Sentinels for the Identification of Pollution in Domains with Missing Data

REZZOUG IMAD, ACHAB FATMA
Department of Mathematics and Computer Science,
University of Oum El Bouaghi,
Laboratory of Dynamics systems and control,
ALGERIA

Abstract: Sentinel method introduced in the study of problems with incomplete data, particularly in the context of distributed systems where pollution terms may arise at the boundary. The idea of sentinel likely involves constructing a surrogate or placeholder value that helps account for missing data or uncertainties in the system. Weakly sentinel appears to be a modification or extension of the concept of a sentinel specifically tailored for estimating pollution terms in distributed systems with missing data. The term weakly might suggest that this sentinel is not as robust or precise as the ideal sentinel, but it serves a similar purpose in providing estimates or approximations in situations where complete data is not available.

Key-Words: Distributed system ; Controllability ; Optimal control ; Pollution term ; Missing term ; Sentinel method.

Received: April 19, 2023. Revised: December 21, 2023. Accepted: December 28, 2023. Published: December 31, 2023.

1 Introduction and Problem Statement

Several domains are modeled by dynamic systems. A dynamic system refers to a system that changes over time, where the behavior of the system is determined by its current state as well as its history. These systems are pervasive in various fields including physics, engineering, biology, economics, and social sciences.

Characteristics of dynamic systems include : Change over Time, State Variables, Feedback Loops, Nonlinearity, Complex Behavior [1], [2], [3], [4], [5].

Examples of dynamic systems include : Mechanical Systems, Electrical Circuits, Biological Systems, Economic Systems, Social Systems [6], [7], [8], [9], [10].

Analyzing dynamic systems often involves mathematical modeling, simulation, and numerical methods to understand their behavior and predict future states. Control theory is a branch of engineering and mathematics that deals specifically with the control and regulation of dynamic systems [11], [12], [13], [14].

Bellow we Present the organization of our work.

In the first section, we present the notion of sentinel and optimal control theory :

The Sentinel Method introduced by Lions provides a powerful framework for solving complex boundary value problems, especially when traditional methods based on fixed boundary conditions may not be applicable or effective. It offers flexibility in handling uncertain or evolving boundary conditions and has found applications in various fields, including fluid dynamics, solid mechanics, heat transfer, and electromagnetic [15], [16], [17], [18], [19], [20], [21], [22].

It's important to note that the Sentinel Method is a sophisticated mathematical technique and may require advanced knowledge of partial differential equations, numerical analysis, and functional analysis for its implementation and understanding [23], [24], [25], [26], [27], [28], [29], [30].

Optimal control theory finds applications in a wide range of fields, including aerospace engineering, robotics, economics, finance, manufacturing, and process control. It provides a powerful framework for designing control strategies that optimize performance, efficiency, and resource utilization in complex dynamical systems [31], [32], [33], [34], [35], [36], [37], [38], [39].

While these two concepts might seem unrelated at first glance, there could be scenarios where optimal control theory could be applied to design control strategies for processes or systems that are being monitored using the sentinel method. For example, in a manufacturing setting, optimal control techniques could be used to adjust process parameters in real-time based on feedback from sentinel units to optimize some performance criterion, such as minimizing defects or maximizing throughput .

In the second section, we introduced the approximate controllability :

Approximate controllability provides a more relaxed notion of controllability that is often more feasible to achieve in practice, especially for systems with inherent uncertainties or limitations. It allows for practical control strategies that can effectively steer a system towards desired states while accounting for real-world constraints and imperfections. [40], [41], [42], [43].

Let be $T > 0$, and Ω an open subset of R^n of smooth boundary $\partial\Omega = \Gamma$ and $\mathcal{D}_0 \subset \Gamma$, Let $\mathcal{O} \subset \Omega$, considered as an observatory. We define Ω_α open "neighbor" of Ω of boundary $\partial\Omega_\alpha = (\Gamma - \mathcal{D}_0) \cup \mathcal{D}_\alpha$.

Where D_α is defined starting from D_0 like the locus of the points $\mathcal{D}_\alpha = \{x + \alpha\beta(x)\nu(x), x \in \mathcal{D}_0\}$.

We denote by ν the outer normal on Γ, α small real parameter, and β is a $\mathcal{C}^1$ function on D_0 with $|\beta(x)| \leq 1$, $\beta = 0$ on ∂D_0.

