Mitigating Multicollinearity in Linear Regression Model with Two Parameter Kibria-Lukman Estimators

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Abstract: - This study delves into the challenges faced by the ordinary least square (OLS) estimator, traditionally regarded as the Best Linear Unbiased Estimator in classical linear regression models. Despite its reliability under specific conditions, OLS falters in the face of multicollinearity, a problem frequently encountered in regression analyses. To combat this issue, various ridge regression estimators have been developed, characterized as one-parameter and two-parameter ridge-type estimators. In this context, our research introduces novel two-parameter estimators, building upon a recently developed one-parameter ridge estimator to mitigate the impact of multicollinearity in linear regression models. Theoretical analysis and simulation experiments were conducted to assess the performance of the proposed estimators. Remarkably, our results reveal that, under certain conditions, these new estimators outperform existing estimators, displaying a significantly reduced mean square error. To validate these findings, real-life data was employed, aligning with the outcomes derived from theoretical analysis and simulations.

Key-Words: - Kibria-Lukman Estimator, New Two-parameter Kibria-Lukman Estimator, Modified Two-parameter Kibria-Lukman Estimator, Monte Carlo Simulation, Multicollinearity, Mean Square Error.

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1 Introduction

Consider the general linear regression model defined in matrix form as:

$$y = X\beta + \varepsilon\ , \tag{1}$$

where y is a $n\times1$ vector of the response variable, X is a known $n\times p$ full-rank matrix of predictor variables, β is a $p\times1$ vector of unknown regression parameters to be estimated, and ε is $n\times1$ vector of random error such that E(ε) = 0 and Cov (ε) = $\sigma^2 I$. Equation (1) can be written in a canonical form as:

$$y = Z\alpha + \varepsilon \tag{2}$$

where $Z = XQ$, $\alpha = Q'\beta$ and Q is the orthogonal matrix whose columns constitute the eigenvectors of $X'X$. Then $Z'Z = Q'X'XQ = \Lambda = diag\left(\lambda_1,...,\lambda_p\right)$, where $\lambda_1 \ge \lambda_2 \ge ...\lambda_p >$ 0 are the ordered eigen values $X'X$. The ordinary least square estimator (OLSE) of β in (1) can be defined as:

$$\hat{\alpha}_{OLS} = \Lambda^{-1}X'y \tag{3}$$

where $\Lambda = X'X$ is the design matrix.

The Ordinary Least Squares (OLS) estimator is considered the Best Linear Unbiased Estimator (BLUE) when all assumptions of the classical linear regression model remain intact and unviolated. This characteristic has established the OLS estimator as the most powerful and popular tool for estimating regression models, [1]. To utilize the OLS estimator for estimating model parameters in linear regression, one essential assumption is the non-correlation of explanatory variables. However, the problem of multicollinearity arises when explanatory variables exhibit strong correlations or linear relationships, [2]. Multicollinearity, if present, severely affects the OLS estimator, leading to unstable and imprecise parameter estimates, questionable predictions, and invalid statistical inferences about model parameters. It also results in regression coefficients with exaggerated absolute values and signs that can reverse with minor changes in the data, [3], [4]. It also impacts t-tests, the extra sum of squares, fitted values, predictions, and coefficient of determination, [5].

To address the issue of multicollinearity, various biased estimators have been developed. Some of these one-parameter estimators include the Stein estimator [6], principal component estimator [7], ordinary ridge regression estimator by [8], modified ridge regression by [9], contraction estimator [10], Liu estimator [11], and KL [12]. Additionally, certain authors have introduced two-parameter estimators to combat multicollinearity, such as [13], [14], [15], [16], [17], [18], [19] and [20]. The ordinary ridge regression estimator (ORRE) is one of the most widely used biased estimators. It intends to overcome the multicollinearity problem by adding a positive value, biasing or shrinkage parameter k, to the ill-conditioned matrix's diagonal elements $X'X$. The selection of k is a major problem of ORRE. This is because the biasing parameter *k* plays a significant role in controlling the regression's bias toward the mean of the dependent variable, [8].

The Ordinary ridge regression estimator, as proposed by [8], stands as one of the most widely used among these estimators. It effectively addresses the issue of multicollinearity by introducing a positive value, denoted as k, to the diagonal elements of the Z'Z matrix. This constant k serves as the biasing parameter. However, a significant challenge associated with ridge regression lies in selecting this biasing parameter k, as it plays a crucial role in controlling the regression's bias towards the mean of the dependent variable. Various authors have put forth proposals for the biasing parameter k. Some notable contributors include, [21], [22], [23], [24], [25], [26], [27], [28], [29], [30] and numerous others. [11], proposed another estimator with d as the biasing parameter. The Liu Estimator gains preference due to its capacity for appropriate d selection, being a linear function of the biasing parameter d. However, selecting a suitable k remains challenging because the ordinary ridge regression estimator relies on a nonlinear function of the biasing parameter k [8].

This paper aims to introduce new two-parameter estimators for regression parameter estimation in scenarios where independent variables exhibit correlation. The performance of the proposed estimators is compared with the OLSE, ridge regression [8], Liu estimator [11], two-parameter estimator (TP) [13], Modified Ridge Type (MRT) [17], and Kibria-Lukman (KL) [12].

2 Some Alternative Biased Estimators and the Proposed Estimator

2.1 Some Estimators as Alternative to OLS

Ridge-type estimators have been proposed as an alternative to the OLSE. The canonical form of OLSE is written in equation (3). Following this, the ordinary ridge regression (RE) proposed by [8] is given as:

$$\hat{\alpha}_{RE}(k)=(\Lambda+kI)^{-1}Z'y=H_0^{-1}Z'y \tag{4}$$

where k is the non-negative constant known as the biasing parameter.

The Liu estimator (LE) [11] is defined as:

$$\hat{\alpha}_{LE}(d)=[\Lambda+I]^{-1}[\Lambda+dI]\hat{\alpha}_{OLS}=D_0D_1\hat{\alpha}_{OLS} \tag{5}$$

where d is the biasing parameter of the Liu Estimator.

The KL estimator [12] is given as:

$$\hat{\alpha}_{KL}(k)=(\Lambda+kI)^{-1}(\Lambda-kI)\hat{\alpha}_{OLS}=H_0^{-1}H_1\hat{\alpha}_{OLS} \tag{6}$$

TP estimator [13] is given as:

$$\hat{\alpha}_{TP}(k,d)=(\Lambda+kI)^{-1}(\Lambda+kdI)\hat{\alpha}_{OLS}=H_0^{-1}H_2\hat{\alpha}_{OLS} \tag{7}$$

where k and d are the biasing parameters of the ridge and Liu Estimator, respectively.

The Modified Ridge Type (MRT) Parameters proposed [17] is given as:

$$\hat{\alpha}_{MRT}(k,d)=(\Lambda+k(1+d)I)^{-1}\Lambda\hat{\alpha}_{OLS}=R_K\hat{\alpha}_{OLS} \tag{8}$$

where $H_0=(\Lambda+kI)$, $H_1=(\Lambda-kI)$, $H_2=(\Lambda+kdI)$, $H_3=(\Lambda-kdI)$, $D_0=(\Lambda+I)^{-1}$, $D_1=(\Lambda+dI)$ and $R_k=\Lambda(\Lambda+k(1+d)I)^{-1}$.

