{32}$$

$$(1+x)^\alpha = b_0 + \frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+\cdots}, \tag{33}$$

where $b_0 = 1, b_2 = 2, b_{i+1} = i+1, a_1 = \alpha x, a_i = (k-\alpha)x, a_{i+1} = (k+\alpha)x$ for $i = 2k, k \geq 1$.
Following this, we have the approximate for $(1+x)^\alpha$:

$$(1+x)^\alpha \approx 1 + \frac{\alpha x}{1+\frac{(1-\alpha)x}{2}} \approx \frac{2+(1+\alpha)x}{2+(1-\alpha)x}. \tag{34}$$

By taking $s = x-1$, we have the first-order and the second-order CFE approximate

$$s^\alpha \approx \frac{(1+\alpha)s+(1-\alpha)}{(1-\alpha)s+(1+\alpha)}. \tag{35}$$

$$s^\alpha \equiv \frac{(\alpha^2+3\alpha+2)s^2+(-2\alpha^2+8)s+(\alpha^2-3\alpha+2)}{(\alpha^2-3\alpha+2)s^2+(-2\alpha^2+8)s+(\alpha^2+3\alpha+2)}. \tag{36}$$

Theorem 3.4. Padé-approximation, [28], [29]: A Padé approximation with real coefficients whose numerator and denominator have degrees L and M is performed

$$[\frac{L}{M}] = \frac{a_0+a_1s+a_2s+\cdots+a_Ls^L}{b_0+b_1s+b_2s+\cdots+b_Ls^M} = \sum_{i=0}^{\infty} c_i s^i. \tag{37}$$

where c_i satisfies the following simultaneous equations

$$\begin{aligned}&a_0 = c_0;\\ &a_1 = c_1+c_0;\\ &a_2 = c_2+b_1c_1+b_2c_0;\\ &\quad\cdots\\ &a_l = c_l + \sum_{i=0}^{min\{L,M\}} b_i c_{L-i}.\end{aligned} \tag{38}$$

For example, the second-order approximation is truncated as

$$\frac{a_0+a_1s+a_2x^2}{1+b_1s+b_2s^2} \approx c_0+c_1s+c_2s^2, \tag{39}$$

where $c_0 = a_0, a_1 = c_1+c_0, a_2 = c_2+b_1c_1+b_2c_0$.
Lemma 3.5. Using Theorem 3.3 and Theorem 3.4 for the second-order expression with combination of equation (36) and equation (39), we derived the approximation

$$\begin{aligned}s^\alpha \approx &\frac{(\alpha^2-3\alpha+2)}{(\alpha^2+3\alpha+2)} - \frac{3(\alpha-2)}{\alpha+2}s \\ &-\frac{6(\alpha^4-3\alpha^3-6\alpha^2+8)}{(\alpha^2+3\alpha+2)^2}s^2.\end{aligned} \tag{40}$$

4 Applications

Example 4.1. Consider the space-time fractional telegraph equation

$$\frac{\partial^\alpha}{\partial t^\alpha} f(x,t) = 2f(x,t), t \geq 0, 1 \leq \alpha \leq 2, \tag{41}$$

$$f(x,0) = e^x, f_t(x,0) = 0, t \geq 0, x > 0. \tag{42}$$

Solution
Taking the NGI transform on both sides equation (41) using Table 1, using initial value conditions (42), and simplify as follows

$$\begin{aligned}[q^\alpha(s)-2]F(x,s) = p(s)q(s)^{\alpha-1}F(x,0).\\ +p(s)q(s)^{\alpha-2}F_t(x,0)\end{aligned} \tag{43}$$

Simplify on both sides of equation (43), we have

$$F(x,s) = \frac{p(s)q(s)^{\alpha-1}e^x}{q^\alpha(s)-2}. \tag{44}$$

Taking inverse NGI transform equation (44) and using propositional Proposition 1, we obtain the solution

$$f(x,t) = E_{\alpha,1}(2t^\alpha)e^x. \tag{45}$$

When $\alpha = 2$ the exact solution becomes

$$f(x,t) = \cosh(\sqrt{2}t)e^x. \tag{46}$$

The solution (46) is depicted in Figure 1, and Figure 2 (Appendix).

