Characterization of Tubular Surfaces in Terms of Finite *III*-type

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Abstract: - In this paper, we first define relations regarding the first and the second Laplace operators corresponding to the third fundamental form *III* of a surface in the Euclidean space E^3. Then, we will characterize the tubular surfaces in terms of their coordinate finite type.

Key-Words: - Surfaces in the Euclidean 3-space, Surfaces of finite Chen-type, Laplace operator, Tubular surfaces, third fundamental form, Anchor ring.

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1 Introduction

Tubular surfaces are a fascinating class of geometric objects that arise in differential geometry. This kind of surface can be seen as the result of sweeping a curve through the Euclidean space, constructing a "tube" around the curve which is considered as the direction of the tube. Studying this class of surfaces, namely the tubular surfaces, is fundamental in many branches of mathematics and is applicable in physics, engineering, and computer graphics.

Normal bundle is one of the essential concepts associated with tubular surfaces. In the Euclidean 3-space, the normal bundle of a curve consists of vectors that are orthogonal to the curve at each point. By extending these vectors, one can create a tubular neighborhood along the curve. The radius of this neighborhood, or the size of the tube, is a crucial parameter that influences the geometry of the tubular surface.

Immersions of finite Chen type, introduced by B.-Y. 50 years ago, [1] and has become a significant topic of active research in the field of differential geometry. Surfaces of finite Chen type encompass diverse surfaces that exhibit certain geometric properties. Examples of surfaces that fall under this category include immersions with vanishing Gauss curvature, minimal surfaces, and various special classes of surfaces, such as tubes [2], quadrics [3], [4], [5], translation surfaces [6], [7], ruled surfaces [8], [9], [10], surfaces of revolution [11], [12], [13], [14], [15], spiral surfaces [16], cyclides of Dupin [17], [18] and helicoidal surfaces [19], [20]. These classes represent various special cases of surfaces that fall under the umbrella of finite Chen type. Each of these classes has its distinctive geometric features.

For a connected surface Q in Euclidean 3-space E^3, described by coordinates v^1, and v^2, the 1^{st}, 2^{nd}, and 3^{rd} fundamental forms are represented by (g_{ij}), (b_{ij}), and (e_{ij}) respectively.

The first fundamental form (g_{ij}) is associated with the metric tensor of the surface, representing lengths and angles on the surface. The second fundamental form (b_{ij}) is related to the shape operator and provides information about the extrinsic curvature of the surface. The third fundamental form (e_{ij}) is associated with the derivatives of the unit normal vector to the surface.

The 1^{st} differential parameter of Beltrami is a mathematical quantity associated with surfaces in differential geometry. Now, let's consider two functions γ and δ defined on the surface Q. The 1^{st} differential parameter of Beltrami concerning the fundamental form $J = I, II, III$ between these two functions is defined as:

$$\nabla^J(\gamma,\delta): = c^{ij}\, \gamma_{/i}\, \delta_{/j},$$

where $\gamma_{/i} := \frac{\partial \gamma}{\partial v^i}$ and (c^{ij}) represents the inverse tensor of (g_{ij}), (b_{ij}), and (e_{ij}). The 2nd differential parameter of Beltrami regarding the fundamental form J of Q is defined by:

$$\Delta^J \gamma := -\frac{1}{\sqrt{c}}(\sqrt{c}c^{ij}\gamma_{/i})_{/j},\ c = \det(c_{ij}).$$

For the position vector $\boldsymbol{z} = \boldsymbol{z}(v^1, v^2)$, of Q in E^3 we have the following relation:

$$\Delta^{III}\boldsymbol{z} = -\nabla^{II}\left(\frac{2H}{K}, \boldsymbol{z}\right) - \frac{2H}{K}\boldsymbol{N},$$

where $\boldsymbol{N}$ is the unit normal vector field, K is the Gauss curvature, and H is the mean curvature of Q. It was subsequently demonstrated that a surface meeting this criterion:

$$\Delta^{III}\boldsymbol{z} = \lambda \boldsymbol{z}, \quad \lambda \in \mathbb{R},$$

i.e. the statement asserts that if Q: $\boldsymbol{z} = \boldsymbol{z}(v^1, v^2)$ satisfies this condition, where all coordinate functions are eigenfunctions of Δ^{III} with eigenvalue λ is the same, then Q is either a part of a sphere (with $\lambda = 2$) or a minimals (with $\lambda = 0$). In other words, this condition provides a geometric characterization of surfaces based on the behavior of their Laplace-Beltrami eigenfunctions. The eigenvalue λ being equal to 0 suggests a minimal surface, which is a surface with mean curvature equal to zero, while λ being equal to 2 suggests a spherical geometry.

