Consistency and feasibility of Haar wavelet collocation method for a nonlinear optimal control problem with application

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Abstract: Haar wavelet-based numerical algorithms have recently been developed for various mathematical problems, including optimal control problems. However, no numerical algorithm is complete without its theoretical analysis. In this paper, we have shown the consistency and feasibility of the Haar wavelet-based collocation method for solving nonlinear optimal control problems that have a single state and a single control variable with constraints. The accuracy of the method has been shown through some application problems.

Key-Words: Haar wavelet, Optimal control, Feasibility, Consistency, Collocation method.

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1 Introduction

Optimal control problems are applied in nearly all engineering and scientific fields. It has many applications in robotics, aeronautics, the chemical, biochemical and medicine industry, etc. In chemical processes, it helps to find an optimal policy that maximizes the yield of a desired product. In the medical sector, researchers use it to find an optimal amount of drug dosage in people of different ages. It helps in the control of thermally unstable batch processes.

The optimal control problems have been solved using various approaches, [1], in the literature, including direct, indirect, and dynamic programming-based methods. In 2020, [2], developed a collocation method based on Legendre wavelet to deal with fractional optimal control problems. To address optimal control problems with non-smooth solutions, in 2021 the standard LGR collocation method, [3], is modified. The authors in [4], used an RBF collocation method to address economic growth model optimal control problems. The authors in [5], have used optimal control techniques to determine the effectiveness of the oncolytic viral therapy for short-term treatment. Using nonlinear delay differential equations that simulate the spread of a computer virus. The authors in [6], introduced Legendre-Gauss-Radau collocation to approximate the optimal control problem. A technique for solving singular optimal control and bang-bang problems was created in 2022 by [7]. It involved the use of adaptive Legendre–Gauss–Radau collocation. Most recently, the Volterra integro-differential equation-governed optimal control problems were solved using a collocation method based on Dickson polynomials, [8].

Haar wavelet-based numerical methods, [9], have attracted significant attention of researchers in recent years. The authors in [10], have highlighted some of the advantages of the Haar wavelet method. The authors in [11], developed a Haar wavelet collocation technique for boundary layer fluid flow problems in 2001. To solve elliptic partial differential equations numerically, two novel and efficient approaches, [12], based on collocation with Haar and Legendre wavelets were introduced in 2013. Two efficient methods based on collocation utilising Haar and Legendre wavelets for the numerical solution of linear as well as nonlinear differential equations are proposed in 2014, [13]. A Haar wavelet-based approach was suggested by [14], in 2014 to solve a time delayed optimal control problem. The convection-diffusion equation can be solved more accurately, simply, quickly, and computationally attractively using the Haar wavelet collocation approach, which was developed in 2015 by [15]. In 2016, [16], developed a Haar wavelet-based collocation algorithm and its application to path planning and obstacle avoidance problems.

In the next section, we define some preliminaries, the Haar wavelets, and how these can be used in the function approximation.

2 Basic definitions and preliminaries

This section contains the introductory material and the definitions used in this article.

Definition 2.1. Function space $C^1[0,1]$
The space $C^1[0,1]$ is defined as the set of all continuously differentiable real-valued functions on $[0,1]$.

Definition 2.2. Function space $L^2[0,1]$
The space $L^2[0,1]$ is defined by the set

$$L^2[0,1] = \left\{ f : [0,1] \to \mathbb{R} : \int_0^1 |f(t)|^2 dt < \infty \right\},$$

with the norm defined as

$$\|f\|_{L^2} = \left(\int_0^1 |f(t)|^2 dt \right)^{\frac{1}{2}}.$$

Definition 2.3. Haar wavelets
The Haar wavelet orthogonal family, [17], $h_i(t)$ defined on $[0,1)$ is defined for $i = 0$ as

$$h_0(t) = \begin{cases} 1, & t \in [0,1), \\ 0, & \text{elsewhere}, \end{cases}$$

and for $i \geq 1$ as

$$h_i(t) = \begin{cases} 1, & t \in \left[\frac{k}{2^j}, \frac{k+\frac{1}{2}}{2^j} \right), \\ -1, & t \in \left[\frac{k+\frac{1}{2}}{2^j}, \frac{k+1}{2^j} \right), \\ 0, & \text{elsewhere}, \end{cases}$$

where $i = 2^j + k,\ j = 0,1,\ldots,\ k = 0,1,\ldots,2^j - 1$. Here, j and k represent the dilation and translation parameters, respectively.

The Haar wavelet family $\{h_i(t)\}_{i=0}^{\infty}$ forms an orthonormal basis for $L^2[0,1]$ as proved by [18]. Hence, any function $z \in L^2[0,1]$ can be written as

$$z(t) = \sum_{i=0}^{\infty} c_i h_i(t),$$

where $c_i = \int_0^1 z(t) h_i(t) dt$.
The approximation $z_M(t)$ of $z(t)$ by considering the first $M = 2^j - 1$ terms is given by

$$z(t) \approx z_M(t) = \sum_{i=0}^{M-1} c_i h_i(t).$$

Similarly, the approximation $\dot{z}_M(t)$ of $\dot{z}(t)$ can be given as

$$\dot{z}(t) \approx \dot{z}_M(t) = \sum_{i=0}^{M-1} d_i h_i(t),$$

from which we can get the approximation for $z(t)$ as

$$z_M(t) = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} d_i p_{i+1,j+1} h_i(t) + z(0),$$

where $P = [p_{i,j}]$, $1 \leq i,j \leq M$, is the Haar wavelet integration matrix, [19].