Example 4.2. Consider the following space-time fractional telegraph equation

$$\frac{\partial^{2\alpha}}{\partial t^{2\alpha}} f(x,t) = 2\frac{\partial^{\alpha}}{\partial t^{\alpha}} f(x,t) - f(x,t), \quad 0 < \alpha \leq 1, \tag{47}$$

$$f(x,0) = e^x, f_t(x,0) = 0, t \geq 0, 0 < x \leq 1. \tag{48}$$

Solution
Taking the NGI transform equation (47) in Table 1, using initial value condition (48), we have

$$\left[(q^\alpha(s)-1)^2\right] F(x,s) = p(s)\left[q(s)^{2\alpha-1} - 2q(s)^{\alpha-1}\right] e^x. \tag{49}$$

Simplify the equation (49), we have

$$\begin{aligned} F(x,s) &= \frac{p(s)\left[q(s)^{2\alpha-1} - 2q(s)^{\alpha-1}\right] e^x}{(q^\alpha(s)-1)^2} \\ &= \frac{p(s)}{q(s)} e^x - \frac{p(s)e^x}{(q^\alpha(s)-1)^2}. \end{aligned} \tag{50}$$

Taking inverse NGI transform equation (50) and apply the Proposition 2:

$$\begin{aligned} f(x,t) = e^x - \sum_{j=1}^{q}\sum_{k=1}^{q} \Big\{ & tE_t\left[(j+k)\alpha - 1, 1\right] \\ & - \left[(j+k)\alpha - 1\right] E_t\left[(j+k)\alpha, 1\right] \Big\} e^x, \end{aligned} \tag{51}$$

where $\alpha = \frac{1}{q}, q \in Z_+$.

When $\alpha = 1$, the exact solution attained is

$$f(x,t) = e^x - e^x[tE_t(1,1) - E_t(2,2)]. \tag{52}$$

or in the form of $f(x,t) = e^{x+t}(1-t)$. The solution (52) is performed in Figure 3, and Figure 4 (Appendix).

Example 4.3. Consider the space-time fractional telegraph equation

$$\frac{\partial^{3\alpha}}{\partial t^{3\alpha}} f(x,t) = 6\frac{\partial^{2\alpha}}{\partial t^{2\alpha}} f(x,t) - 12\frac{\partial^{\alpha}}{\partial t^{\alpha}} f(x,t) + 8f(x,t), \quad 0 < \alpha \leq 1. \tag{53}$$

$$f(x,0) = e^x, f_t(x,0) = 0, f_{tt}(x,0) = 0, t \geq 0, 0 < x \leq 1. \tag{54}$$

Solution
Taking NGT on both sides equation (53) using condition (54), we have

$$\mathbb{T}\left[\frac{\partial^{3\alpha}}{\partial t^{3\alpha}} f(x,t)\right] = \mathbb{T}\left[6\frac{\partial^{2\alpha}}{\partial t^{2\alpha}} f(x,t) - 12\frac{\partial^{\alpha}}{\partial t^{\alpha}} f(x,t) + 8f(x,t)\right]. \tag{55}$$

Simplify the equation (55), we have

$$\left[(q^\alpha(s)-2)^3\right] F(x,s) = p(s)\left[q(s)^{3\alpha-1} - 6q(s)^{2\alpha-1} + 12q(s)^{\alpha-1}\right] e^x. \tag{56}$$

Equation (56) can be reduced as follows

$$\begin{aligned} F(x,s) &= \frac{p(s)\left[q(s)^{3\alpha} - 6q(s)^{2\alpha} + 12q(s)^{\alpha}\right] e^x}{q(s)(q^\alpha(s)-2)^3} \\ &= \frac{p(s)}{q(s)} e^x + \frac{8p(s)e^x}{q(s)(q^\alpha(s)-2)^3}. \end{aligned} \tag{57}$$

Using Proposition 3, from equation (57), we get the solution:

$$\begin{aligned} f(x,t) = e^x + & \sum_{j_1=1}^{q}\sum_{j_2=1}^{q}\sum_{j_3=1}^{q} \frac{2^{J-3}}{2\Gamma(J\alpha-2)} \times \\ & \sum_{k=0}^{2} (-1)^k C_2^k \Gamma(J\alpha+k-2) \\ & \times t^{2-k} E_t(J\alpha+k-2, 2^q) e^x. \end{aligned} \tag{58}$$

where $J = \sum_{i=1}^{3} j_i$; $\alpha = \frac{1}{q}, q \in Z_+, C_2^k = \frac{2!}{k!(2-k)!}, 0 \leq k \leq 2$.
When $\alpha = 1$ the exact solution becomes

$$f(x,t) = e^x + 4e^x[t^2E_t(1,2) - 2tE_t(2,2) + 2E_t(3,2)]. \tag{59}$$

The solution (59) is portrayed in Figure 5, and Figure 6 (Appendix).