2 Fundamentals

Consider the parametric representation

$$\boldsymbol{r}(x, y) = \{r_1(x, y), r_2(x, y), r_3(x, y)\},\ (x, y) \in \mathrm{B} \subset \mathbb{R}^2$$

Of a surface Q. Denote by:

$$\boldsymbol{r}_x = \frac{\partial \boldsymbol{r}}{\partial x},\quad \boldsymbol{r}_y = \frac{\partial \boldsymbol{r}}{\partial y},\quad \boldsymbol{r}_{xx} = \frac{\partial^2 x}{\partial x^2},\ \ldots$$

For the metric I of Q, it's known that:

$$I = Edx^2 + 2Fdxdy + Gdy^2.$$

Applying the Laplacian operator Δ^I, to a sufficiently differentiable function $\varphi(x, y)$ defined on the same region $\mathrm{D} \subset \mathbb{R}^2$ gives, [21]:

$$\Delta^I \varphi = -\frac{1}{\sqrt{EG - F^2}}\left[\left(\frac{G\varphi_x - F\varphi_y}{\sqrt{EG - F^2}}\right)_x - \left(\frac{F\varphi_x - E\varphi_y}{\sqrt{EG - F^2}}\right)_y\right].$$

The metric II of Q is:

$$II = Ldx^2 + 2Mdxdy + Ndy^2.$$

The Laplacian Δ^{II} is given by [21]:

$$\Delta^{II} \varphi = -\frac{1}{\sqrt{LN - M^2}}\left[\left(\frac{N\varphi_x - M\kappa_y}{\sqrt{LN - M^2}}\right)_x - \left(\frac{M\varphi_x - L\varphi_y}{\sqrt{LN - M^2}}\right)_y\right].$$

The metric III of Q is:

$$III = e_{11}\, dx^2 + 2e_{12}\, dxdy + e_{22}\, dy^2.$$

The Laplacian Δ^{III} is given by:

$$\Delta^{III}\varphi := -e^{ik}\nabla_k^{III}\varphi_{/i}.$$

For any vector-valued function $\boldsymbol{r} = \{r_1, r_2, r_3\}$, defined on $\mathrm{B} \subset \mathbb{R}^2$, we have:

$$\Delta^J \boldsymbol{r} = \{\Delta^J r_1, \Delta^J r_2, \Delta^J r_3\},\ J = I, II, III.$$

Certainly, let's elaborate on the definition of immersions of coordinate finite type, Subsequently, we can extend this investigation to a significant category of surfaces known as tubular surfaces.

Definition 1. A surface Q is termed to be of coordinate finite type concerning the metric III if the position vector $\boldsymbol{r}$ of Q adheres to a specific relation of the form

$$\Delta^{III}\boldsymbol{r} = A\boldsymbol{r}, \tag{1}$$

where A is a square matrix of order 3.

3 Tubular Surfaces

A tubular surface is a surface that is formed by sweeping a regular unit speed curve C: $\boldsymbol{c} = \boldsymbol{c}(v)$, $v \in (a, b)$ of finite length in space along a given direction. It can be thought of as a surface "wrapped around" a curve. Let $\boldsymbol{T}$, $\boldsymbol{N}$, $\boldsymbol{B}$ be the Frenet frame of the curve C and let $\kappa > 0$ be its curvature. Then a regular parametric representation of a tubular surface $Ƀ$ of radius s satisfies $0 < s < \min \frac{1}{|\kappa|}$ is given by [7]:

$$Ƀ : \boldsymbol{r}(v, \psi) = \boldsymbol{c} + s\cos\psi\, \boldsymbol{N} + s\sin\psi\, \boldsymbol{B}. \tag{2}$$