3 Error bounds at the collocation points

In this section, using existing results in the literature, we have derived the error bounds for the function approximated by Haar wavelets at the collocation points.

Theorem 3.1. *Let $z(t) \in C^1[0,1]$ and let $z_M(t)$ be the Haar wavelet approximation, [20], of $z(t)$. Then*

$$\|z - z_M\|_{L^2} \leq \frac{K}{\sqrt{3}M},$$

where K is the bound for $\dot{z}(t)$.

Remark 3.2. Let $z : [0,1] \longrightarrow \mathbb{R}$ be a function having bounded third-order derivative and $\ddot{z}_M$ be the Haar wavelet approximation of $\ddot{z}$, then by Theorem 3.1, we have

$$\|\ddot{z} - \ddot{z}_M\|_{L^2} \leq \frac{k'}{\sqrt{3}M}, \tag{1}$$

where k' is the bound for $\dddot{z}(t)$. Thus, we have, $\dot{z}_M(t) = \int_0^t \ddot{z}_M(s) ds + \dot{z}(0)$. Hence, we get

$$\begin{aligned} |\dot{z}(t) - \dot{z}_M(t)| &= \left| \int_0^t (\ddot{z}(s) - \ddot{z}_M(s)) ds \right| \\ &\leq \int_0^t |\ddot{z}(s) - \ddot{z}_M(s)| \, ds \\ &\leq \left(\int_0^1 |\ddot{z}(s) - \ddot{z}_M(s)|^2 ds \right)^{\frac{1}{2}} \\ &= \|\ddot{z} - \ddot{z}_M\|_{L^2}. \end{aligned}$$

Thus, we have, for all $t \in [0,1]$

$$|\dot{z}(t) - \dot{z}_M(t)| \leq \|\ddot{z} - \ddot{z}_M\|_{L^2}. \tag{2}$$

Thus, at the collocation points $t_j = \frac{j+1/2}{M}, j = 0,1,2,\ldots,M-1$, we have, from (1) and (2)

$$|\dot{z}(t_j) - \dot{z}_M(t_j)| \leq \frac{k'}{\sqrt{3}M}.$$

Again, from (1) and (2) , it follows that

$$\|\dot{z} - \dot{z}_M\|_{L^2} \leq \frac{k'}{\sqrt{3}M}.$$

Using a similar argument, we get

$$|z(t_j) - z_M(t_j)| \le \frac{k'}{\sqrt{3}M},$$
$$\|z - z_M\|_{L^2} \le \frac{k'}{\sqrt{3}M}. \tag{3}$$

4 Optimal control problem and its corresponding discretization

In this section, the optimal control problem and its discretization are discussed.
The continuous optimal control problem is to determine the state $x \in L^2[0,1]$ and control $u \in L^2[0,1]$, that minimizes the cost functional

$$I[x,u] = \int_0^1 F(x,u)dt + E(x(0), x(1)), \tag{4}$$

with dynamics

$$\dot{x} = f(x,u), \tag{5}$$

endpoint conditions

$$e_p(x(0), x(1)) = 0, \;\; p = 1,2,\ldots,N_e, \tag{6}$$

and mixed constraint

$$g_q(x,u) \le 0, \;\; q = 1,2,\ldots,N_g. \tag{7}$$

It is assumed that $F : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$, $E : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$, $f : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$, with

$$e_p : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}, \;\; p = 1,2,\ldots,N_e,$$
$$g_q : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}, \;\; q = 1,2,\ldots,N_g,$$

are Lipschitz continuous with respect to each argument. Additionally, it is assumed that an optimal solution (x^*, u^*) exists.

Next, the discretized version of the continuous optimal control problem (4)-(7) by using Haar wavelet-based collocation method is described below:

Determine x_M and u_M that minimize

$$I[x_M, u_M] = \frac{1}{M}\sum_{j=0}^{M-1} F(x_M(t_j), u_M(t_j)) + E(x_M(0), x_M(1)), \tag{8}$$

subject to

$$|\dot{x}_M(t_j) - f(x_M(t_j), u_M(t_j))| \le \delta_1, \quad j = 0,1,\ldots,M-1, \tag{9}$$

for each $p = 1,2,\ldots,N_p$

$$|e_p(x_M(0), x_M(1))| \le \delta_2, \tag{10}$$

for each $q = 1,2,\ldots,N_g$

$$g_q(x_M(t_j),(u_M(t_j)) \le \delta_3, \quad j = 0,1,\ldots,M-1, \tag{11}$$

where δ_1, δ_2 and δ_3 are the relaxation bounds, which are positive constants dependent on M.

5 Feasibility and consistency of the approximation

This section contains two results, the feasibility and consistency of the Haar wavelet approximation (8)-(11).