We consider the parabolic evolution equation :

$$\begin{cases} y' + Ay + h(y) \mid_{\mathcal{Q}_\alpha = \Omega_\alpha \times]0,T[} = 0 \\ y \mid_{\Sigma_0 = \Gamma_0 \times]0,T[} = f + \lambda\widehat{f} \\ y \mid_{\Sigma\setminus\Sigma_0} = 0 \\ y(0) \mid_{\Omega_\alpha} = y_0 + \tau\widehat{y}_0 \end{cases} \tag{1}$$

where $y = y(x,t;\lambda,\tau)$, and where $\Gamma_0 \cap \mathcal{D}_\tau =$. We assume here that $h : R \longrightarrow R$ is of class $\mathcal{C}^1$, the functions y_0 and f are known with $y_0 \in L^2(\Omega_\alpha)$. But, the terms : $\tau\widehat{y}_0$ (so-called missing term) and $\lambda\widehat{f}$ (so-called pollution term) are unknown, $\widehat{y}_0$ and $\widehat{f}$ are renormalized and represent the size of missing and pollution $\|\widehat{y}_0\|_{L^2(\Omega_\alpha)} \leq 1$, $\tau \in R$ "small".

The observation is y on $\mathcal{O}$, for the time T. we denote by y_{obs} this observation

$$y_{obs} = m_0 \in L^2(\mathcal{O} \times]0,T[) . \tag{2}$$

We suppose that (1) has a unique solution denoted by $y(\lambda,\tau) := y(x,t;\lambda,\tau)$ in some relevant space. The question is

(q) : how to calculate the pollution term, independently from the variation missing term ?.

Least squares. Least squares is a powerful and versatile technique that is widely used in data analysis and regression modeling due to its simplicity, robustness, and efficiency. It provides a systematic way to estimate parameters of mathematical models from noisy or imperfect data, making it a fundamental tool in statistical analysis and scientific research.

Sentinels. The Sentinel Method introduced by Jacques-Louis Lions, a renowned French mathematician, is a mathematical approach used primarily in the field of partial differential equations (PDEs) for solving boundary value problems (BVPs). Jacques-Louis Lions made significant contributions to various areas of mathematics, including functional analysis, control theory, and numerical analysis, and his work laid the foundation for modern PDE theory and its applications.

In the context of partial differential equations, the Sentinel Method introduced by Lions involves a technique for solving problems where the boundary conditions are unknown or partially known. The method is particularly useful when the boundary conditions depend on the solution itself or when the boundary conditions are uncertain.

The sentinel concept relies on the following three objects: some state equation (1), some observation function (2), and some control function u to be determined.

J.L.Lions calls a "sentinel", a functional $\mathcal{S}(.)$ which is the scalar product of the measure y_{obs} and a function u. It is built to get some information on the pollution term.

2 Presentation of the Method

Proposition 1 *(definition, existence and uniqueness of the sentinel)*

We now consider the sentinel method of Lions which is an other attempt and brings better answer to question (q), as we will explain now :

Let h_0 be some function in $L^2(\mathcal{O} \times]0,T[)$. Let on the other hand ω be some open and non empty subset of Ω.

For a control function $u_\epsilon \in L^2(\omega \times]0;T[)$, we define the functional

$$\mathcal{S}(\lambda,\tau) = \int_{\mathcal{Q}} (h_0\chi_{\mathcal{O}} + u_\epsilon\chi_\omega)\, y(\lambda,\tau)\, dxdt \tag{3}$$

where $y(\lambda,\tau) = y(x,t;\lambda,\tau)$ is the solution of (3), and the function u_ϵ are to be found in such a way that

for all $\epsilon > 0$ there exists $u_\epsilon \in L^2(\omega \times]0;T[)$ such as

$$\|u_\epsilon\|_{L^2(\omega\times]0;T[)} = \min \|v\| ; v \in \mathcal{U}, \tag{4}$$

where $\mathcal{U} = \left\{ v \in L^2(\omega \times]0;T[) ; \frac{\partial}{\partial\tau}\mathcal{S}(0,0) = 0 \right\}$.

$$\left| \frac{\partial}{\partial\tau}\mathcal{S}(0,0) \right| \leq \epsilon, \forall\widehat{y}_0, 05 \tag{5}$$

$$\frac{\partial}{\partial\tau}\mathcal{S}(0,0) = 0, \forall\widehat{y}_0; (\epsilon \longrightarrow 0) . \tag{6}$$

Then $\mathcal{S}(\lambda,\tau)$ defined by (3) (4) (6) exists and is unique (that means the existence and uniqueness of the function u_ϵ).