For the components g_{ij} of the first fundamental form $I = g_{ij}dv^i d\psi^j$ we have:

$$g_{ij} = \begin{pmatrix} \delta^2 + s^2\tau^2 & s^2\tau \\ s^2\tau & s^2 \end{pmatrix},$$

while the components b_{ij} of the second fundamental form are given by:

$$b_{ij} = \begin{pmatrix} s\tau^2 - \kappa\delta\cos\psi & s\tau \\ s\tau & s \end{pmatrix},$$

where τ is the torsion of the curve $\boldsymbol{c}$, and $\delta := (1 - s\kappa\cos\psi)$. For the Gauss curvature of $Ƀ$, we have:

$$K_G = -\frac{K\cos\psi}{s\delta} \tag{3}$$

As we note before the Gauss curvature never vanishes, so we must have $K \neq 0$. The Beltrami operator corresponding to the metric III of $Ƀ$ can be found as follows:

$$\Delta = \frac{1}{(K\cos\psi)^2}\Big[-\frac{\partial^2}{\partial v^2} + \frac{\beta}{K\cos\psi}\frac{\partial}{\partial v} + 2\tau\frac{\partial^2}{\partial v\partial\psi} - (\tau^2 + K^2\cos^2\psi)\frac{\partial^2}{\partial\psi^2} + (\tau' + K^2\cos\psi\sin\psi - \frac{\tau\beta}{K\cos\psi})\frac{\partial}{\partial\psi}\Big]. \tag{4}$$

where $\beta := K'\cos\psi + K\tau\sin\psi$, and $' := \frac{d}{dv}$.

Inserting the position vector of (2) in relation (4) we get:

$$\Delta\boldsymbol{r} = \frac{\beta}{(K\cos\psi)^3}\boldsymbol{T} + (2s\cos\psi - \frac{1}{K\cos^2\psi})\boldsymbol{N} + 2s\sin\psi\,\boldsymbol{B}. \tag{5}$$

Let r_1, r_2, r_3 the component functions of the parametric representation (5). We will examine when will the surface $Ƀ$ satisfies the relation (1). Analytically, we have:

$$\begin{bmatrix}\Delta r_1\\ \Delta r_2\\ \Delta r_3\end{bmatrix} = \begin{bmatrix}\lambda_{11} & \lambda_{12} & \lambda_{13}\\ \lambda_{21} & \lambda_{22} & \lambda_{23}\\ \lambda_{31} & \lambda_{32} & \lambda_{33}\end{bmatrix}\begin{bmatrix}r_1\\ r_2\\ r_3\end{bmatrix}. \tag{6}$$

Let c_i, T_i, N_i, B_i, $i = 1, 2, 3$, the component functions of the vectors $\boldsymbol{c}$, $\boldsymbol{T}$, $\boldsymbol{N}$, and $\boldsymbol{B}$ respectively. From (2) and system (6) we have:

$$\frac{\beta}{(K\cos\psi)^3}T_i + (2s\cos\psi - \frac{1}{K\cos^2\psi})N_i + 2s\sin\psi\,B_i =$$
$$= \lambda_{i1}(c_1 + s\cos\psi\,N_1 + s\sin\psi\,B_1) + \lambda_{i2}(c_2 + s\cos\psi\,N_2 + s\sin\psi\,B_2) + \lambda_{i3}(c_3 + s\cos\psi\,N_3 + s\sin\psi\,B_3), \tag{7}$$
$$i = 1, 2, 3.$$

We have the following two cases:

Case I. $\beta = 0$. Then $K' = 0$ and $K\tau = 0$. Thus $\tau = 0$ and K = const. $\neq 0$, therefore the curve $\boldsymbol{c}$ is a plane circle and so, $Ƀ$ is an anchor ring. In this case, a regular parametric representation of an anchor ring is:

$$Ƀ: \boldsymbol{r}(u,v) = \{(a + s\cos u)\cos v,\ (a + s\cos u)\sin v,\ s\sin u\}, \tag{8}$$
$$a > s,\ a, s \in \mathbb{R},\quad 0 \le u < 2\pi,\quad 0 \le v < 2\pi.$$