Theorem 5.1. *Let $(x(t), u(t))$ be any given feasible solution to the problem (4)-(7) such that $x \in L^2[0,1]$ has third order bounded derivative and $u \in L^2[0,1]$ has second order bounded derivative, then the problem (8)-(11) has a Haar wavelet feasible solution (x_M, u_M) such that for $j = 0,1,\ldots,M-1$*

$$|x(t_j) - x_M(t_j)| \le \frac{N_1}{\sqrt{3}M},$$
$$|u(t_j) - u_M(t_j)| \le \frac{N_2}{\sqrt{3}M},$$

where $N_1, N_2 > 0$ are the bounds for the third-order derivative of x and the second-order derivative of u respectively.

Proof. Since x has a third-order bounded derivative, we have, from Theorem 3.1

$$\|\ddot{x} - \ddot{x}_M\|_{L^2} \le \frac{N_1}{\sqrt{3}M}, \tag{12}$$

where $\ddot{x}_M$ is the Haar wavelet approximation of $\ddot{x}$ and N_1 is the bound for $\dddot{x}$.

Then, from the Remark 3.2 it follows that, for each $j = 0,1,\ldots,M-1$, we obtain

$$|x(t_j) - x_M(t_j)| \le \frac{N_1}{\sqrt{3}M},$$
$$|u(t_j) - u_M(t_j)| \le \frac{N_2}{\sqrt{3}M},$$
$$|\dot{x}(t_j) - \dot{x}_M(t_j)| \le \frac{N_1}{\sqrt{3}M}.$$

For the dynamic constraint, we get

$$\begin{aligned}
&|\dot{x}_M(t_j) - f(x_M(t_j), u_M(t_j))| \\
&\le |\dot{x}(t_j) - \dot{x}_M(t_j)| \\
&+ |f(x(t_j), u(t_j)) - f(x_M(t_j), u_M(t_j))| \\
&\le \frac{N_1}{\sqrt{3}M} + l_1|x(t_j) - x_M(t_j)| + l_2|u(t_j) - u_M(t_j)| \\
&\le \frac{N_1}{\sqrt{3}M} + l_1\frac{N_1}{\sqrt{3}M} + l_2\frac{N_2}{\sqrt{3}M},
\end{aligned}$$

where l_1 and l_2 are Lipschitz constants of f with respect to x and u, respectively, which are independent of M.
It follows that, for $j = 0, 1, \dots, M-1$, we have

$$\left|\dot{x}_M(t_j) - f(x_M(t_j), u_M(t_j))\right| \le \delta_1,$$

where

$$\delta_1 = \frac{N_1 + l_1 N_1 + l_2 N_2}{\sqrt{3}M}.$$

For each of the endpoint conditions with $p = 1, 2, \dots, N_e$, we get

$$\begin{aligned}&|e_p(x_M(0), x_M(1))| \\ &= |e_p(x_M(0), x_M(1)) - e_p(x(0), x(1))| \\ &\le l_{1p}|x(0) - x_M(0)| + l_{2p}|x(1) - x_M(1)| \\ &\le (l_{1p} + l_{2p})\frac{N_1}{\sqrt{3}M},\end{aligned}$$

where l_{1p} and l_{2p} are Lipschitz constants.
Thus,

$$|e_p(x_M(0), x_M(1))| \le \delta_2,$$

where, $\delta_2 = (l_{1p} + l_{2p})\frac{N_1}{\sqrt{3}M}$.
For each of the path constraints $g_q, q = 1, \dots, N_g$

$$\begin{aligned}&g_q(x_M(t_j), u_M(t_j)) \\ &\le g_q(x_M(t_j), u_M(t_j)) - g_q(x(t_j), u(t_j)) \\ &\le |g_q(x_M(t_j), u_M(t_j)) - g_q(x(t_j), u(t_j))| \\ &\le l''_{1q}\,|x(t_j) - x_M(t_j)| + l''_{2q}\,|u(t_j) - u_M(t_j)| \\ &\le l''_{1q}\frac{N_1}{\sqrt{3}M} + l''_{2q}\frac{N_2}{\sqrt{3}M},\end{aligned}$$

where l''_{1q} and l''_{2q} are Lipschitz constants of g_q for x and u respectively. Thus we have

$$g_q(x_M(t_j), u_M(t_j)) \le \delta_3, \quad j = 0, 1, \dots, M-1,$$

where

$$\delta_3 = \frac{l''_{1q}N_1 + l''_{2q}N_2}{\sqrt{3}M}.$$

Thus, (x_M, u_M) is a feasible solution to the problem (8)-(11). ☐

Remark 5.2. The set of feasible solutions to the problems (8)-(11) is non-empty as a consequence of Theorem 5.1.