It will take two steps :

1/ The conditions (4) (6) will be rewritten into a control problem,

2/ An weakly controllability result will be proved,

First step :

We consider the functions y_0 which solve problem (1) for $\lambda = 0$ and $\tau = 0$:

$$\begin{cases} \frac{\partial}{\partial t} y_0 + Ay_0 + h(y_0) \mid_{\mathcal{Q}_\alpha = \Omega_\alpha \times]0,T[} = 0, \\ y_0 \mid_{\Sigma_0 = \Gamma_0 \times]0,T[} = f, \\ y_0 \mid_{\Sigma\setminus\Sigma_0} = 0, \\ y_0(0) \mid_{\Omega_\alpha} = y_0, \end{cases} \tag{7}$$

because of (3) we can write

$$\mathcal{S}(0,0)=\int_{\mathcal{Q}}\left(h_0\chi_{\mathcal{O}}+u_\epsilon\chi_\omega\right)y_0(x,t)\,dxdt,$$

is known. One carries out a development of Taylor of $\mathcal{S}$ in the vicinity of $(0,0)$

$$\mathcal{S}(\lambda,\tau)\simeq\mathcal{S}(0,0)+\lambda\frac{\partial\mathcal{S}}{\partial\lambda}(0,0)+\tau\frac{\partial\mathcal{S}}{\partial\tau}(0,0),$$

for τ small.
And

$$\frac{\partial\mathcal{S}}{\partial\lambda}(0,0)=\int_{\mathcal{Q}}\left(h_0\chi_{\mathcal{O}}+u_\epsilon\chi_\omega\right)y_\lambda(x,t)\,dxdt,$$

$$\frac{\partial\mathcal{S}}{\partial\tau}(0,0)=\int_{\mathcal{Q}}\left(h_0\chi_{\mathcal{O}}+u_\epsilon\chi_\omega\right)y_\tau(x,t)\,dxdt,$$

and $y_\lambda(x,t)$ is the solution of

$$\begin{cases}\frac{\partial}{\partial t}y_\lambda+Ay_\lambda+h'(y_0)\,y_\lambda=0,\\ y_\lambda\mid_{\Sigma_0}=\widehat{f},\\ y_\lambda\mid_{\Sigma\setminus\Sigma_0}=0,\\ y_\lambda(0)=0,\end{cases}\tag{8}$$

and $y_\tau(x,t)$ is the solution of

$$\begin{cases}\frac{\partial}{\partial t}y_\tau+Ay_\tau+h'(y_0)\,y_\tau\mid_{\Omega\times]0,T[}=0,\\ y_\tau\mid_{\mathcal{D}_0\times]0,T[}=-\beta\frac{\partial y_0}{\partial\nu},\\ y_\tau\mid_{\Sigma\setminus\mathcal{D}_0\times]0,T[}=0,\\ y_\tau(0)=0.\end{cases}\tag{9}$$

To build the sentinel, one must determine u_ϵ which ensures the condition (4), (6) for a given positive ϵ.

Adjoint state :

Assume that $\frac{\partial y}{\partial\tau}$ can be defined for $\lambda=\tau=0$. Then, the y_τ solves the problem (9).

If y_τ and y_0 solve respectively (9) and (7), then the insensibility condition (6) is equivalent to

$$\frac{\partial\mathcal{S}}{\partial\tau}(0,0)=\int_{\mathcal{Q}}\left(h_0\chi_{\mathcal{O}}+u_\epsilon\chi_\omega\right)y_\tau(x,t)\,dxdt,\tag{10}$$

$\forall\widehat{y}_0,\|\widehat{y}_0\|_{L^2(\Omega_\alpha)}\leq 1.$

Let $q=q(x,t)$ be the solution of the following adjoint problem :

$$\begin{cases}-q'+A^*q+h'(y_0)q=h_0\chi_{\mathcal{O}}+u_\epsilon\chi_\omega,\\ q\mid_\Sigma=0,\\ q(T)=0.\end{cases}\tag{11}$$

As for the problem (9), the problem (11) has a unique solution q. The function q depends on the control u_ϵ that we shall determine :

Indeed, if we multiply the first equation in (11) by y_τ, and we integrate by parts, lead to

It is seen that the conditions (10), and for $\epsilon\longrightarrow 0$ one gets :

$$\begin{aligned}\frac{\partial\mathcal{S}}{\partial\tau}(0,0)&=-\int_\Sigma\frac{\partial q}{\partial\nu_*}y_\tau d\Sigma\qquad(12)\\&=\int_{\mathcal{D}_0\times]0,T[}\beta\frac{\partial y_0}{\partial\nu}\frac{\partial q}{\partial\nu_*}d\Sigma\\&=0.\end{aligned}$$

This equality must take place for any regular function α, with $|\alpha(x)|\leq 1$, $\alpha=0$ on ∂D_0. That is equivalent to

$$\int_0^T\frac{\partial y_0}{\partial\nu}\frac{\partial q}{\partial\nu_*}dt=0;\forall x\in\mathcal{D}_0.\tag{13}$$

The problem thus now to find u_ϵ in $\mathcal{U}=L^2(\omega\times]0;T[)$.