2.2 New Two parameter Kibria-Lukman (NTPKL) Estimator

In this article, the ridge estimator was grafted into the Kibria-Lukman estimator, [12], to propose new parameter estimators. The first proposed two-parameter estimator is defined as follows:

$$\hat{\alpha}_{NTPKL}(k,d)=(\Lambda+kd_1I)^{-1}(\Lambda-kd_2I)\hat{\alpha}_{RE} \tag{9}$$

$$\hat{\alpha}_{NTPKL}(k,d)=H_2^{-1}H_3\hat{\alpha}_{RE} \tag{10}$$

where $k>0$, $0<d<1$ and $d_1=d_2$

The following are the properties of the NTPKL Estimator

$$E(\hat{\alpha}_{NTPKL}(k,d))=\Lambda H_2^{-1}H_3H_0^{-1}\alpha \tag{11}$$

$$B(\hat{\alpha}_{NTPKL}(k,d))=(\Lambda H_2^{-1}H_3H_0^{-1}-I)\alpha \tag{12}$$

$$D[\hat{\alpha}_{NTPKL}(k,d)]=\sigma^2H_2^{-1}H_3H_0^{-1}\Lambda H_2^{-1}H_3H_0^{-1} \tag{13}$$

The Mean Square Error Matrix (MSEM) of the proposed estimator is given as:

$$MSEM\left[\hat{\alpha}_{NTPKL}(k,d)\right]=\sigma^2H_2^{-1}H_3H_0^{-1}\Lambda H_2^{-1}H_3H_0^{-1}+\left(\Lambda H_2^{-1}H_3H_0^{-1}-I\right)\alpha\alpha'\left(\Lambda H_2^{-1}H_3H_0^{-1}-I\right)' \tag{14}$$

2.3 Modified Two Parameter Kibria-Lukman (MTPKL) Estimators

The Modified Two Parameter Kibria-Lukman Estimators are special cases of the New Two Parameter Kibra-Lukman Estimator in equation (9). The first Modified estimator (MTPKL1) is obtained by equating d_2 to one (i.e., $d_2=1$) in equation (9).

$$\hat{\alpha}_{MTPKL1}(k,d)=(\Lambda+kdI)^{-1}(\Lambda-kI)\hat{\alpha}_{RE} \tag{15}$$

$$\hat{\alpha}_{MTPLK1}(k,d)=H_2^{-1}H_1\hat{\alpha}_{RE} \tag{16}$$

where $k>0$ and $0<d<1$.

The following are the properties of the MTPKL1 Estimator:

$$E(\hat{\alpha}_{MTPKL1}(k,d))=\Lambda H_0^{-1}H_1H_2^{-1}\alpha \tag{17}$$

$$B(\hat{\alpha}_{MTPKL1}(k,d))=(\Lambda H_0^{-1}H_1H_2^{-1}-I)\alpha \tag{18}$$

$$D[\alpha_{MTPKL1}(k,d)]=\sigma^2H_0^{-1}H_1H_2^{-1}\Lambda H_0^{-1}H_1H_2^{-1} \tag{19}$$

The Mean Square Error Matrix (MSEM) of the proposed MTPKL1 estimator is given as:

$$MSEM\left[\hat{\alpha}_{MTPKL1}(k,d)\right]=\sigma^2H_0^{-1}H_1H_2^{-1}\Lambda H_0^{-1}H_1H_2^{-1}+\left(\Lambda H_0^{-1}H_1H_2^{-1}-I\right)\alpha\alpha'\left(\Lambda H_0^{-1}H_1H_2^{-1}-I\right)' \tag{20}$$

The second Modified estimator (MTPKL2) is obtained by equating d_1 to one (i.e., $d_1=1$) in equation (9). It was also obtained by modifying, [13], following the idea of the (k – d) class estimator proposed by [31]. Thus, the MTPKL2 estimator can be defined as:

$$\hat{\alpha}_{MTPKL2}(k,d)=(\Lambda+kI)^{-1}(\Lambda-kdI)\hat{\alpha}_{RE} \tag{21}$$

$$\hat{\alpha}_{MTPKL2}(k,d) = H_0^{-1} H_3 \hat{\alpha}_{RE} \quad (22)$$

where *k>0* and *0<d<1*.

The following are the properties of the MTPKL2 Estimator:

$$E(\hat{\alpha}_{MTPKL2}(k,d)) = \Lambda H_0^{-1} H_3 H_0^{-1} \alpha \quad (23)$$

$$B(\hat{\alpha}_{MTPKL2}(k,d)) = (\Lambda H_0^{-1} H_3 H_0^{-1} - I)\alpha \quad (24)$$

$$D[\hat{\alpha}_{MTPKL2}(k,d)] = \sigma^2 H_0^{-1} H_3 H_0^{-1} \Lambda H_0^{-1} H_3 H_0^{-1} \quad (25)$$

The Mean Square Error Matrix (MSEM) of the proposed MTPKL2 estimator is given as:

$$MSEM\left[\hat{\alpha}_{MTPKL2}(k,d)\right] = \sigma^2 H_0^{-1} H_3 H_0^{-1} \Lambda H_0^{-1} H_3 H_0^{-1} + \left(\Lambda H_0^{-1} H_3 H_0^{-1} - I\right)\alpha\alpha'\left(\Lambda H_0^{-1} H_3 H_0^{-1} - I\right)' \quad (26)$$

Since the MTPKL1 and MTPKL2 estimators are special cases of NTPKL, Lemmas 1, 2 and 3 will be used to make some theoretical comparisons between NTPKL and six existing estimators and to prove the statistical properties of the NTPKL Estimator.

Lemma 1. *Let n x n matrices M > 0 and N > 0 (or $N \geq 0$), Then M > N if and only if $\lambda_i(NM^{-1}) < 1$ where $\lambda_i(NM^{-1})$ is the largest eigenvalue of the matrix NM^{-1}*, [32].

Lemma 2. *Let M be an n x n positive definite matrix, that is, M > 0, and α be some vector, then, $M - \alpha\alpha' \geq 0$ if and only if $\alpha' M^{-1} - \alpha \leq 1$*, [33].

Lemma 3. *Let $\hat{\alpha}_i = A_i y$, i = 1, 2, be two linear estimators of α. Suppose that $D = Cov(\hat{\alpha}_1) - Cov(\hat{\alpha}_2) > 0$, where $Cov(\hat{\alpha}_i), i = 1,2$ denotes the covariance matrix of $\hat{\alpha}_i$ and $b_i = Bias(\hat{\alpha}_i) = (A_i X - I)\alpha, i = 1,2$. Consequently,*

$$\Delta(\hat{\alpha}_1 - \hat{\alpha}_1) = MSEM(\hat{\alpha}_1) - MSEM(\hat{\alpha}_2) = \sigma^2 D + b_1 b_2' - b_2 b_2' > 0 \quad (27)$$

if and only if $b_2'\left[\sigma^2 D + b_1 b_1'\right]^{-1} b_2 < 1$ [34]

where $MSEM(\hat{\alpha}_i) = Cov(\hat{\alpha}_i) + b_i b_i'$

2.4 Comparison of NTPKL Estimator with Existing Estimators

In this section, the theoretical comparison is carried out among the estimators to examine the performance of the proposed estimator, $\hat{\alpha}_{NTPKL}(k,d)$ over other estimators; $\hat{\alpha}_{OLS}$, $\hat{\alpha}_{RE}$, $\hat{\alpha}_{LE}$, $\hat{\alpha}_{NTP}$, $\hat{\alpha}_{MRT}$, $\hat{\alpha}_{KL}$.