Lemma 5.3. *Let (x_M, u_M) be any feasible solution to problem (8)-(11) such that x_M and u_M converge uniformly to x^f and u^f, respectively, with $\dot{x}^f$ and u^f continuous on $[0,1]$. Then, (x^f, u^f) is a feasible solution to problem (4)-(7).*

Proof. To prove that (x^f, u^f) is a feasible solution to problem (4)-(7), first we show that it satisfies the dynamic constraint (5). By the contradiction argument, suppose that (x^f, u^f) does not satisfy the differential equation (5). Then there exists some $t' \in [0,1]$ such that

$$|\dot{x}^f(t') - f(x^f(t'), u^f(t'))| > 0. \tag{13}$$

Since the collocation points $t_j, j = 0, 1, \dots, M-1$ are dense in $[0,1]$, there exists a sequence of indices $\{j_M\}_{M=0}^{\infty}$ such that

$$\lim_{M\to\infty} t_{j_M} = t'.$$

Thus, we have

$$\begin{aligned}&|\dot{x}^f(t') - f(x^f(t'), u^f(t'))| \\ &= \lim_{M\to\infty} |\dot{x}_M(t_{j_M}) - f(x_M(t_{j_M}), u_M(t_{j_M}))| \\ &\le \lim_{M\to\infty} \delta_1 = 0,\end{aligned}$$

which implies that

$$\dot{x}^f(t') - f(x^f(t'), u^f(t')) = 0,$$

a contradiction to our assumption.
Hence, we have

$$\dot{x}^f(t) = f(x^f(t), u(t)).$$

By a similar argument, it follows that (x^f, u^f) satisfies the endpoint constraints (6) and path constraints (7). Hence, (x^f, u^f) is a feasible solution to problem (4)-(7). ☐

Next, we prove the consistency of the approximation.

Theorem 5.4. *Suppose that (x^*_M, u^*_M) is a solution to problem (8)-(11) and there exist $(\tilde{x}, \tilde{u})$ with $\dot{\tilde{x}}(t)$ and $\tilde{u}(t)$ continuous on $[0,1]$ such that $x^*_M(t) \to \tilde{x}(t)$ and $u^*_M(t) \to \tilde{u}(t)$ uniformly. Then*

$$\lim_{M\to\infty} I[x^*_M, u^*_M] = I[\tilde{x}, \tilde{u}], \tag{14}$$

and $(\tilde{x}, \tilde{u})$ is an optimal solution to problem (4)-(7).

Proof. The cost functional of the continuous problem is given by

$$I[\tilde{x}, \tilde{u}] = \int_0^1 F(\tilde{x}(t), \tilde{u}(t))dt + E(\tilde{x}(0), \tilde{x}(1)),$$

while the cost functional of the approximation problem is

$$\begin{aligned}I[x^*_M, u^*_M] =& \frac{1}{M}\sum_{j=0}^{M-1} F(x^*_M(t_j), u^*_M(t_j)) \\ &+ E(x^*_M(0), x^*_M(1)).\end{aligned}$$

To prove (14), we first show that

$$\int_0^1 F(\tilde{x}(t),\tilde{u}(t))dt = \lim_{M\to\infty}\left(\frac{1}{M}\sum_{j=0}^{M-1} F(x_M^*(t_j),u_M^*(t_j))\right).$$

Since $F(\tilde{x}(t),\tilde{u}(t))$ is continuous in t, we have, [21]

$$\int_0^1 F(\tilde{x}(t),\tilde{u}(t))dt = \lim_{M\to\infty}\left(\frac{1}{M}\sum_{j=0}^{M-1} F(\tilde{x}(t_j),\tilde{u}(t_j))\right).$$

Thus, we have

$$\int_0^1 F(\tilde{x}(t),\tilde{u}(t))dt = \lim_{M\to\infty}\left(\frac{1}{M}\sum_{j=0}^{M-1} F(x_M^*(t_j),u_M^*(t_j))\right) + \lim_{M\to\infty}\left(\frac{1}{M}\sum_{j=0}^{M-1} (F(\tilde{x}(t_j),\tilde{u}(t_j)) - F(x_M^*(t_j),u_M^*(t_j)))\right).$$

Now by the Lipschitz continuity of $F(x,u)$, we have

$$|F(\tilde{x}(t_j),\tilde{u}(t_j)) - F(x_M^*(t_j),u_M^*(t_j))| \le L(|\tilde{x}(t_j)-x_M^*(t_j)| + |\tilde{u}(t_j)-u_M^*(t_j)|),$$

for some $L>0$ and for all $0\le j\le M-1$. Since $x_M^*(t)\to\tilde{x}(t)$ and $u_M^*(t)\to\tilde{u}(t)$ uniformly, we have

$$\lim_{M\to\infty}|\tilde{x}(t_j)-x_M^*(t_j)| = 0,$$

and

$$\lim_{M\to\infty}|\tilde{u}(t_j)-u_M^*(t_j)| = 0.$$

for all $j=0,1,\ldots,M-1$. Thus,

$$\lim_{M\to\infty}\left|\frac{1}{M}\sum_{j=0}^{M-1}(F(\tilde{x}(t_j),\tilde{u}(t_j)) - F(x_M^*(t_j),u_M^*(t_j)))\right| \le \lim_{M\to\infty}\frac{L}{M}\sum_{j=0}^{M-1}(|\tilde{x}(t_j)-x_M^*(t_j)| + |\tilde{u}(t_j)-u_M^*(t_j)|) = 0.$$