Such that one has (12), et (6).

This is a controllability problem.

Equivalent controllability problem :

For that one breaks up the system (11) into two systems:

$$\begin{cases}-q_0'+A^*q_0+h'(y_0)q_0=h_0\chi_{\mathcal{O}},\\ q_0(T)=0,\\ q_0\mid_\Sigma=0,\end{cases}\tag{14}$$

and

$$\begin{cases}-z'+A^*z+h'(y_0)z=u_\epsilon\chi_\omega,\\ z(T)=0,\\ z\mid_\Sigma=0.\end{cases}\tag{15}$$

Thus $q=q_0+z$ such as q_0 is thus given. Then one seeks u_ϵ so that $z=z(u_\epsilon)$ who checks

$$\int_0^T\frac{\partial y_0}{\partial\nu}\frac{\partial q}{\partial\nu_*}dt=-\int_0^T\frac{\partial y_0}{\partial\nu}\frac{\partial q_0}{\partial\nu_*}dt,\tag{16}$$

on $\mathcal{D}_0$.

If it is considered here that

$u_\epsilon=$ function of control.

$z=$ state of one (new) system.

That is to say $q_0(0)$ the desired state given by the resolution of the system (14), the problem of regional controllability consists in finding, for all $\epsilon>0$ a control u_ϵ of the space of control $\mathcal{U}=L^2(\mathcal{O}\times(0,T))$ allowing to approach with ϵ meadows, in a time finished, the state $z(t)$ of the system (15) of an initial state $z(T)=0$, in a desired final state $q_0(0)$ on Ω (see: [44]).

Second step :

Penalization and system of optimality

For $\vartheta>0$, consider the function J_ϑ defined by

$$\begin{aligned}J_\vartheta(u_\epsilon,z)&=\frac{1}{2}\int_0^T\int_\omega u_\epsilon^2dxdt+\qquad(17)\\&\frac{1}{2\vartheta}\|\Xi\|^2_{L^2(\Omega\times]0,T[)},\end{aligned}$$

such that $\Xi = -z' + A^* z + h'(y_0) z - u_\epsilon \chi_\omega$.
Where one posed $z' = \partial z / \partial t$
In (17), one considers all z such that

$$\begin{cases} -z' + A^* z + h'(y_0) z \in L^2(\Omega \times (0,T)), \\ z(T) = 0; z \mid_\Sigma = 0, \\ \int_0^T \frac{\partial y_0}{\partial \nu} \frac{\partial q}{\partial \nu_*} dt \mid_{\mathcal{D}_0} = -\int_0^T \frac{\partial y_0}{\partial \nu} \frac{\partial q_0}{\partial \nu_*} dt. \end{cases} \quad (18)$$

Let $u_\epsilon^\vartheta, z^\vartheta$ the solution of

$$\inf J_\vartheta(u_\epsilon, z).$$

One poses moreover
$\rho^\vartheta = \frac{1}{\vartheta}\left(-\partial z^\vartheta / \partial t + A^* z^\vartheta + h'(y_0) z^\vartheta - \chi_\omega u_\epsilon^\vartheta\right)$.
The couple $u_\epsilon^\vartheta, z^\vartheta$ is characterized by :

$$\begin{aligned} & \int\int_{\omega \times (0,T)} u_\epsilon^\vartheta . \widehat{u}_\epsilon dx dt + \\ & \int\int_{\Omega \times (0,T)} \rho^\vartheta (\Xi) dx dt \\ = \; & 0, \end{aligned} \quad (19)$$

such that $\Xi = -\partial \widehat{z} / \partial t + A^* \widehat{z} + h'(y_0) \widehat{z} - \chi_\omega \widehat{u}_\epsilon$.
$\forall \widehat{u}_\epsilon$ and $\forall \widehat{z}$ such that

$$\int_0^T \frac{\partial y_0}{\partial \nu} \frac{\partial \widehat{z}}{\partial \nu_*} dt \mid_{\mathcal{D}_0} = 0. \quad (20)$$

One thus has

$$\left| \begin{array}{l} -\partial \rho^\vartheta / \partial t + A \rho^\vartheta + h'(y_0) \rho^\vartheta = 0, \\ \rho^\vartheta(0) = 0, \\ \rho^\vartheta \mid_{\Sigma \setminus \mathcal{D}_0 \times]0,T[} = 0. \end{array} \right. \quad (21)$$

And

$$\frac{\partial \rho^\vartheta}{\partial \nu} = \sigma^\vartheta \frac{\partial y_0}{\partial \nu}(x.t) \mid_{\mathcal{D}_0 \times]0,T[} . \quad (22)$$