The first fundamental form becomes:

$$I := s^2du^2 + (a + s\cos u)^2dv^2,$$

while the second is:

$$II := sdu^2 + (a + s\cos u)\cos u\,dv^2.$$

The Laplacian corresponding to the metric III of $Ƀ$ can be found as follows:

$$\Delta = -\frac{\partial^2}{\partial u^2} + \frac{\sin u}{\cos u}\frac{\partial}{\partial u} - \frac{1}{\cos^2 u}\frac{\partial^2}{\partial v^2}. \tag{9}$$

Let r_1, r_2, r_3 the component functions of the parametric representation (2). Applying relation (9) for the functions r_1, r_2, and r_3 we get:

$$\Delta r_1 = \Delta[(a + s\cos u)\cos v] = \frac{a\cos v}{\cos^2 u} + 2s\cos u\cos v,$$
$$\Delta r_2 = \Delta[(a + r\cos u)\sin v] = \frac{a\sin v}{\cos^2 u} + 2s\cos u\sin v,$$
$$\Delta r_3 = \Delta(s\sin u) = 2s\sin u.$$

From the last three equations and system (6) we have:

$$\frac{a\cos v}{\cos^2 u} + 2s\cos u\cos v = \lambda_{11}(a + s\cos u)\cos v + \lambda_{12}(a + s\cos u)\sin v + \lambda_{13}s\sin u, \tag{10}$$

$$\frac{a\sin v}{\cos^2 u} + 2s\cos u\sin v = \lambda_{21}(a + s\cos u)\cos v + \lambda_{22}(a + s\cos u)\sin v + \lambda_{23}s\sin u, \tag{11}$$

$$2s\sin u = \lambda_{31}(a + s\cos u)\cos v + \lambda_{32}(a + s\cos u)\sin v + \lambda_{33}s\sin u. \tag{12}$$

From (12) it can be easily seen that:

$$\lambda_{31} = \lambda_{32} = 0,\qquad \lambda_{33} = 2.$$

Deriving relation (10) twice with respect to the parameter v we get:

$$\frac{a\cos v}{\cos^2 u} + 2s\cos u\cos v = \lambda_{11}(a + s\cos u)\cos v + \lambda_{12}(a + s\cos u)\sin v. \tag{13}$$

From (10) and (13) we find that $\lambda_{13} = 0$.

Similarly, we will get $\lambda_{23} = 0$, and relations (10) and (11) finally become

$$a\cos v = \lambda_{11}a\cos^2u\cos v + \lambda_{12}a\cos^2u\sin v + (\lambda_{11} - 2)s\cos^3u\cos v + \lambda_{12}s\cos^3u\sin v, \tag{14}$$

$$a\sin v = \lambda_{21}a\cos^2u\cos v + \lambda_{22}a\cos^2u\sin v + (\lambda_{22} - 2)s\cos^3u\sin v + \lambda_{21}s\cos^3u\cos v. \tag{15}$$

Deriving (14) and (15) with respect to the parameter v we have:

$$2\lambda_{11}a\cos v + 2\lambda_{12}a\sin v + 3(\lambda_{11} - 2)s\cos u\cos v + 3\lambda_{12}s\cos u\sin v = 0, \tag{16}$$
$$2\lambda_{21}a\cos v + 2\lambda_{22}a\sin v + 3(\lambda_{22} - 2)s\cos u\sin v + 3\lambda_{21}s\cos u\cos v = 0.$$

Deriving (16) with respect to the parameter v we have:

$$-2\lambda_{11} a\sin v + 2\lambda_{12} a\cos v - 3(\lambda_{11} - 2)s\cos u\sin v + 3\lambda_{12} s\cos u\cos v = 0. \quad (17)$$