Hence, we get

$$\int_0^1 F(\tilde{x}(t),\tilde{u}(t))dt = \lim_{M\to\infty}\frac{1}{M}\sum_{j=0}^{M-1} F(x_M^*(t_j),u_M^*(t_j)). \quad (15)$$

It is straightforward to prove that

$$\lim_{M\to\infty} E(x_M^*(0),x_M^*(1)) = E(\tilde{x}(0),\tilde{x}(1)). \quad (16)$$

Thus, from (15) and (16), we have

$$\lim_{M\to\infty} I[x_M^*,u_M^*] = I[\tilde{x},\tilde{u}].$$

Next, to show that $(\tilde{x},\tilde{u})$ is an optimal solution to problem (4)-(7), we first show that it is a feasible solution. By Lemma 5.3, it follows that $(\tilde{x},\tilde{u})$ is a feasible solution to the problem (4)-(7).
Finally, we prove that $(\tilde{x},\tilde{u})$ is an optimal solution to problem (4)-(7) by using the contradiction argument. Suppose $(\tilde{x},\tilde{u})$ is not optimal and there exist optimal $(\hat{x},\hat{u})$ so that

$$I[\hat{x},\hat{u}] < I[\tilde{x},\tilde{u}].$$

Now by Theorem 5.1, $(\hat{x}_M,\hat{u}_M)$ is a feasible solution to problem (8)-(11). But since (x_M^*,u_M^*) is assumed to be optimal solution to problem (8)-(11), we have

$$I[x_M^*,u_M^*] < I[\hat{x}_M,\hat{u}_M].$$

Letting $M\to\infty$, we get

$$I[\tilde{x},\tilde{u}] < I[\hat{x},\hat{u}],$$

a contradiction.
Hence $(\tilde{x},\tilde{u})$ is an optimal solution to the problem (4)-(7). □

6 Applications

In this section, the Haar wavelet collocation method is applied to problems in fluid dynamics and economics. The accuracy of the method has been shown by evaluating the maximum absolute error L_∞ defined as

$$L_\infty = \max_{j=0,1,\ldots,M-1}|y(t_j)-y_M(t_j)|.$$

For the examples where the exact solution is unknown, $M=128$ has been considered as exact, and the L_∞ error has been calculated with respect to it.

Example 6.1. (**Temperature control of CSTR**)
Consider the temperature control problem, [22], of a continuous-stirred tank reactor (CSTR) by cooling-rate manipulation. The optimal control problem formulation of the temperature is as follows

$$\min\ I[x,u] = \int_0^{0.5}[(x(t)-1.3)^2 + \mu u^2(t)]dt,$$

subject to $\dot{x} = 1 - x(t) + ae^{-\gamma/x(t)} - u(t),$
$x(0)=1.5,\quad x(0.5)=1.3,$

where $a=1000, \gamma=10$ and $\mu=0.25$.

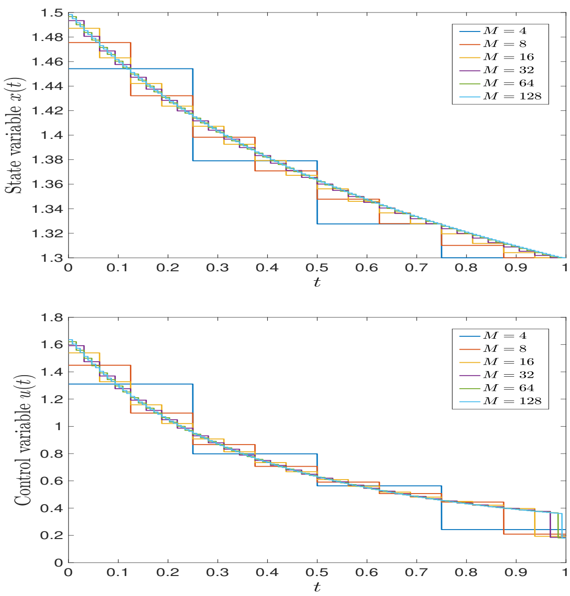

Fig. 1: Comparison of approximate states and controls for various values of M in case of Example 6.1.

Table 1. Comparison of values of cost functional and error for various values of M in case of Example 6.1

M	$I[x^*_M(\cdot), u^*_M(\cdot)]$	L_∞ for state	L_∞ for control
4	0.0892	0.0061	0.1659
8	0.0885	0.0040	0.0711
16	0.0882	0.0020	0.0314
32	0.0880	0.0009	0.0131
64	0.0879	0.0003	0.0043

The Haar wavelet approximations for the state and control variables for $M = 4, 8, 16, 32, 64, 128$ are shown in Figure 1. The values of the approximate cost functional and the L_∞ error in the state and control variables decrease with the increasing value of M, as shown in Table 1.