For any σ^ϑ. So that (19) becomes

$$u_\epsilon^\vartheta = \chi_\omega \rho^\vartheta. \quad (23)$$

System of optimality : $(\vartheta \to 0)$:
For $\rho^0 \in L^2(\Omega)$ and for σ regular function, one defines ρ solution of

$$\left| \begin{array}{l} \rho' + A\rho + h'(y_0) \rho \mid_{\Omega \times (0,T)} = 0, \\ \rho(0) = 0, \\ \rho \mid_{\mathcal{D}_0 \times]0,T[} = \sigma(x) \frac{\partial y_0}{\partial \nu}, \\ \rho \mid_{\Sigma \setminus \mathcal{D}_0 \times]0,T[} = 0, \end{array} \right. \quad (24)$$

One defines then z by

$$\left| \begin{array}{l} -z' + A^* z + h'(y_0) z = \chi_\omega \rho, \\ z(T) = 0, \\ z \mid_\Sigma = 0. \end{array} \right. \quad (25)$$

One seeks σ so that

$$\int_0^T \frac{\partial y_0}{\partial \nu} \frac{\partial z}{\partial \nu_*} dt \mid_{\mathcal{D}_0} = -\int_0^T \frac{\partial y_0}{\partial \nu} \frac{\partial q_0}{\partial \nu_*} dt. \quad (26)$$

We now define a linear operator Λ by

$$\Lambda \sigma \mid_{\mathcal{D}_0} = \int_0^T \frac{\partial y_0}{\partial \nu} \frac{\partial z}{\partial \nu_*} dt. \quad (27)$$

And

$$\mathcal{M} h_0 = \int_0^T \frac{\partial y_0}{\partial \nu} \frac{\partial q_0}{\partial \nu_*} dt.$$

It remains to solve (26).
Multiplying (25) by ρ, we obtain after integrating by part

$$\langle \Lambda \sigma, \sigma \rangle = \int\int_{\omega \times (0,T)} \rho^2 dx dt.$$

What results in introducing

$$\|\sigma\|_F = \left(\int\int_{\omega \times (0,T)} \rho^2 dx dt \right)^{\frac{1}{2}}. \quad (28)$$

One indicates by F the space of Hilbert separate and supplemented regular functions σ for the norm (28).
$\Lambda \in \mathcal{L}(F, F')$ is an isomorphism of F on F', and $\Lambda^* = \Lambda$; F' being the dual space of F.
The equation (26) is written

$$\Lambda \sigma = -\mathcal{M} h_0,$$

from where

$$\sigma = -\Lambda^{-1} \mathcal{M} h_0, \quad (29)$$

subject checking that

$$\mathcal{M} h_0 \in F'. \quad (30)$$

But if one multiplies (14) by ρ, one sees that

$$\langle \mathcal{M} h_0, \sigma \rangle = (h_0, \chi_\omega \rho)_{L^2(\mathcal{Q})} . \quad (31)$$

from where (30) with

$$\|\mathcal{M} h_0\|_{F'} \leq \|h_0\|_{L^2(\mathcal{Q})} .$$

therefore, the sought sentinel is given by

$$\begin{aligned} \mathcal{S}(\lambda, \tau) &= \int_{\mathcal{Q}} (h_0 \chi_{\mathcal{O}} + u_\epsilon \chi_\omega) y(\lambda, \tau) dx dt \\ &= \int_{\mathcal{Q}} \Xi y(\lambda, \tau) dx dt, \end{aligned}$$

such that $\Xi = h_0 \chi_{\mathcal{O}} - \mathcal{M}^* \Lambda^{-1} \mathcal{M} h_0 \chi_\omega$.
In what follows we apply the preceding result to estimate the term of pollution of the system (1).

3 A use of the concept of sentinel: The identification of the unknown pollution term

Remark 2 *If the semigroup $S^*(t)$ generated by the operator A^* is compact in $L^2(\Omega)$, the system (15) is not exactly controllable [44].*

Remark 3 *There are systems which are weakly controllable but they are not exactly controllable.*

Example 4 *Ω an open subset of R^n of smooth boundary $\partial\Omega$, we consider here the state equation:*

$$\begin{cases} y' - \Delta y \mid_Q = v, \\ y(x,0) \mid_\Omega = 0, \\ y(x,t) \mid_\Sigma = 0. \end{cases}$$

The system above is a particular case of system (15); indeed, it is enough to take $A^* = \Delta$ when $y \in D(\Omega) = H^2(\Omega) \cap H_0^1(\Omega), \mathcal{O} = \Omega, v = u_\epsilon \in L^2(\mathcal{U})$. This system cannot be exactly controllable in $L^2(\Omega)$ because the semigroup $S^*(t)$ generated by $A^* = \Delta$ is compact, but it is exactly controllable in $H_0^1(\Omega)$ [44].