Example 4.4. Consider the approximate solution of the space-fractional telegraph equations , [30], [31], [32], [33]:

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} f(x,t) = \frac{\partial^{2}}{\partial t^{2}} f(x,t) + \frac{\partial}{\partial t} f(x,t) + f(x,t), \quad t \geq 0, 1 < \alpha \leq 2, \tag{60}$$

subjected to the initial condition

$$f(x,0) = e^x, f_t(x,0) = -e^{-t}, 0 < x \leq 1. \tag{61}$$

Solution
Taking NGI transform on both sides equation (60), and condition (61), the equation is turned into

$$\begin{aligned} [q(s)^\alpha - q(s)^2 - q(s) - 1]F(x,s) = p(s)[q(s)^{\alpha-1} \\ - q(s)^{\alpha-2} - s]f(x,0). \end{aligned} \tag{62}$$

$$F(x,s) = \frac{p(s)[q(s)^{\alpha-1} - q(s)^{\alpha-2} - s]}{q(s)^\alpha - q(s)^2 - q(s) - 1} e^x. \tag{63}$$

Taking inverse NGI transformation on both sides equation (63), and applying the Theorem 3.2 to find the solution for the general value α:

$$f(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{q(s)t+x}}{p(s)} \frac{p(s)[q(s)^{\alpha-1} - q(s)^{\alpha-2} - s]}{[q(s)^{\alpha} - q(s)^2 - q(s) - 1]} e^x ds \tag{64}$$

$$= \sum_{j=1}^{m} Res[f(x,q(s)),q(s)_j]. \tag{65}$$

where $q(s_j), j = 1,\cdots,m$ be the isolated singularities in the domain and f(x,q(s)) is the function under the integration.
We find the solution of the equation $q(s)^{\alpha} - q(s)^2 - q(s) - 1$ by apply Lemma 3.5, and getting two solutions $q(s_1), q(s_2)$ from the approximate expression:

$$h[q(s)] = -\frac{(\alpha-2)^2\left(7\alpha^2+16\alpha+13\right)}{\left(\alpha^2+3\alpha+2\right)^2}s^2 - 4\frac{\alpha-1}{\alpha+2}s - 6\frac{\alpha}{\alpha^2+3\alpha+2}. \tag{66}$$

Finally, the approximate solution is gathered from the equation (64) shown in Table 2 (Appendix). The expressions of $q(s_1), q(s_2)$ is shown in Table 3 (Appendix).
When $\alpha = 2$, we gain the exact solution:

$$f(x,t) = e^{x-t}. \tag{67}$$

The solution (73) is demonstrated in Figure 7, and Figure 8 (Appendix).

Example 4.5. Considering the approximate solution of the space-fractional telegraph equations, [30], [31], [32], [33]:

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} f(x,t) = \frac{\partial^2}{\partial t^2} u(x,t) + \frac{\partial}{\partial t} f(x,t) + f(x,t) - x^2 - t + 1, \tag{68}$$

subjected to the initial condition

$$f(x,0) = x^2, f_t(x,0) = 0, 0 < x \leq 1, t \geq 0, 0 < \alpha \leq 1. \tag{69}$$

Solution
Taking NGI transform on both sides of the equation (68), using equation (69), the equation becomes

$$[q(s)^{\alpha} - q(s)^2 - q(s) - 1]F(x,s) = p(s)[q(s)^{\alpha-1} - q(s) - 1]f(x,0) - \frac{p(s)}{q(s)^2} + (1-x^2)\frac{p(s)}{q(s)}. \tag{70}$$

Simplify on both sides of equation (70), we attain

$$F(x,s) = \frac{p(s)}{q(s)}x^2 + \frac{p(s)[q(s)-1]}{[q(s)^{\alpha} - q(s)^2 - q(s) - 1]q(s)^2}. \tag{71}$$