Example 6.2. (Isothermal reaction in the presence of catalyst)

Consider an isothermal liquid-phase reaction, [23], in a Continuous stirred tank reactor (CSTR) $A \longrightarrow B$ in the presence of a solid catalyst. It is required to find the $u(t)$, which represents the time-dependent volumetric throughput per unit reactor volume, that minimizes the deviation in the concentration x of species A and u from the reference condition (x_s, u_s), in a given time t_f. The corresponding optimal control problem can be formulated as

$$I[x, u] = \int_0^{t_f} [(x - x_s)^2 + (u - u_s)^2] dt,$$

with dynamics

$$\dot{x} = u(x_f - x) - kx^2,$$
$$x(0) = x_0,$$

where, x_f is the x in the feed and k is the reaction rate coefficient.
The parameters taken to get the numerical solution are $x_0 = 5\ g/cm^3$, $x_s = 8\ g/cm^3$, $x_f = 10\ g/cm^3$, $u_s = 5\ min^{-1}$, $k = 10^{-3}\ cm^3/(g \cdot min)$ and $t_f = 1\ min$.

For $M = 4, 8, 16, 32, 64, 128$, the approximate state and control variables are plotted and shown in Figure 2. It can be seen that the approximation improves as the value of M increases. The value of the cost functional, L_∞ error in the state and control variables also decreases with increasing value of M and is shown in Table 2.

Table 2. Comparison of values of cost functional and error for various values of M in case of Example 6.2

M	$I[x^*_M(\cdot), u^*_M(\cdot)]$	L_∞ for state	L_∞ for control
4	2.4464	0.3273	0.0954
8	2.4450	0.1420	0.0431
16	2.4443	0.0634	0.0195
32	2.4442	0.0266	0.0083
64	2.4441	0.0088	0.0027

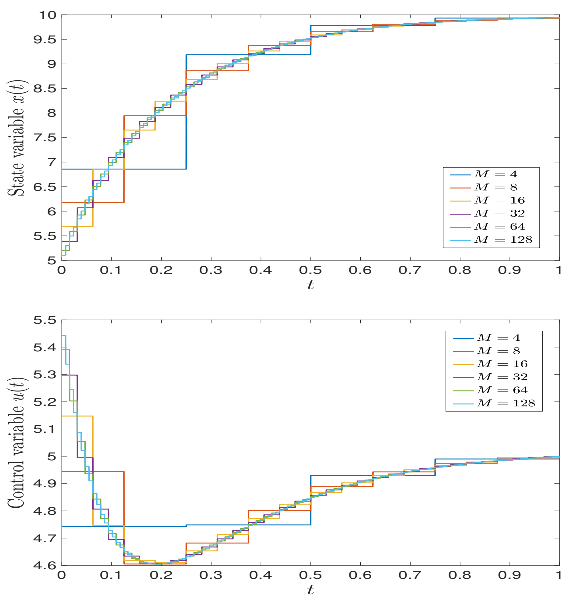

Fig. 2: Comparison of approximate states and controls for various values of M in case of Example 6.2.

Example 6.3. (Product quality control via pH value in a chemical reaction)

The pH value of a chemical reaction is an important factor that decides the quality of the product of the reaction. In this example, [24], we consider a chemical reaction over a fixed interval $[0, T]$. An ingredient with strength $u(t)$ is added to the chemical reaction to control the pH value $x(t)$ at time t. The rate of change in pH is assumed to be proportional to the current pH value and the strength of the ingredient u. The dynamics of the reaction is given by

$$\dot{x}(t) = \alpha x(t) + \beta u(t), \quad x(0) = x_0,$$

where $\alpha, \beta \in \mathbb{R}^+$ are known and x_0 is the initial pH value.
The cost functional to be minimized for this model is given by

$$I[x(\cdot), u(\cdot)] = \frac{1}{2}\int_0^T (ax^2(t) + u^2(t))dt,$$

where $\int_0^T x^2(t)dt$ is the decrease in the yield due to changes in pH, and the cost rate of maintaining the strength u is proportional to u^2.

The exact solution to this problem is given by

$$x^*(t) = \frac{1}{a\beta}[c_1(r+\alpha)e^{rt} - c_2(r-\alpha)e^{-rt}],$$
$$u^*(t) = c_1 e^{rt} + c_2 e^{-rt},$$

where

$$r = \sqrt{\alpha^2 + a\beta^2}, \quad c_1 = \frac{a\beta x_0}{(r+\alpha) + (r-\alpha)e^{2rT}},$$
$$c_2 = -\frac{a\beta x_0 e^{2rT}}{(r+\alpha) + (r-\alpha)e^{2rT}}.$$

Table 3. Comparison of values of cost functional and error for different values of M in case Example 6.3

M	$I[x_M^*(\cdot), u_M^*(\cdot)]$	L_∞ for state	L_∞ for control
4	15.9187	0.0127	1.3826
8	15.7378	0.0073	0.5832
16	15.6924	0.0059	0.2565
32	15.6811	0.0030	0.1069
64	15.6782	0.0011	0.0352
128	15.6775	0.0001	0.0001

The numerical solution is obtained by taking $\alpha = 2$, $\beta = 0.7$, $x_0 = 2$, $a = 3$ and $T = 1$. The optimal cost is 15.6773.