2.4.1 Comparison between $\hat{\alpha}_{OLS}$ and $\hat{\alpha}_{NTPKL}(k,d)$

The MSEM of the estimator $\hat{\alpha}_{OLS} = \Lambda^{-1} Z' y$ is as follows:

$$MSEM[\hat{\alpha}_{OLS}] = \sigma^2 \Lambda^{-1} \quad (28)$$

The difference between (14) and (28)

$$MESM(\hat{\alpha}_{OLS}) - MSEM[\hat{\alpha}_{NTPKL}(k,d)] = \sigma^2 \Lambda^{-1} - \sigma^2 H_2^{-1} H_3 H_0^{-1} \Lambda H_2^{-1} H_3 H_0^{-1} + (\Lambda H_2^{-1} H_3 H_0^{-1} - I)\alpha\alpha'(\Lambda H_2^{-1} H_3 H_0^{-1} - I)' \quad (29)$$

Let *k > 0* and 0 < *d* < 1. Thus, the following theorem holds.

Theorem: The proposed estimator $\hat{\alpha}_{NTPKL}(k,d)$ is superior to $\hat{\alpha}_{OLS}$ if and only if

$$\alpha'\left(\Lambda H_2^{-1} H_3 H_0^{-1} - I\right)' \sigma^2 \left[\left(\Lambda^{-1} - H_2^{-1} H_3 H_0^{-1} \Lambda H_2^{-1} H_3 H_0^{-1}\right)\right]^{-1} \left(\Lambda H_2^{-1} H_3 H_0^{-1} - I\right)\alpha < 1 \quad (30)$$

Proof

$$D(\hat{\alpha}_{OLS}) - D(\hat{\alpha}_{NTPKL}(k,d)) = \sigma^2\left(\Lambda^{-1} - H_2^{-1} H_3 H_0^{-1} \Lambda H_2^{-1} H_3 H_0^{-1}\right) = \sigma^2 diag\left\{\frac{1}{\lambda_i} - \frac{\lambda_i(\lambda_i - kd)^2}{(\lambda_i + kd)^2(\lambda_i + k)^2}\right\}_{i=1}^{p} \quad (31)$$

$\Lambda^{-1} - H_2^{-1} H_3 H_0^{-1} \Lambda H_2^{-1} H_3 H_0^{-1}$ will be pdf if and only if $(\lambda_i + kd)^2(\lambda_i + k)^2 - \lambda_i^2(\lambda_i - kd)^2 > 0$. By lemma 3, the proof is completed.

2.4.2 Comparison between $\hat{\alpha}_{RE}(k)$ and $\hat{\alpha}_{NTPKL}(k,d)$

The bias vector, covariance matrix, and MSEM of the estimator $\hat{\alpha}_{RE}(k)=(\Lambda+kI)^{-1}Z'y$ are as follows:

$$E[\hat{\alpha}_{RE}(k)]=(\Lambda+kI)^{-1}\Lambda\alpha \quad (32)$$

$$D[\hat{\alpha}_{RE}(k)]=\sigma^2(\Lambda+kI)^{-1}\Lambda(\Lambda+kI)^{-1} \quad (33)$$

and the Mean Square Error Matrix is given as: $MSEM[\hat{\alpha}_{RE}(k)]=\sigma^2 H_0^{-1}\Lambda H_0^{-1}+k^2 H_0^{-1}\alpha\alpha' H_0^{-1}$ (34)

The difference between (14) and (34)

$$MESM(\hat{\alpha}_{RE}(k))-MSEM[\hat{\alpha}_{NTPKL}(k,d)]$$
$$=\sigma^2 H_0^{-1}\Lambda H_0^{-1}-\ \sigma^2 H_2^{-1}H_3H_0^{-1}\Lambda H_2^{-1}H_3H_0^{-1}+k^2H_0^{-1}\alpha\alpha' H_0^{-1}$$
$$-\left(\Lambda H_2^{-1}H_3H_0^{-1}-I\right)\alpha\alpha'\left(\Lambda H_2^{-1}H_3H_0^{-1}-I\right)' \quad (35)$$

Let $k > 0$ and $0 < d < 1$. Thus, the following theorem holds.

Theorem: The proposed estimate $\hat{\alpha}_{NTPKL}(k,d)$ is superior to $\hat{\alpha}_{RE}(k)$ if and only if $MSEM[\hat{\alpha}_{RE}(k)]-MSEM[\hat{\alpha}_{NTPKL}(k,d)]>0$ if and only if

$$\alpha'\left(\Lambda H_2^{-1}H_3H_0^{-1}-I\right)'$$
$$\left[\sigma^2\left(H_0^{-1}\Lambda H_0^{-1}-H_2^{-1}H_3H_0^{-1}\Lambda H_2^{-1}H_3H_0^{-1}\right)+k^2H_0^{-1}\alpha\alpha' H_0^{-1}\right]^{-1} \quad (36)$$
$$\left(\Lambda H_2^{-1}H_3H_0^{-1}-I\right)\alpha<1$$

Proof: Considering the dispersion matrix difference between $D[\hat{\alpha}_{RE}(k)]$ and $D[\hat{\alpha}_{NTPKL}(k,d)]$

$$D_d=\sigma^2 H_0^{-1}\Lambda H_0^{-1}-\sigma^2 H_2^{-1}H_3H_0^{-1}\Lambda H_2^{-1}H_3H_0^{-1}$$
$$D_d=\sigma^2(\Lambda+kI)^{-1}\Lambda(\Lambda+kI)^{-1}-\sigma^2(\Lambda+kdI)^{-1}(\Lambda-kdI)$$
$$(\Lambda+kI)^{-1}\Lambda(\Lambda+kdI)^{-1}(\Lambda-kdI)(\Lambda+kI)^{-1}$$
$$D_d=\sigma^2(\Lambda+kI)^{-1}(\Lambda+kdI)^{-1}$$
$$\left[\Lambda(\Lambda+kdI)^2-\Lambda(\Lambda-kdI^2)\right](\Lambda+kI)^{-1}(\Lambda+kdI)^{-1}$$
$$D_d=\sigma^2(\Lambda+kI)^{-1}(\Lambda+kdI)^{-1}$$
$$\left[4\Lambda^2kdI\right](\Lambda+kI)^{-1}(\Lambda+kdI)^{-1} \quad (37)$$

It is observed that D_d is positive definite. By lemma 3, the proof is completed.

2.4.3 Comparison between $\hat{\alpha}_{LE}(d)$ and $\hat{\alpha}_{NTPKL}(k,d)$

The bias vector, covariance matrix, and MSEM of the estimator $\hat{\alpha}_{Liu}=(\Lambda+I)^{-1}(\Lambda+dI)\hat{\alpha}_{OLS}$ are as follows:

$$B[\hat{\alpha}_{LE}(d)]=(d-1)(\Lambda+I)^{-1}\alpha \quad (38)$$

$$D[\hat{\alpha}_{LE}]=\sigma^2(\Lambda+I)^{-1}(\Lambda+dI)^1\ \Lambda^{-1}(\Lambda+I)^{-1}(\Lambda+dI)^1 \quad (39)$$

$$MSEM[\hat{\alpha}_{LE}]=\sigma^2D_0D_1\Lambda^{-1}D'_0D'_1+$$
$$(d-1)D_0\alpha\alpha'(d-1)'D'_0{}' \quad (40)$$