We find the solution of the equation $q(s)^{\alpha} - q(s)^2 - q(s) - 1$ by apply Lemma 3.5 to get the solutions $q(s_1), q(s_2)$, two simple simple poles, given by equations (74) and equation (75), and $q(s) = 0$ is the pole of order 2 of the denominator of equation (71):

$$\begin{aligned} f(x,t) &= x^2 \\ &+ \frac{-\left(\alpha^2+3\alpha+2\right)^2 (q(s_1)-1)e^{q(s_1)t}}{(\alpha-2)^2\left(7\alpha^2+16\alpha+13\right)[q(s_1)-q(s_2)]q(s_1)^2} \\ &+ \frac{-\left(\alpha^2+3\alpha+2\right)^2 (q(s_2)-1)e^{q(s_2)t}}{(\alpha-2)^2\left(7\alpha^2+16\alpha+13\right)[q(s_2)-q(s_1)]q(s_2)^2} \\ &+ \frac{\partial}{\partial q(s)}\left[\frac{q(s)-1}{h[q(s)]}\right]_{q(s)=0}. \end{aligned} \tag{72}$$

where $h[q(s)]$ given by equation (66).
When $\alpha = 2$, from equation (71) we have an exact solution $f(x,t) = x^2 + t^2$, [30], [31], [32], [33].
The solution (72) is shown in Figure 9, and Figure 10 (Appendix).

5 Conclusion

In this paper, a new general integral transform (NGI) for solving space-time fractional telegraph equations has been constructed by taking an inverse NGI transform based on the present formulas. Some particular functions are cumbersome to use by inverse NGI transform, so we apply the Padé approximate procedure including continued fraction expansion to evaluate the polynomial expansions. The limitation of the Laplace-typed transform is the polynomial expression and it is hard to take inverse transform directly. However, they are also useful in the case of using inverse integral transform by establishing the sequence formula of the transform. The results showed the approximate solutions approaching exact solutions including the order CFE model of s^{α} and the [L/M] Padé approximation.

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APPENDIX

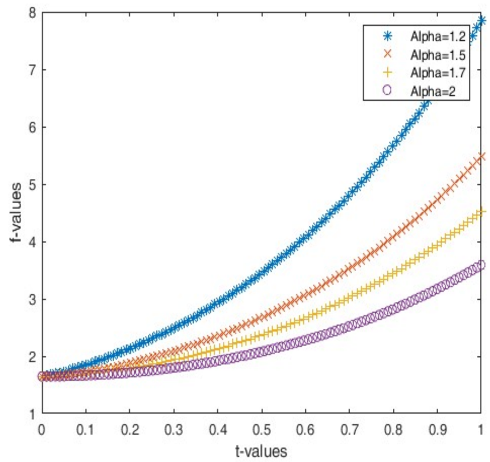

Figure 1: 2D solution performance of example 4.1.

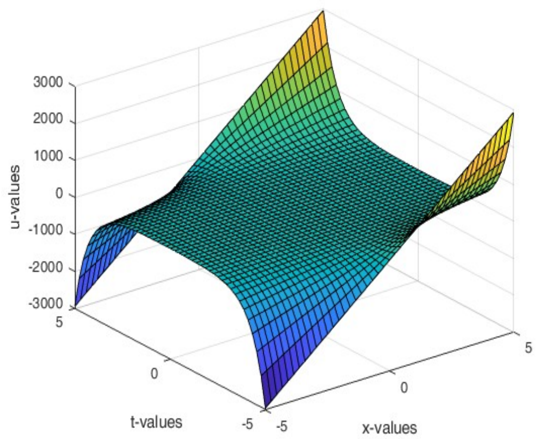

Figure 2: 3D solution of example 4.1 when $\alpha = 2$.

Table 2: Solution of example 4.4.

$$f(x,t) = \frac{\left(\alpha^2+3\alpha+2\right)^2 e^{q(s_1)t+x}[q(s_1)^{\alpha-1}+q(s_1)^{\alpha-2}-q(s_1)]}{(\alpha-2)^2\,(7\alpha^2+16\alpha+13)\,[q(s_1)-q(s_2)]} + \frac{\left(\alpha^2+3\alpha+2\right)^2 e^{q(s_2)t+x}[q(s_2)^{\alpha-1}+q(s_2)^{\alpha-2}-q(s_2)]}{(\alpha-2)^2\,(7\alpha^2+16\alpha+13)\,[q(s_2)-q(s_1)]}. \tag{73}$$

Table 3: Form of $q(s_1), q(s_2)$.