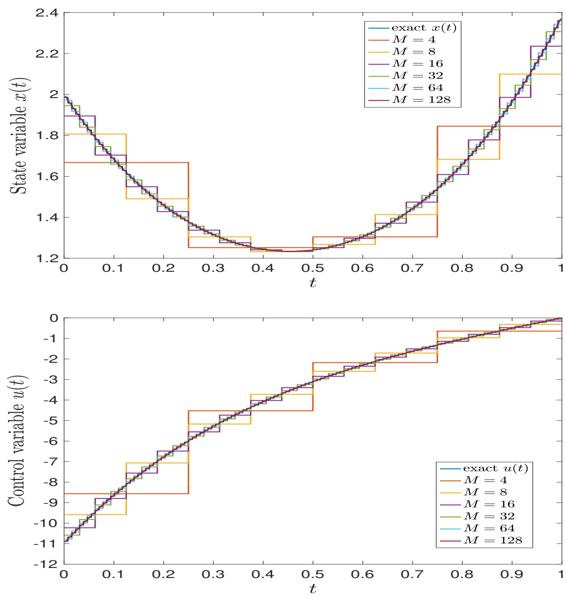

Fig. 3: Comparison of approximate states and controls for various values of M in case of Example 6.3.

Figure 3 shows that as the level of approximation increases with $M = 4, 8, 16, 32, 64$, and 128, the Haar wavelet approximations for the state and control variables become more accurate. The values of the estimated cost functional and the L_∞ error in the state and control variables decrease with increasing values of M, as shown in Table 3.

Example 6.4. (**Various investment problems**) In this example, we consider three types of investment problems, [25], [26], namely, the unbounded investment problem, the bounded investment problem, and the minimum control effort investment problem. Let $x(t)$ denote the available capital, $u(t)$ the gross capital expenditures, and $\dot{x}(t)$ the variation in the capital stock. In the case of unbounded investment, we want to maximize the profit performance measure, which can be formulated as an optimal control problem to maximize

$$I[x, u] = \int_0^1 \left(x - \frac{1}{2}u^2\right) dt, \qquad (17)$$
with dynamics $\dot{x} = u - \delta x,$
$$x(0) = 0, \quad x(1) = \text{free},$$

where δ is the depreciation rate.
The optimal solution to this problem is given by

$$x(t) = 1 - \frac{1}{2}e^{t-1} + \left(\frac{1}{2e} - 1\right)e^{-t}, \qquad (18)$$
$$u(t) = 1 - e^{t-1},$$

and the optimal cost is 0.0840.

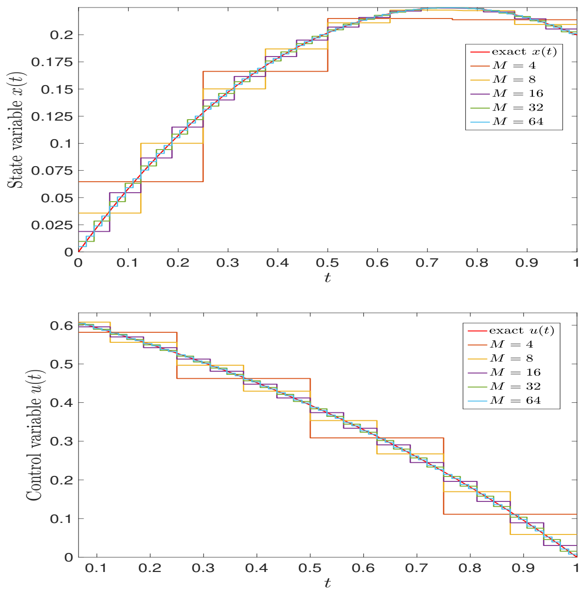

Fig. 4: Comparison of approximate states and controls for various values of M in case of unbounded investment problem.

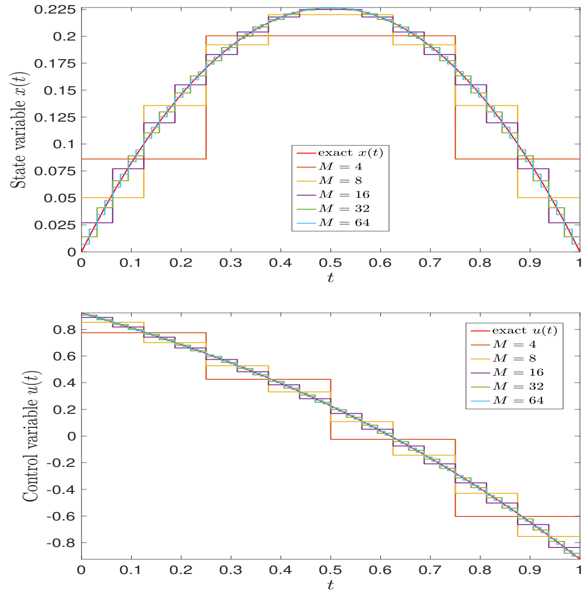

Fig. 5: Comparison of approximate states and controls for various values of M in case of bounded investment problem.