Theorem: The proposed estimator $\hat{\alpha}_{NTPKL}(k,d)$ is superior to $\hat{\alpha}_{LE}(d)$ if and only if $MSEM[\hat{\alpha}_{LE}(d)]-MSEM[\hat{\alpha}_{NTPKL}(k,d)]>0$ if and only if

$$\alpha'\left(\Lambda H_2^{-1}H_3H_0^{-1}-I\right)'\begin{bmatrix}\sigma^2\begin{pmatrix}D_0D_1\Lambda^{-1}D'_0D'_1-\\ \sigma^2H_2^{-1}H_3H_0^{-1}\Lambda H_2^{-1}H_3H_0^{-1}\end{pmatrix}+\\ (d-1)D_0\alpha\alpha'(d-1)'D'_0\end{bmatrix}^{-1}$$
$$\left(\Lambda H_2^{-1}H_3H_0^{-1}-I\right)\alpha<1$$

Proof: Considering the dispersion matrix difference between $D[\hat{\alpha}_{LE}(d)]$ and $D[\hat{\alpha}_{NTPKL}(k,d)]$

$$D_d=\sigma^2D_0D_1\Lambda^{-1}D'_0D'_1-\sigma^2H_2^{-1}H_3H_0^{-1}\Lambda H_2^{-1}H_3H_0^{-1}$$
$$D_d=\sigma^2(\Lambda+I)^{-1}(\Lambda+dI)^1\ \Lambda^{-1}(\Lambda+I)^{-1}(\Lambda+dI)^1-$$
$$\sigma^2(\Lambda+kdI)^{-1}(\Lambda-kdI)(\Lambda+kI)^{-1}\Lambda$$
$$(\Lambda+kdI)^{-1}(\Lambda-kdI)(\Lambda+kI)^{-1}$$
$$D_d=\sigma^2diag\left\{\frac{(\lambda_i+d)^2}{\lambda_i(\lambda_i+1)^2}-\frac{\lambda_i(\lambda_i-kd)^2}{(\lambda_i+kd)^2(\lambda_i+k)^2}\right\}_i^p \quad (41)$$

will be pdf if and only if $(\lambda_i+d)^2(\lambda_i+kd)^2(\lambda_i+k)^2-\lambda^2(\lambda_i-kd)^2(\lambda_i+1)^2>0$

For 0 < d < 1 and k > 0, it was observed that $(\lambda_i+d)^2(\lambda_i+kd)^2(\lambda_i+k)^2-\lambda^2(\lambda_i-kd)^2(\lambda_i+1)^2>0$

By lemma 3, the proof is completed.

2.4.4 Comparison between $\hat{\alpha}_{KL}(k)$ and $\hat{\alpha}_{NTPKL}(k,d)$

The bias vector, covariance matrix, and MSEM of the estimator $\hat{\alpha}_{KL}(k)=(\Lambda+kI)^{-1}(\Lambda-kI)\hat{\alpha}$ are as follows:

$$B[\hat{\alpha}_{KL}(k)]=[H_0^{-1}H_1-I]\alpha \tag{42}$$

$$D[\hat{\alpha}_{KL}(k)]=\sigma^2 H_0^{-1}H_1\Lambda^{-1}H_0^{-1}H_1' \tag{43}$$

$$MSEM[\hat{\alpha}_{KL}(k)]=\sigma^2 H_0^{-1}H_1\Lambda^{-1}H_0^{-1}H_1 + [H_0^{-1}H_1-I]\alpha\alpha'[H_0^{-1}H_1-I]' \tag{44}$$

Theorem: The proposed estimator $\hat{\alpha}_{NMKL2}(k,d)$ is superior to $\hat{\alpha}_{KL}(k)$ if and only if: $MSEM[\hat{\alpha}_{KL}(k)]-MSEM[\hat{\alpha}_{NTPKL}(k,d)]>0$ if and only if

$$\alpha'\left(\Lambda H_2^{-1}H_3H_0^{-1}-I\right)'\begin{bmatrix}\sigma^2\begin{pmatrix}H_0^{-1}H_1\Lambda^{-1}H_0^{-1}H_1- \\ H_2^{-1}H_3H_0^{-1}\Lambda H_2^{-1}H_3H_0^{-1}\end{pmatrix} \\ +H_0^{-1}H_1\alpha\alpha'H_0^{-1}H_1\end{bmatrix}^{-1}\left(\Lambda H_2^{-1}H_3H_0^{-1}-I\right)\alpha<1 \tag{45}$$

Proof: Considering the dispersion matrix difference between $D[\hat{\alpha}_{KL}(k)]$ and $D[\hat{\alpha}_{NTPKL}(k,d)]$

$$D_d=\sigma^2H_0^{-1}H_1\Lambda^{-1}H_0^{-1}H_1-\sigma^2H_2^{-1}H_3H_0^{-1}\Lambda H_2^{-1}H_3H_0^{-1}$$

$$D_d=\sigma^2(\Lambda-kI)(\Lambda+kI)^{-1}\Lambda^{-1}(\Lambda-kI)(\Lambda+kI)^{-1}-\sigma^2(\Lambda+kdI)^{-1}(\Lambda-kdI)(\Lambda+kI)^{-1}\Lambda(\Lambda+kdI)^{-1}(\Lambda-kdI)(\Lambda+kI)^{-1}$$

$$D_d=\sigma^2 diag\left\{\frac{(\lambda_i-k)^2}{\lambda_i(\lambda_i+k)^2}-\frac{\lambda_i(\lambda_i-kd)^2}{(\lambda_i+kd)^2(\lambda_i+k)^2}\right\}_i^p$$

will be pdf if and only if $(\lambda_i-k)^2(\lambda_i+kd)^2-\lambda_i^2(\lambda_i-kd)^2>0$. For 0 < d < 1 and k > 0, it was observed that $(\lambda_i-k)^2(\lambda_i+kd)^2-\lambda_i^2(\lambda_i-kd)^2>0$. By lemma 3, the proof is completed.

2.4.5 Comparison between $\hat{\alpha}_{MRT}(k,d)$ and $\hat{\alpha}_{NTPKL}(k,d)$

The bias vector, covariance matrix, and MSEM of the estimator $\hat{\alpha}_{MRT}(k,d)=(\Lambda+k(1+d)I)^{-1}\Lambda\hat{\alpha}_{OLS}=R_K\hat{\alpha}_{OLS}$ are as follows:

$$B[\hat{\alpha}_{MRT}(k,d)]=[R_k-I]\alpha \tag{46}$$

$$D[\hat{\alpha}_{MRT}(k,d)]=\sigma^2R_k\Lambda^{-1}R'_k \tag{47}$$

$$MSEM[\hat{\alpha}_{MRT}(k,d)]=\sigma^2R_k\Lambda^{-1}R'_k+[R_k-I]\alpha\alpha'[R_k-I]' \tag{48}$$

Where $R_k=\Lambda(\Lambda+k(1+d)I)^{-1}$. Let k > 0 and 0 < d < 1. Thus, the following theorem holds.