$$q(s_1) = \frac{\left(3\alpha+\alpha^2+2\right)^2}{(\alpha-2)^2\,(\alpha+2)\,(16\alpha+7\alpha^2+13)} \times \left(\begin{array}{c} -2\alpha+8\sqrt{2}\sqrt{\frac{1}{(\alpha+1)^3(\alpha+2)^7}\,(-150\alpha-42\alpha^2+57\alpha^3+36\alpha^4-15\alpha^5+2\alpha^6+4)} \\ +12\sqrt{2}\alpha\sqrt{\frac{1}{(\alpha+1)^3(\alpha+2)^7}\,(-150\alpha-42\alpha^2+57\alpha^3+36\alpha^4-15\alpha^5+2\alpha^6+4)} \\ +6\sqrt{2}\alpha^2\sqrt{\frac{1}{(\alpha+1)^3(\alpha+2)^7}\,(-150\alpha-42\alpha^2+57\alpha^3+36\alpha^4-15\alpha^5+2\alpha^6+4)} \\ +\sqrt{2}\alpha^3\sqrt{\frac{1}{(\alpha+1)^3(\alpha+2)^7}\,(-150\alpha-42\alpha^2+57\alpha^3+36\alpha^4-15\alpha^5+2\alpha^6+4)}+2 \end{array} \right) \tag{74}$$

and

$$q(s_2) = -\frac{\left(3\alpha+\alpha^2+2\right)^2}{(\alpha-2)^2\,(\alpha+2)\,(16\alpha+7\alpha^2+13)} \times \left(\begin{array}{c} 2\alpha+8\sqrt{2}\sqrt{\frac{1}{(\alpha+1)^3(\alpha+2)^7}\,(-150\alpha-42\alpha^2+57\alpha^3+36\alpha^4-15\alpha^5+2\alpha^6+4)} \\ +12\sqrt{2}\alpha\sqrt{\frac{1}{(\alpha+1)^3(\alpha+2)^7}\,(-150\alpha-42\alpha^2+57\alpha^3+36\alpha^4-15\alpha^5+2\alpha^6+4)} \\ +6\sqrt{2}\alpha^2\sqrt{\frac{1}{(\alpha+1)^3(\alpha+2)^7}\,(-150\alpha-42\alpha^2+57\alpha^3+36\alpha^4-15\alpha^5+2\alpha^6+4)} \\ +\sqrt{2}\alpha^3\sqrt{\frac{1}{(\alpha+1)^3(\alpha+2)^7}\,(-150\alpha-42\alpha^2+57\alpha^3+36\alpha^4-15\alpha^5+2\alpha^6+4)}-2 \end{array} \right) \tag{75}$$

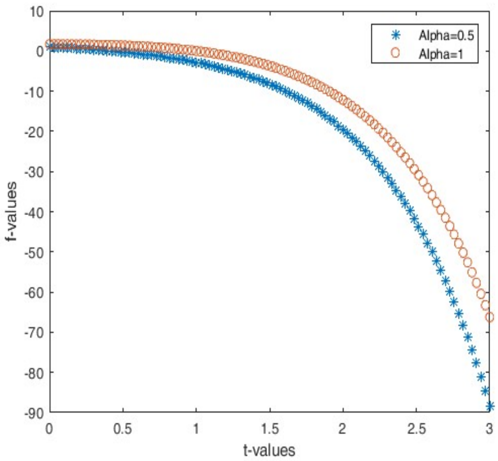

Figure 3: 2D solution performance of example 4.2.

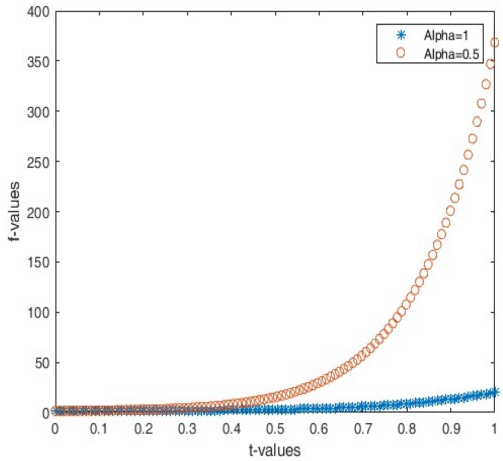

Figure 5: 2D solution of example 4.3.

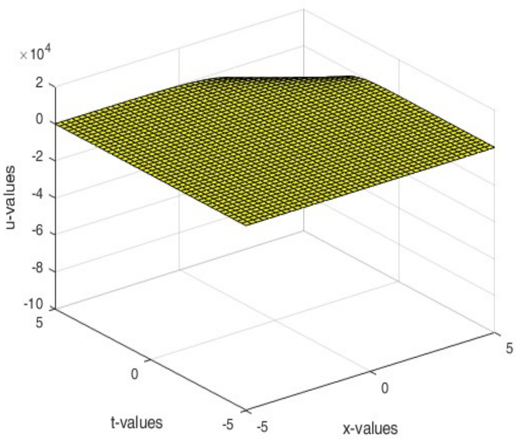

Figure 4: 3D solution of example 4.2 when $\alpha = 1$.