Multiplying (16) by $\sin v$ and (17) by $\cos v$ and adding the resulting equations, we obtain:

$$\lambda_{12}(2a + 3r\cos u) = 0 \Rightarrow \lambda_{12} = 0.$$

Following the same procedure, we also get $\lambda_{21} = 0$. Thus relations (14) and (15) become:

$$a = \lambda_{11} a\cos^2 u + (\lambda_{11} - 2)s\cos^3 u,$$
$$a = \lambda_{22} a\cos^2 u + (\lambda_{22} - 2)s\cos^3 u.$$

From the last two equations, we conclude that λ_{11}, and λ_{22} depend on the parameter u and are not constants, and hence relation (1) cannot be satisfied so we proved

Proposition 1. The position vector of a parametric representation of an anchor ring (8) does not satisfy the relation $\Delta^{III}\boldsymbol{r} = A\boldsymbol{r}$.

Case II. $\beta \neq 0$.

Recalling equations (7), then we can write these equations as follows:

$$\beta T_i + 2s\kappa^3\cos^4\psi N_i - \kappa^2\cos\psi N_i + 2s\kappa^3\sin\psi\cos^3\psi B_i -$$
$$-\lambda_{i1}\kappa^3(c_1\cos^3\psi + s\cos^4\psi N_1 + s\cos^3\psi\sin\psi B_1) -$$
$$-\lambda_{i2}\kappa^3(c_2\cos^3\psi + s\cos^4\psi N_2 + s\cos^3\psi\sin\psi B_2) -$$
$$-\lambda_{i3}\kappa^3(c_3\cos^3\psi + s\cos^4\psi N_3 + s\cos^3\psi\sin\psi B_3) = 0,$$
$$i = 1, 2, 3,$$

We also rewrite it in terms of $\cos\psi$ as follows:

$$\beta T_i - \kappa^2\cos\psi N_i + s\kappa^3(2N_i - \lambda_{i1}N_1 - \lambda_{i2}N_2 - \lambda_{i3}N_3)\cos^4\psi +$$
$$+ s\kappa^3(2B_i - \lambda_{i1}B_1 - \lambda_{i2}B_2 - \lambda_{i3}B_3)\cos^3\psi\sin\psi -$$
$$\kappa^3(\lambda_{i1}c_1 + \lambda_{i2}c_2 + \lambda_{i3}c_3)\cos^3\psi = 0,$$
$$i = 1, 2, 3.$$

The above equations for $i = 1, 2, 3$, are polynomials of the variables $\cos\psi$, $\sin\psi$ with coefficients functions of the variable v. To be the last equations satisfied for all $i = 1, 2, 3$, then the coefficients functions of these polynomials must equal zeros. So we must have:

$$\lambda_{i1}c_1 + \lambda_{i2}c_2 + \lambda_{i3}c_3 = 0,$$
$$2N_i - \lambda_{i1}n_1 - \lambda_{i2}N_2 - \lambda_{i3}N_3 = 0,$$
$$2B_i - \lambda_{i1}B_1 - \lambda_{i2}B_2 - \lambda_{i3}B_3 = 0,$$
$$\beta T_i - \kappa^2\cos\psi N_i = 0, \quad (18)$$
$$i = 1, 2, 3.$$

Since relation (18) holds for all $i = 1, 2, 3$, then we write (18) in vector notation as follows:

$$\beta\mathbf{T} + \kappa^2\cos\psi\,\mathbf{N} = \mathbf{0},$$

from which we obtain that $\beta = 0$ and $\kappa = 0$. Hence Ƀ is an anchor ring, a case that has been investigated previously. So we proved:

Theorem 1. There are no tubular surfaces in the three-dimensional Euclidean space whose position vector satisfies the relation $\Delta^{III}\boldsymbol{r} = A\boldsymbol{r}$.

4 Conclusion

This research article was divided into three sections, where after the introduction, the needed definitions and relations regarding this interesting field of study were given. Then a formula for the Laplace operator corresponding to the first, second, and third fundamental forms of a surface Q were defined. Finally, we classified the tubular surfaces satisfying the relation $\Delta\boldsymbol{r} = A\boldsymbol{r}$, for a real square matrix A of order 3. It is also interesting if this type of research can be applied to other families of surfaces that have not been studied yet such as spiral surfaces, or cyclides of Dupin.

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Conflict of Interest

The author has no conflicts of interest to declare.