Theorem: The proposed estimator $\hat{\alpha}_{NTPKL}(k,d)$ is superior to $\hat{\alpha}_{MRT}(k,d)$ if and only if $MSEM[\hat{\alpha}_{MRT}(k,d)]-MSEM[\hat{\alpha}_{NTPKL}(k,d)]>0$ if and only if

$$\alpha'\left(\Lambda H_2^{-1}H_3H_0^{-1}-I\right)'\left[\sigma^2\left(R_k\Lambda^{-1}R'_k-\sigma^2H_2^{-1}H_3H_0^{-1}\Lambda H_2^{-1}H_3H_0^{-1}\right)+R_k\alpha\alpha'R'_k\right]^{-1}\left(\Lambda H_2^{-1}H_3H_0^{-1}-I\right)\alpha<1 \tag{49}$$

Proof: Considering the dispersion matrix difference between $D[\hat{\alpha}_{MRT}(k,d)]$ and $D[\hat{\alpha}_{NTPKL}(k,d)]$

$$D_d=\sigma^2R_k\Lambda^{-1}R'_k-\sigma^2H_2^{-1}H_3H_0^{-1}\Lambda H_2^{-1}H_3H_0^{-1}$$

$$D_d=\sigma^2(\Lambda+k(1+d)I)^{-1}\Lambda(\Lambda+k(1+d)I)^{-1}-\sigma^2(\Lambda+kdI)^{-1}(\Lambda-kdI)(\Lambda+kI)^{-1}\Lambda(\Lambda+kdI)^{-1}(\Lambda-kdI)(\Lambda+kI)^{-1}$$

$$D_d=\sigma^2 diag\left\{\frac{\lambda_i}{(\lambda_i+k(1+d))^2}-\frac{\lambda_i(\lambda_i-kd)^2}{(\lambda_i+kd)^2(\lambda_i+k)^2}\right\}_{i=1}^p \tag{50}$$

will be pdf if and only if $\lambda_i(\lambda_i+kd)^2(\lambda_i+k)^2-\lambda_i(\lambda_i-kd)^2(\lambda_i+k(1+d))^2>0.$ For 0 < d < 1 and k > 0, it was observed that

$$\lambda_i(\lambda_i+kd)^2(\lambda_i+k)^2-\lambda_i(\lambda_i-kd)^2(\lambda_i+k(1+d))^2>0.$$

By lemma 3, the proof is completed.

2.4.6 Comparison between $\hat{\alpha}_{TP}(k,d)$ and $\hat{\alpha}_{NTPKL}(k,d)$

The bias vector, covariance matrix, and MSEM of the estimator $\hat{\alpha}_{TP}(k,d)=(\Lambda+kI)^{-1}(\Lambda+kdI)\hat{\alpha}_{OLS}$ are as follows:

$$B[\hat{\alpha}_{TP}(k,d)]=[H_0^{-1}H_2-I]\alpha \quad (51)$$

$$D[\hat{\alpha}_{TP}(k,d)]=\sigma^2H_0^{-1}H_2\Lambda^{-1}H_0^{-1}H'_2 \quad (52)$$

$$MSEM[\hat{\alpha}_{TP}(k,d)]=\sigma^2H_0^{-1}H_2\Lambda^{-1}H_0^{-1}H_2+[H_0^{-1}H_2-I]\alpha\alpha'[H_0^{-1}H_2-I]' \quad (53)$$

Let k > 0 and 0 < d < 1. Thus, the following theorem holds.

Theorem: The proposed estimator $\hat{\alpha}_{NTPKL}(k,d)$ is superior to $\hat{\alpha}_{TP}(k,d)$ if and only if

$$MSEM[\hat{\alpha}_{TP}(k,d)]-MSEM[\hat{\alpha}_{NTPKL}(k,d)]>0 .$$

That is, if and only if,

$$\alpha'(\Lambda H_2^{-1}H_3H_0^{-1}-I)'\left[\sigma^2\begin{pmatrix}H_0^{-1}H_2\Lambda^{-1}H_0^{-1}H_2\\-H_2^{-1}H_3H_0^{-1}\Lambda H_2^{-1}H_0^{-1}\end{pmatrix}+(H_0^{-1}H_3-I)\alpha\alpha'(H_0^{-1}H_3-I)'\right]^{-1}(\Lambda H_2^{-1}H_3H_0^{-1}-I)\alpha<1 \quad (54)$$

Proof: Considering the dispersion matrix difference between $D[\hat{\alpha}_{TP}(k,d)]$ and $D[\hat{\alpha}_{NTPKL}(k,d)]$

$$D_d=\sigma^2H_0^{-1}H_2\Lambda^{-1}H_0^{-1}H_2-\sigma^2H_2^{-1}H_3H_0^{-1}\Lambda H_2^{-1}H_3H_0^{-1}$$

$$D_d=\sigma^2(\Lambda+kdI)(\Lambda+kI)^{-1}\Lambda^{-1}(\Lambda+kdI)(\Lambda+kI)^{-1}-\sigma^2(\Lambda+kdI)^{-1}(\Lambda-kdI)(\Lambda+kI)^{-1}\Lambda(\Lambda+kdI)^{-1}(\Lambda-kdI)(\Lambda+kI)^{-1}$$

$$D_d=\sigma^2\, diag\left\{\frac{(\lambda_i+kd)^2}{\lambda(\lambda_i+k)^2}-\frac{\lambda_i(\lambda_i-kd)^2}{(\lambda_i+kd)^2(\lambda_i+k)^2}\right\}_{i=1}^{p} \quad (55)$$

will be pdf if and only if $(\lambda_i+kd)^4-\lambda_i^2(\lambda_i-kd)^2>0$. For 0 < d < 1 and k > 0, it was observed that $(\lambda_i+kd)^4-\lambda_i^2(\lambda_i-kd)^2>0$. By lemma 3, the proof is completed.

2.5 Determination of Biasing Parameters k and d

Finding the appropriate ridge and Liu parameters, *k* and *d*, respectively, is a critical issue in the study of the ridge and Liu regression. These parameters may either be non-stochastic or may depend on the observed data. The choice of values for these ridge parameters has been one of the most difficult problems confronting the study of the generalized ridge regression, [35]. The biasing parameters k proposed by [12] in equation (6) will be used to determine and evaluate the performance of the proposed estimators compared to the OLS estimator and other estimators. It is given as:

$$k=\frac{\hat{\sigma}^2}{2\hat{\alpha}_i^2+(\hat{\sigma}^2/\lambda_i)} \quad (56)$$

In determining the optimal value of *d* for $\hat{\alpha}_{NTPKL}(k,d)$, *k* is fixed. The optimal value of *d* can be regarded as to be that *d* that minimizes

$$MSEM[\hat{\alpha}_{NTPKL}(k,d)]=\sigma^2H_2^{-1}H_3H_0^{-1}\Lambda H_2^{-1}H_3H_0^{-1}+(\Lambda H_2^{-1}H_3H_0^{-1}-I)\alpha\alpha'(\Lambda H_2^{-1}H_3H_0^{-1}-I)'$$

$$f(k,d)=MSEM[\hat{\alpha}_{NTPKL}(k,d)]=tr[MSEM(\hat{\alpha}_{NTPKL}(k,d))]$$

$$f(k,d)=\sigma^2\sum_i^p\frac{\lambda_i(\lambda_i-kd)^2}{(\lambda_i+kd)^2(\lambda_i+k)^2}+\sum_i^p\alpha_i^2\frac{(2\lambda_ikd+\lambda_ik+k^2d)^2}{(\lambda_i+kd)^2(\lambda_i+k)^2} \quad (57)$$

Differentiating $f(k,d)$ with respect to *d* and equate to zero gives

$$d=\sum_i^p\left[\frac{\lambda_i(\sigma^2-\alpha^2k)}{2\alpha^2\lambda k+\alpha^2k^2+\sigma^2k}\right] \quad (58)$$

For practical purposes, σ^2 and α_i^2 are replaced with $\hat{\sigma}^2$ and $\hat{\alpha}_i^2$, respectively. Consequently, (58) becomes

$$\hat{d} = \sum_{i}^{p}\left[\frac{\lambda_i\left(\hat{\sigma}^2 - \hat{\alpha}_i^2 k\right)}{2\hat{\alpha}_i^2\lambda k + \hat{\alpha}_i^2 k^2 + \hat{\sigma}^2 k}\right] \qquad (59)$$

3 Simulation Study

To support the theoretical comparison of the estimators, a Monte Carlo simulation study was conducted using R 4.0.2 to examine the performance of these estimators. Also, in this section, the results of the simulation will be discussed.