These two remarks led us to introduce the notion of the sentinel to estimate the term of pollution independently of the missing term. It is supposed that the system (15) is not exactly controllable thus the following theorem shows the interest of weakly controllability in the construction industry of the sentinels.

Theorem 5 *If the system (15) is weakly controllable then for all ϵ positive it exists a function $u_\epsilon \in L^2(\omega \times (0,T))$ who checks the conditions (4), (6) of the proposition (1).*

its shows already.

Theorem 6 *Since the system (15) is weakly controllable on Ω then one has*
$\int_{\Sigma_0} \frac{\partial}{\partial \nu_*} q(h_0) \left\{\lambda \widehat{f}\right\} d\Sigma \leq$
$\int_Q (h_0 \chi_\mathcal{O} + u_\epsilon \chi_\omega) |m_0 - y_0| \, dxdt + \tau\epsilon,$
where $y_0(x,t)$ is the solution of (7) and m_0 is the state observed on $\mathcal{O}$ during the interval of time $(0,T)$.

that is to say $\mathcal{S}(\lambda,\tau)$ the sentinel defined by h_0 thus
$\lambda \frac{\partial \mathcal{S}}{\partial \lambda}(0,0)$
$= \lambda \int_Q (h_0 \chi_\mathcal{O} + u_\epsilon \chi_\omega) y_\lambda(x,t) \, dxdt$
$= \mathcal{S}(\lambda,\tau) - \mathcal{S}(0,0) - \tau \frac{\partial \mathcal{S}}{\partial \tau}(0,0)$.
And on the observatory $\mathcal{O}$ one poses $y = m_0$ then
$\lambda \frac{\partial \mathcal{S}}{\partial \lambda}(0,0)$
$= \int_Q (h_0 \chi_\mathcal{O} + u_\epsilon \chi_\omega)(m_0 - y_0) \, dxdt - \tau \frac{\partial \mathcal{S}}{\partial \tau}(0,0),$
where $y_\lambda(x,t)$ is the solution of (8).

Now, we designate as $q(h_0)$ the unique solution of (11) depending on h_0.

Multiplying (11) by y_λ, we obtain after integrating by part
$\int_{\Sigma_0} \frac{\partial}{\partial \nu_*} q(h_0) \left\{\lambda \widehat{f}\right\} d\Sigma$
$= \lambda \int_Q (h_0 \chi_\mathcal{O} + u_\epsilon \chi_\omega) y_\lambda(x,t) \, dxdt,$
and in addition one has
$\left| \frac{\partial \mathcal{S}}{\partial \tau}(0,0) \right|$
$= \left| \int_Q (h_0 \chi_\mathcal{O} + u_\epsilon \chi_\omega) y_\tau(x,t) \, dxdt \right|$
$= 0$, for $\epsilon \longrightarrow 0$.

It results that the unknown pollution term $\lambda \widehat{f}$ can be defined as follows
$\int_{\Sigma_0} \frac{\partial}{\partial \nu_*} q(h_0) \left\{\lambda \widehat{f}\right\} d\Sigma$
$= \mathcal{S}(\lambda,\tau) - \mathcal{S}(0,0) - \tau \frac{\partial \mathcal{S}}{\partial \tau}(0,0)$
$\leq \int_Q (h_0 \chi_\mathcal{O} + u_\epsilon \chi_\omega) |m_0 - y_0| \, dxdt + \tau\epsilon$
$\leq \int_Q (h_0 \chi_\mathcal{O} - \mathcal{M}^* \Lambda^{-1} \mathcal{M} h_0 \chi_\omega) |m_0 - y_0| \, dxdt + \tau\epsilon,$
thus, the proof of Theorem.

Acknowledgements

The authors are grateful to the reviewers for their valuable and insightful comments.

Availability of data and materials

This paper does not use data and materials.

Ethics approval and consent to participate

The authors reveal that there is no ethical problem in the production of this paper.

Consent for publication

The authors want to publish this paper in this journal.

References:

[1] M. Rutter. Incidence of autism spectrum disorders: Changes over time and their meaning. Acta Paediatrica. 2007. https://doi.org/10.1111/j.1651-2227.2005.tb01779.x

[2] D.C. Bernard and E.G. Morton. Thermodynamics with Internal State Variables. J. Chem. Phys. 47, 597–613 (1967). https://doi.org/10.1063/1.1711937

[3] S. Harris and A. Levine. The p53 pathway: positive and negative feedback loops. Oncogene 24, 2899–2908 (2005). https://doi.org/10.1038/sj.onc.1208615

[4] K. Emancipator and M.H. Kroll. A quantitative measure of nonlinearity. Clinical Chemistry, Volume 39, Issue 5, 1 May 1993, Pages 766–772, https://doi.org/10.1093/clinchem/39.5.766