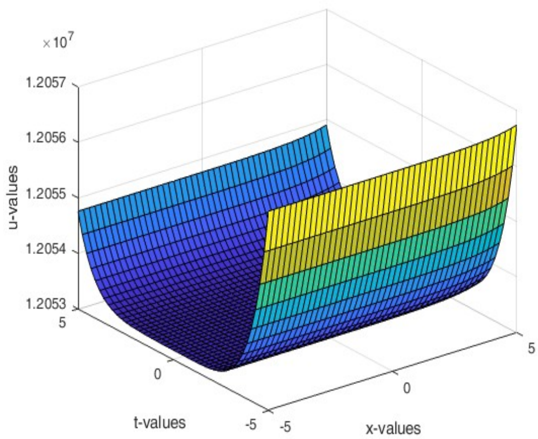

Figure 6: 3D solution of example 4.3 when $\alpha = 1$.

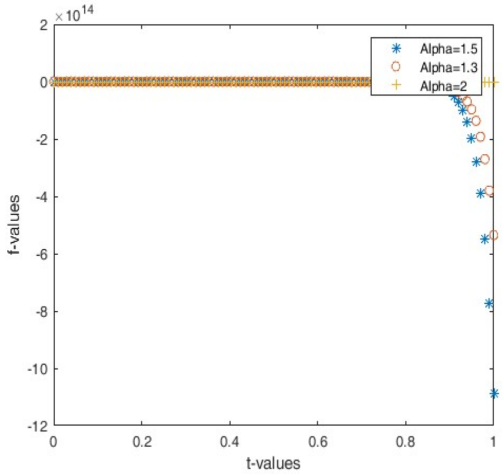

Figure 7: 2D solution performance for example 4.4.

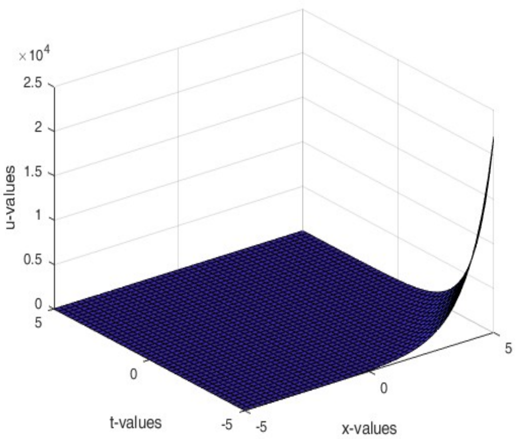

Figure 8: 3D solution for example 4.4 when $\alpha = 2$.

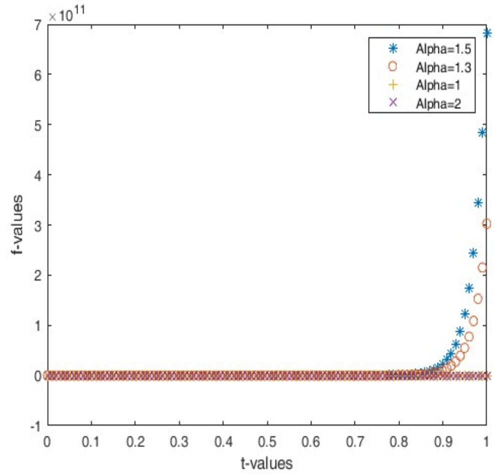

Figure 9: 2D solution performance for example 4.5.

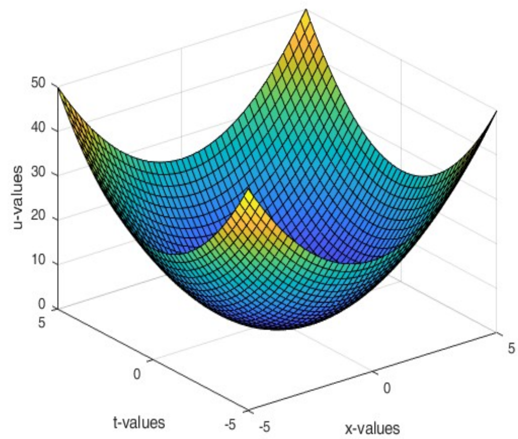

Figure 10: 3D solution for example 4.5 when $\alpha = 2$.