3.1 Simulation Technique

The simulation procedure used by [22], [25], [35] and [36], were used to generate the explanatory variables in this study: This is given as:

$$x_{ij} = \left(1-\rho^2\right)^{1/2} z_{ij} + \rho z_{i,p+1},\ i = 1,2,\ldots,n,\ j = 1,2,\ldots,p. \qquad (60)$$

where z_{ij} is an independent standard normal pseudo-random number, ρ is the correlation between any two explanatory variables considered to be 0.8, 0.9, 0.95, and 0.99. Also, p is the number of explanatory variables, and p=3 is considered for the simulation study. The variables are standardized so that $X'X$ and $X'y$ are in correlation forms. The values of $\beta'\beta = 1$, [37]. The standard deviations in this simulation study were σ = 1, 3, 5, and 10. In comparing the estimators, *k* was chosen to be 0.3, 0.6, and 0.9, which lies between 0 and 1, following [35] and [38]. We also chose d = 0.2, 0.5, 0.8 for sample sizes 50 and 100. The replication for the study is 1000 times. The mean square error was obtained using:

$$MSE(\hat{\alpha}) = \frac{1}{1000}\sum_{j=1}^{1000}\left(\hat{\alpha}_{ij} - \alpha_i\right)'\left(\hat{\alpha}_{ij} - \alpha_i\right) \qquad (61)$$

Table 1 (Appendix) shows that when k=0.3 and d=0.2, $\hat{\alpha}_{LE}(d)$ and $\hat{\alpha}_{MTPKL1}(k,d)$ consistently perform better than $\hat{\alpha}_{NTPKL}(k,d)$and other existing estimators across the four levels of sigma (1, 3, 5, and 10) and at the two sample sizes (n = 50 and 100), particularly when rho is 0.8, 0.9, 0.95. At sample size 50, $\hat{\alpha}_{KL}(k)$and $\hat{\alpha}_{MTPKL1}(k,d)$dominates other estimators, while at n= 100, the proposed estimators, $\hat{\alpha}_{MTPKL1}(k,d)$ and $\hat{\alpha}_{MTPKL2}(k,d)$perform better than other estimators when rho is 0.99.

From Table 2, Table 3, Table 9 and Table 12 in Appendix (when k=0.3 and d=0.5, k=0.3 and d=0.8, k=0.7 and d=0.8, k=0.9 and d=0.8), the simulation results show that two of the proposed estimators; $\hat{\alpha}_{MTPKL1}(k,d)$ and $\hat{\alpha}_{MTPKL2}(k,d)$consistently outperform $\hat{\alpha}_{NTPKL}(k,d)$ and other existing estimators in this study at sample size 50 and 100 across the four levels of sigma (1, 3, 5 and 10) and at rho is 0.8, 0.9, 0.95, and 0.99

From Table 4 and Table 7 in Appendix (when k=0.6 and d=0.2, k=0.7 and d=0.2), the simulation results show that two of the proposed estimators; $\hat{\alpha}_{MTPKL1}(k,d)$ and $\hat{\alpha}_{MTPKL2}(k,d)$consistently dominate $\hat{\alpha}_{NTPKL}(k,d)$ and other existing estimators in this study except for when rho is 0.99, where $\hat{\alpha}_{MTPKL1}(k,d)$ and $\hat{\alpha}_{KL}(k)$ dominates $\hat{\alpha}_{NTPKL}(k,d)$ and other existing estimators at sample size 50 and 100.

From Table 5 and Table 8 in Appendix (when k=0.6 and d=0.5, k=0.7 and d=0.5), the simulation results show that two of the proposed estimators; $\hat{\alpha}_{MTPKL1}(k,d)$ and $\hat{\alpha}_{MTPKL2}(k,d)$consistently dominate $\hat{\alpha}_{NTPKL}(k,d)$ and other existing estimators in this study except for when rho is 0.99, where $\hat{\alpha}_{MTPKL1}(k,d)$ and $\hat{\alpha}_{KL}(k)$ dominates $\hat{\alpha}_{NTPKL}(k,d)$ and other existing estimators at sample size 50 only.

From Table 6 in Appendix (when k=0.6 and d=0.8), the simulation results show that two of the proposed estimators; $\hat{\alpha}_{MTPKL1}(k,d)$ and $\hat{\alpha}_{MTPKL2}(k,d)$consistently dominate $\hat{\alpha}_{NTPKL}(k,d)$ and other existing estimators in this study except for when rho is 0.99, sigma = 10 and n=50 where $\hat{\alpha}_{MTPKL1}(k,d)$ and $\hat{\alpha}_{NTPKL}(k,d)$ dominates $\hat{\alpha}_{MTPKL2}(k,d)$ and other existing estimators.

From Table 10 in Appendix (when k=0.9 and d=0.2), the simulation results show that two of the proposed estimators; $\hat{\alpha}_{MTPKL1}(k,d)$ and $\hat{\alpha}_{MTPKL2}(k,d)$consistently dominate $\hat{\alpha}_{NTPKL}(k,d)$ and other existing estimators in this study except for when rho is 0.95 and 0.99, where $\hat{\alpha}_{MTPKL1}(k,d)$ and $\hat{\alpha}_{KL}(k)$ dominates $\hat{\alpha}_{MTPKL2}(k,d)$ and other existing estimators at sample size 50. However, when the sample size is 100, $\hat{\alpha}_{MTPKL1}(k,d)$ and $\hat{\alpha}_{MTPKL2}(k,d)$ consistently dominates $\hat{\alpha}_{NTPKL}(k,d)$ and other existing estimators except for when rho 0.99, where $\hat{\alpha}_{MTPKL1}(k,d)$ and $\hat{\alpha}_{KL}(k)$ dominates $\hat{\alpha}_{MTPKL2}(k,d)$ and other existing estimators.

From Table 11 in Appendix (when k=0.9 and d=0.5), the simulation results show that two of the proposed estimators; $\hat{\alpha}_{MTPKL1}(k,d)$ and $\hat{\alpha}_{MTPKL2}(k,d)$ consistently dominate $\hat{\alpha}_{NTPKL}(k,d)$ and other existing estimators in this study while $\hat{\alpha}_{MTPKL1}(k,d)$ has the least MSE value among the three proposed estimators across the sample sizes, sigma and rho levels used in this study.

4 Numerical Example

In this section, Longley data was used to demonstrate the performance of the proposed estimator. Longley data were used by [39] and [40]. The regression model for these data is defined as:

$$y = \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_6 X_6 + \varepsilon \qquad (62)$$

For more details on the data set, [39]. The variance inflation factors are VIF_1 = 135.53, VIF_2 = 1788.51, VIF_3 = 33.62, VIF_4 = 3.59, VIF_5 = 399.15 and VIF_6 = 758.98. Eigenvalues of $X'X$ matrix are $\lambda_1 = 2.76779 \times 10^{12}$, λ_2 = 7,039,139,179, λ_3 = 11,608,993.96, λ_4 = 2,504,761.021, λ_5 = 1738.356, λ_6 = 13.309 and the condition number of $X'X$ is 456,070. The VIFs, the eigenvalues, and the condition number all indicate that severe multicollinearity exists. The estimated parameters and the MSE values of the estimators are presented in Table 13 (Appendix). Two of the proposed estimators perform best among other estimators as they given the smallest MSE value
Though, $\hat{\alpha}_{MTPKL1}(k,d)$and $\hat{\alpha}_{MTPKL2}(k,d)$ which are special cases of $\hat{\alpha}_{NTPKL}(k,d)$ perform better than $\hat{\alpha}_{NTPKL}(k,d)$ and other existing estimators, as they give smaller MSE values compared with all the existing estimators considered in this study, $\hat{\alpha}_{MTPKL1}(k,d)$ performs best in all. This is consistent with the results of the simulation study.