[5] S. Dymond and R.A. Rehfeldt. Understanding complex behavior: The transformation of stimulus functions. BEHAV ANALYST 23, 239–254 (2000). https://doi.org/10.1007/BF03392013

[6] Y. Liu and H. Yu. A survey of underactuated mechanical systems. IET Control Theory and Applications. 2013. https://doi.org/10.1049/iet-cta.2012.0505

[7] F. Cipparrone and D. Consonni. Calculation of Step Discontinuities in Electric Circuits: A Time-Domain Analysis. IEEE Transactions on Education. 65:4. (575-583). Online publication date: 1-Nov-2022. https://doi.org/10.1109/TE.2022.3151376

[8] W.J. Stark. Nanoparticles in Biological Systems. Angewandte Chemie International Edition. 2011. https://doi.org/10.1002/anie.200906684

[9] P. Ghisellini, C. Cialani and S. Ulgiati. A review on circular economy: the expected transition to a balanced interplay of environmental and economic systems. Journal of Cleaner Production. Volume 114, 15 February 2016, Pages 11-32. https://doi.org/10.1016/j.jclepro.2015.09.007

[10] A. R. Radcliffe-Brown. On Social Structure. Royal Anthropological Institute of Great Britain and Ireland. Vol. 70, No. 1 (1940), pp. 1-12. https://doi.org/10.2307/2844197

[11] S. He, Y. Li and R.Z. Wang. Progress of mathematical modeling on ejectors. Renewable and Sustainable Energy Reviews. Volume 13, Issue 8, October 2009, Pages 1760-1780. https://doi.org/10.1016/j.rser.2008.09.032

[12] H. Chen and C.F. Lee. Numerical simulation of debris flows. Canadian Geotechnical Journal. February 2000. https://doi.org/10.1139/t99-089

[13] S. Pirozzoli. Numerical Methods for High-Speed Flows. Annual Review of Fluid Mechanics. Vol. 43:163-194. 2011. https://doi.org/10.1146/annurev-fluid-122109-160718

[14] S. Schaal and P. Mohajerian and A. Ijspeert. Dynamics systems vs. optimal control-a unifying view. Progress in Brain Research Volume 165, 2007, Pages 425-445. https://doi.org/10.1016/S0079-6123(06)65027-9

[15] G. L. Lövei and M. Ferrante. A review of the sentinel prey method as a way of quantifying invertebrate predation under field conditions. Insect Science. 2016. https://doi.org/10.1111/1744-7917.12405

[16] V. L. Rvachev and T. I. Sheiko. R-Functions in Boundary Value Problems in Mechanics. Appl. Mech. Rev. Apr 1995, 48(4): 151-188 (38 pages). https://doi.org/10.1115/1.3005099

[17] R.E. Behrend, P.A. Pearce and D.L. O'Brien. Interaction-round-a-face models with fixed boundary conditions: The ABF fusion hierarchy. J Stat Phys 84, 1–48 (1996). https://doi.org/10.1007/BF02179576

[18] C. Busse, A.P. Kach, and S.M. Wagner. Boundary Conditions: What They Are, How to Explore Them, Why We Need Them, and When to Consider Them. Volume 20, Issue 4, 2016. https://doi.org/10.1177/1094428116641191

[19] O.F. Andreas, P.W. Walsh and J.E. Moore Jr. Computational Fluid Dynamics and Stent Design. Artificial Organs. 2002. https://doi.org/10.1046/j.1525-1594.2002.07084.x

[20] R. Hill. Aspects of Invariance in Solid Mechanics. Advances in Applied Mechanics. Volume 18, 1979, Pages 1-75. https://doi.org/10.1016/S0065-2156(08)70264-3

[21] V. K. Dhir. Bolling Heat Transfer. Annual Review of Fluid Mechanics. Vol. 30:365-401, 1998. https://doi.org/10.1146/annurev.fluid.30.1.365

[22] E.J. Rothwell and M.J. Cloud. Electromagnetics. Journals. Taylor Francis. 2009. https://doi.org/10.1201/9781315222578

[23] B. Elhamza and A. Hafdallah. Identification of the potential coefficient in the wave equation with incomplete data: a sentinel method. Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika, 2022, Number 12, Pages 113–122. DOI: https://doi.org/10.26907/0021-3446-2022-12-113-122

[24] N. Bouafi, N.L. Merabti and I. Rezzoug. Comparison Between the two HUM and No-regret Control Methods. WSEAS Transactions on Systems and Control, Volume 18, 2023. DOI: 10.37394/23203.2023.18.37