Table 4. Comparison of values of cost functional and error for various values of M in case of unbounded investment problem

M	$I[x_M^*(\cdot), u_M^*(\cdot)]$	L_∞ for state	L_∞ for control
4	0.0824	0.0067	0.0063
8	0.0836	0.0018	0.0017
16	0.0839	0.0004	0.0004
32	0.0840	0.0001	0.0001
64	0.0840	0.0001	0.0000

The bounded investment problem can be formulated to maximize

$$\begin{aligned} I[x,u] &= \int_0^1 \left(2x - \frac{1}{2}u^2\right) dt, \\ \text{with dynamics} \quad \dot{x} &= u - \delta x, \\ x(0) &= 0, \quad x(1) = 0, \end{aligned} \tag{19}$$

where the capital path is required to be zero at $T = 1$. The analytical solution is given by

$$\begin{aligned} x(t) &= 2 - \frac{2e^t + 2e^{1-t}}{e+1}, \\ u(t) &= 2 - \frac{4e^t}{e+1}, \end{aligned} \tag{20}$$

and the optimal cost is 0.1515.

Table 5. Comparison of values of cost functional and error for various values of M in case of bounded investment problem

M	$I[x_M^*(\cdot), u_M^*(\cdot)]$	L_∞ for state	L_∞ for control
4	0.1433	0.0140	0.0224
8	0.1495	0.0037	0.0061
16	0.1510	0.0009	0.0016
32	0.1514	0.0002	0.0004
64	0.1515	0.0000	0.0001

Finally, the minimum control effort investment problem, so called because of the performance measure of type $\int_0^T u^2 dt$, is to maximize

$$\begin{aligned} I[x,u] &= \int_0^1 -u^2 dt, \\ \text{with dynamics} \quad \dot{x} &= u - \delta x, \\ x(0) &= 0, \quad x(1) = 1. \end{aligned} \tag{21}$$

The exact solution is given by

$$\begin{aligned} x(t) &= \frac{e^{1-t} - e^{1+t}}{1 - e^2}, \\ u(t) &= \frac{-2e^{1+t}}{1 - e^2}, \end{aligned}$$

and the optimal cost is 2.3130.
By using an optimal investment strategy and minimizing the total cost in the functional, which is di-

rectly proportionate to the investments made, the objective is to accumulate a unit of capital over the interval $[0, 1]$.
For all three problems, the rate of depreciation, δ, is assumed to be 1.

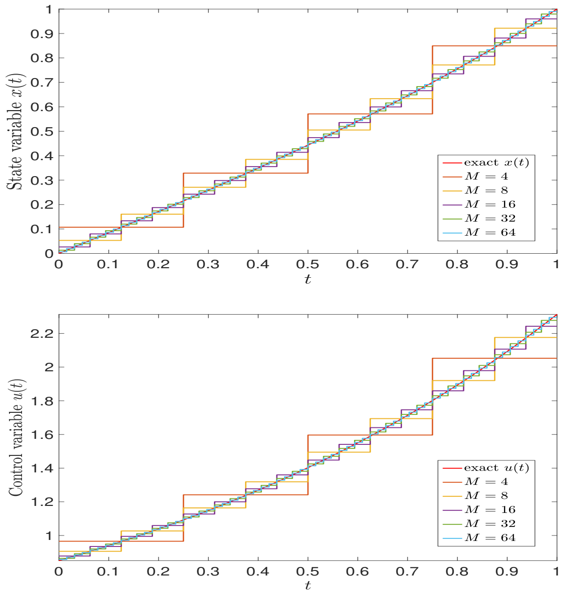

Fig. 6: Comparison of approximate states and controls for various values of M in case of minimum control effort investment problem.

Table 6. Comparison of values of cost functional and error for various values of M in case of minimum control effort investment problem

M	$I[x^*_M(\cdot), u^*_M(\cdot)]$	L_∞ for state	L_∞ for control
4	2.3092	0.0063	0.0114
8	2.3120	0.0017	0.0031
16	2.3127	0.0004	0.0008
32	2.3129	0.0001	0.0002
64	3.3130	0.0000	0.0000

Figure 4, Figure 5 and Figure 6 show the Haar wavelet approximations for $M = 4, 8, 16, 32, 64$ along with exact state and exact control for the unbounded investment problem, the bounded investment problem, and the minimum control effort investment problem, respectively. It can be seen through the graph that the approximations get better with increasing levels of approximation. Also, Table 4, Table 5, and Table 6 show that the Haar wavelet cost approaches the optimal cost as the value of M increases in all three cases.

7 Conclusion

The consistency and feasibility of the Haar wavelet collocation algorithm are proved for a nonlinear optimal control problem with mixed state and control constraints. Through the consistency result, it has been shown that the finite-dimensional nonlinear programming problem (8)-(11), consistently approximates the infinite-dimensional continuous problem (4)-(7). The implementation of the algorithm has been shown to solve various problems in fluid dynamics and economics. In the future, the current study can be extended further for optimal control problems with multiple states and multiple control varibles. The extended study can be applied to wide-ranging fields like aerospace, robotics, healthcare, and more while also incorporating machine learning, deep learning, and reinforcement learning.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

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-Amit Setia carried out the conceptualization, supervision, and writing review and editing.
-Muniyasamy M. carried out the conceptualization, formal analysis and validation.
-Ravi P. Agarwal carried out the supervision and writing review and editing.

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Conflicts of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.