5 Summary and Concluding Remarks

This paper proposes new two-parameter estimators to solve the multicollinearity problem for linear regression models. The proposed estimators were theoretically compared with six other existing estimators. A simulation study was then conducted to compare the performance of the proposed estimators; $\hat{\alpha}_{MTPKL1}(k,d)$, $\hat{\alpha}_{MTPKL2}(k,d)$ and $\hat{\alpha}_{NTPKL}(k,d)$ with six existing estimators. From the theoretical comparison, simulation study, and application of life data, the proposed estimators, especially the two special cases of $\hat{\alpha}_{NTPKL}(k,d)$ which are $\hat{\alpha}_{MTPKL1}(k,d)$and $\hat{\alpha}_{MTPKL2}(k,d)$ give better results in terms of MSE. Hopefully, this paper will be helpful to researchers in different fields. The proposed estimators are as a result of this, recommended for use by researchers in various fields.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Idowu, J.I. Ayinde, K., Oladapo, O. J. and Owolabi, A.T. conceived and designed the study. Oladapo, O. J., Alao, N. A., Idowu, J.I. and Oshuporu, O. A. conducted the data analysis. Ayinde K. supervised the study. Idowu, J. I., Oladapo, O. J., Owolabi, A.T., Ayinde K., and Alao, N. A. interpreted the study results. Owolabi, A. T., Oladapo, O.J., Idowu, J. I. and Oshuporu, O. A. wrote the first draft of the manuscript. Ayinde K. and Owolabi, A. T. reviewed and corrected the draft manuscript. All authors read through and approved the final manuscript.

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The authors declare that they have no conflicts of interest.

APPENDIX

Table 1. Estimated MSE when k=0.3, d=0.2

N	σ^2	rho	OLS	RIDGE	LIU	K-L	MRT	TP	NTPKL	MTPKL1	MTPKL2
50	1	0.8	0.1363	0.1311	**0.1234**	0.1261	0.1301	0.1322	0.1291	***0.1251***	0.1252
		0.9	0.2462	0.2288	***0.2046***	0.2120	0.2255	0.2322	0.2220	**0.2089**	0.2094
		0.95	0.4682	0.4064	**0.3333**	0.3492	0.3956	0.4184	0.3833	***0.3393***	0.3429
		0.99	2.2417	1.2173	0.7223	**0.5101**	1.0998	1.3968	0.9166	***0.4469***	0.5771
	3	0.8	1.2270	1.1802	***1.1098***	1.1344	1.1712	1.1895	1.1618	**1.1255**	1.1264
		0.9	2.2158	2.0587	***1.8412***	1.9076	2.0293	2.0896	1.9980	**1.8794**	1.8847
		0.95	4.2136	3.6579	***2.9999***	3.1428	3.5604	3.7658	3.4501	**3.0533**	3.0857
		0.99	20.1751	10.9561	6.5009	**4.5913**	9.8985	12.5716	8.2494	***4.0221***	5.1935
	5	0.8	3.4082	3.2782	***3.0825***	3.1509	3.2532	3.3040	3.2272	**3.1264**	3.1288
		0.9	6.1551	5.7185	***5.1142***	5.2987	5.6370	5.8045	5.5499	**5.2205**	5.2351
		0.95	11.7045	10.1609	***8.3330***	8.7300	9.8901	10.4606	9.5836	**8.4815**	8.5716
		0.99	56.0420	30.4336	18.0581	**12.7536**	27.4958	34.9210	22.9151	***11.1726***	14.4263
	10	0.8	13.6328	13.1127	***12.3294***	12.6032	13.0124	13.2158	12.9084	**12.5051**	12.5148
		0.9	24.6204	22.8740	***20.4565***	21.1943	22.5479	23.2179	22.1993	**20.8816**	20.9400
		0.95	46.8180	40.6437	***33.3323***	34.9200	39.5605	41.8426	38.3346	**33.9260**	34.2864
		0.99	224.1680	121.7344	72.2327	**51.0147**	109.9835	139.6840	91.6605	***44.6905***	57.7054
100	1	0.8	0.0622	0.0611	***0.0594***	0.0600	0.0609	0.0613	0.0607	**0.0598**	**0.0598**
		0.9	0.1131	0.1093	***0.1036***	0.1056	0.1086	0.1101	0.1078	**0.1049**	0.1050
		0.95	0.2163	0.2023	***0.1828***	0.1889	0.1997	0.2051	0.1970	**0.1864**	0.1868
		0.99	1.0445	0.7667	0.5345	0.5329	0.7248	0.8187	0.6710	***0.4995***	**0.5275**
	3	0.8	0.5596	0.5499	***0.5345***	0.5403	0.5480	0.5518	0.5461	**0.5384**	0.5385
		0.9	1.0178	0.9839	***0.9322***	0.9506	0.9774	0.9907	0.9706	**0.9442**	0.9448
		0.95	1.9464	1.8211	***1.6447***	1.7002	1.7976	1.8458	1.7726	**1.6775**	1.6813
		0.99	9.4002	6.9000	4.8107	4.7956	6.5235	7.3684	6.0393	***4.4950***	**4.7478**
	5	0.8	1.5543	1.5275	***1.4848***	1.5009	1.5222	1.5328	1.5168	**1.4957**	1.4959
		0.9	2.8273	2.7332	***2.5896***	2.6407	2.7149	2.7519	2.6961	**2.6228**	2.6243
		0.95	5.4066	5.0586	***4.5687***	4.7227	4.9932	5.1273	4.9238	**4.6598**	4.6704
		0.99	26.1116	19.1667	13.3631	13.3212	18.1207	20.4678	16.7757	***12.4861***	**13.1882**
	10	0.8	6.2173	6.1100	***5.9393***	6.0036	6.0889	6.1314	6.0674	**5.9827**	5.9837
		0.9	11.3093	10.9326	***10.3582***	10.5627	10.8597	11.0074	10.7844	**10.4911**	10.4973
		0.95	21.6263	20.2345	***18.2747***	18.8909	19.9729	20.5090	19.6951	**18.6390**	18.6815
		0.99	104.4463	76.6669	53.4523	53.2847	72.4828	81.8710	67.1027	***49.9442***	**52.7527**

Two estimators with Minimum MSE values are bolded in each row. The smaller of the two is italicized.