[25] I. Rezzoug, A. Necib and T.E. Oussaeif. No Regret Control for Heat Equation With Delay and Incomplete Data. WSEAS Transactions on Systems and Control, Volume 17, 2022. DOI: 10.37394/23203.2022.17.53

[26] M. Pinto and F. Poblete and D. Sepúlveda. Approximation of mild solutions of delay differential equations on Banach spaces. Journal of Differential Equations. Volume 303, 5 December 2021, Pages 156-182. https://doi.org/10.1016/j.jde.2021.09.008

[27] I. Rezzoug and T.E. Oussaeif. Approximate Controllability. Wseas Transcations on Systems. Volume 19, (2020). DOI: 10.37394/23202.2020.19.3

[28] B. Elhamza and A. Hafdallah. Identification of the bulk modulus coefficient in the acoustic equation from boundary observation: a sentinel method. Bound Value Probl 2023, 23 (2023). https://doi.org/10.1186/s13661-023-01708-3

[29] Y. Lev and A. Faye and K.Y. Volokh. Thermoelastic deformation and failure of rubberlike materials.Journal of the Mechanics and Physics of Solids Volume 122, January 2019, Pages 538-554. https://doi.org/10.1016/j.jmps.2018.09.033

[30] Z. Wang and J. Liu. Identification of the pollution source from one-dimensional parabolic equation models. Applied Mathematics and Computation Volume 219, Issue 8, 15 December 2012, Pages 3403-3413. https://doi.org/10.1016/j.amc.2008.03.014

[31] J.P. Nunes and J.F. Silva. 5 - Sandwiched composites in aerospace engineering. Advanced Composite Materials for Aerospace Engineering. Processing, Properties and Applications. 2016, Pages 129-174. https://doi.org/10.1016/B978-0-08-100037-3.00005-5

[32] G.M. Whitesides. Soft Robotics. Angewandte Chemie. 2018. https://doi.org/10.1002/anie.201800907

[33] G. Avi and C. Tucker. 2019. ”Digital Economics.” Journal of Economic Literature, 57 (1): 3-43. DOI: 10.1257/jel.20171452

[34] D. Hirshleifer. Behavioral Finance. Annual Review of Financial Economics. Vol. 7:133-159, 2015. https://doi.org/10.1146/annurev-financial-092214-043752

[35] M.G. Mehrabi, A.G. Ulsoy and Y. Koren. Reconfigurable manufacturing systems: Key to future manufacturing. Journal of Intelligent Manufacturing 11, 403–419 (2000). https://doi.org/10.1023/A:1008930403506

[36] A. Benbrahim and I. Rezzoug and A. Necib. A revision of M. Asch notion of Discrete control. Wseas Transcations on Applied and Theorical Mechanics. Volume 17, (2022). DOI: 10.37394/232011.2022.17.29

[37] E. Casas and K. Kunisch and F. Tröltzsch. Chapter 4 - Optimal control of PDEs and FE-approximation. Handbook of Numerical Analysis Volume 23, 2022, Pages 115-163. https://doi.org/10.1016/bs.hna.2021.12.004

[38] A. Akkouche and A. Maidi and M. Aidene. Optimal control of partial differential equations based on the Variational Iteration Method. Computers Mathematics with Applications Volume 68, Issue 5, September 2014, Pages 622-631. https://doi.org/10.1016/j.camwa.2014.07.007

[39] R.W.H. Sargent. Optimal control. Journal of Computational and Applied Mathematics. Volume 124, Issues 1–2, 1 December 2000, Pages 361-371. https://doi.org/10.1016/S0377-0427(00)00418-0

[40] H. L. Trentelman. Controllability and Observability. In: Baillieul, J., Samad, T. (eds) Encyclopedia of Systems and Control. Springer, London. 2014. https://doi.org/10.1007/978-1-4471-5102-9-192-1

[41] J.L. Lions. Controllability, Penalty and stiffness, Anal, Nor, Serie 4, Tome 25, N3-4. PP. 547-610, 1997.

[42] E. Casas and M. Mateos. Optimal Control of Partial Differential Equations. In: Mateos, M., Alonso, P. (eds) Computational Mathematics, Numerical Analysis and Applications. SEMA SIMAI Springer Series, vol 13. Springer, Cham. 2017. https://doi.org/10.1007/978-3-319-49631-3-1

[43] O. Bodart and P. Demeestere. Sentinels for the identification of an unknown boundary. University of Compiegne (France). M3AS, 7(6), pp. 871-885. 1997.

[44] L. Afifi, A.El Jai and E. Zerrik. Systems Theory : Modelling, Analysis and Control. FES2009 Proceedings. PUP, ISBN : 978-2-35412-043-6, 2009.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution and all authors read and approved the final manuscript.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflicts of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.