Table 2. Estimated MSE when k=0.3, d=0.5

N	σ^2	rho	OLS	RIDGE	LIU	K-L	MRT	TP	NTPKL	MTPKL1	MTPKL2
		0.8	0.1363	0.1311	0.1281	0.1261	0.1287	0.1337	0.1261	***0.1237***	***0.1237***
	1	0.9	0.2462	0.2288	0.2197	0.2120	0.2207	0.2374	0.2123	***0.2043***	**0.2048**
		0.95	0.4682	0.4064	0.3811	0.3492	0.3802	0.4367	0.3512	***0.3252***	**0.3278**
		0.99	2.2417	1.2173	1.1925	0.5101	0.9535	1.6899	0.5978	***0.3726***	**0.4549**
		0.8	1.2270	1.1802	1.1530	1.1344	1.1578	1.2034	1.1348	***1.1125***	**1.1132**
	3	0.9	2.2158	2.0587	1.9774	1.9076	1.9865	2.1365	1.9104	***1.8384***	**1.8424**
		0.95	4.2136	3.6579	3.4302	3.1428	3.4215	3.9307	3.1608	***2.9264***	**2.9506**
50		0.99	20.1751	10.9561	10.7325	4.5913	8.5816	15.2088	5.3805	***3.3531***	**4.0942**
		0.8	3.4082	3.2782	3.2026	3.1509	3.2161	3.3429	3.1522	***3.0902***	**3.0920**
		0.9	6.1551	5.7185	5.4928	5.2987	5.5181	5.9347	5.3065	***5.1066***	**5.1177**
		0.95	11.7045	10.1609	9.5284	8.7300	9.5043	10.9186	8.7801	***8.1288***	**8.1961**
	5	0.99	56.0420	30.4336	29.8126	12.7536	23.8378	42.2467	14.9459	***9.3141***	**11.3728**
		0.8	13.6328	13.1127	12.8101	12.6032	12.8643	13.3714	12.6084	***12.3602***	**12.3677**
		0.9	24.6204	22.8740	21.9710	21.1943	22.0722	23.7388	21.2258	***20.4259***	**20.4704**
		0.95	46.8180	40.6437	38.1136	34.9200	38.0173	43.6745	35.1204	***32.5154***	**32.7845**
	10	0.99	224.1680	121.7344	119.2504	51.0147	95.3513	168.9870	59.7836	***37.2565***	**45.4912**
		0.8	0.0622	0.0611	0.0604	0.0600	0.0606	0.0616	0.0600	***0.0595***	**0.0595**
	1	0.9	0.1131	0.1093	0.1071	0.1056	0.1075	0.1112	0.1057	***0.1039***	**0.1039**
		0.95	0.2163	0.2023	0.1950	0.1889	0.1959	0.2092	0.1891	***0.1827***	**0.1830**
		0.99	1.0445	0.7667	0.7055	0.5329	0.6683	0.9001	0.5497	***0.4551***	**0.4749**
		0.8	0.5596	0.5499	0.5438	0.5403	0.5452	0.5547	0.5404	***0.5356***	**0.5357**
	3	0.9	1.0178	0.9839	0.9639	0.9506	0.9677	1.0008	0.9509	***0.9347***	**0.9351**
		0.95	1.9464	1.8211	1.7548	1.7002	1.7631	1.8832	1.7022	***1.6444***	**1.6473**
100		0.99	9.4002	6.9000	6.3497	4.7956	6.0150	8.1006	4.9469	***4.0958***	**4.2742**
		0.8	1.5543	1.5275	1.5107	1.5009	1.5144	1.5409	1.5010	***1.4879***	**1.4881**
	5	0.9	2.8273	2.7332	2.6774	2.6407	2.6879	2.7800	2.6415	***2.5963***	**2.5975**
		0.95	5.4066	5.0586	4.8743	4.7227	4.8976	5.2311	4.7284	***4.5677***	**4.5758**
		0.99	26.1116	19.1667	17.6381	13.3212	16.7084	22.5017	13.7414	***11.3772***	**11.8727**
		0.8	6.2173	6.1100	6.0427	6.0036	6.0574	6.1635	6.0041	***5.9516***	**5.9523**
	10	0.9	11.3093	10.9326	10.7097	10.5627	10.7517	11.1201	10.5660	***10.3851***	**10.3899**
		0.95	21.6263	20.2345	19.4972	18.8909	19.5904	20.9244	18.9138	***18.2708***	**18.3033**
		0.99	104.4463	76.6669	70.5524	53.2847	66.8336	90.0070	54.9657	***45.5089***	**47.4907**

Two estimators with Minimum MSE values are bolded in each row. The smaller of the two is italicized.

Table 3. Estimated MSE when k=0.3, d=0.8

N	σ^2	rho	OLS	RIDGE	LIU	K-L	MRT	TP	NTPKL	MTPKL1	MTPKL2
		0.8	0.1363	0.1311	0.1330	0.1261	0.1272	0.1353	0.1232	***0.1223***	***0.1223***
	1	0.9	0.2462	0.2288	0.2354	0.2120	0.2162	0.2427	0.2030	***0.1999***	**0.2001**
		0.95	0.4682	0.4064	0.4323	0.3492	0.3657	0.4555	0.3218	***0.3119***	**0.3132**
		0.99	2.2417	1.2173	1.7822	0.5101	0.8349	2.0114	0.3858	***0.3158***	**0.3479**
		0.8	1.2270	1.1802	1.1971	1.1344	1.1448	1.2175	1.1085	***1.0997***	**1.1000**
	3	0.9	2.2158	2.0587	2.1188	1.9076	1.9451	2.1839	1.8268	***1.7988***	**1.8007**
		0.95	4.2136	3.6579	3.8903	3.1428	3.2909	4.0992	2.8962	***2.8074***	**2.8185**
50		0.99	20.1751	10.9561	16.0396	4.5913	7.5144	18.1030	3.4725	***2.8421***	**3.1315**
		0.8	3.4082	3.2782	3.3251	3.1509	3.1798	3.3820	3.0790	***3.0546***	**3.0555**
		0.9	6.1551	5.7185	5.8855	5.2987	5.4031	6.0665	5.0742	***4.9965***	**5.0017**
		0.95	11.7045	10.1609	10.8064	8.7300	9.1413	11.3868	8.0450	***7.7985***	**7.8293**
	5	0.99	56.0420	30.4336	44.5544	12.7536	20.8735	50.2860	9.6458	***7.8948***	**8.6985**
		0.8	13.6328	13.1127	13.3005	12.6032	12.7188	13.5279	12.3156	***12.2179***	**12.2214**
		0.9	24.6204	22.8740	23.5419	21.1943	21.6120	24.2658	20.2964	***19.9855***	**20.0064**
		0.95	46.8180	40.6437	43.2258	34.9200	36.5654	45.5471	32.1800	***31.1940***	**31.3174**
	10	0.99	224.1680	121.7344	178.2178	51.0147	83.4941	201.1442	38.5833	***31.5795***	**34.7943**
		0.8	0.0622	0.0611	0.0615	0.0600	0.0603	0.0620	0.0594	***0.0592***	**0.0592**
	1	0.9	0.1131	0.1093	0.1107	0.1056	0.1065	0.1123	0.1035	***0.1028***	**0.1028**
		0.95	0.2163	0.2023	0.2076	0.1889	0.1922	0.2134	0.1817	***0.1792***	**0.1793**
		0.99	1.0445	0.7667	0.9008	0.5329	0.6183	0.9854	0.4500	***0.4166***	**0.4252**
		0.8	0.5596	0.5499	0.5532	0.5403	0.5424	0.5576	0.5347	***0.5329***	**0.5329**
	3	0.9	1.0178	0.9839	0.9961	0.9506	0.9581	1.0110	0.9317	***0.9253***	**0.9255**
		0.95	1.9464	1.8211	1.8685	1.7002	1.7297	1.9210	1.6348	***1.6123***	**1.6136**
100		0.99	9.4002	6.9000	8.1072	4.7956	5.5649	8.8685	4.0503	***3.7489***	**3.8264**
		0.8	1.5543	1.5275	1.5368	1.5009	1.5066	1.5489	1.4854	***1.4802***	**1.4803**
	5	0.9	2.8273	2.7332	2.7668	2.6407	2.6614	2.8084	2.5881	***2.5702***	**2.5708**
		0.95	5.4066	5.0586	5.1902	4.7227	4.8048	5.3360	4.5411	***4.4785***	**4.4823**
		0.99	26.1116	19.1667	22.5199	13.3212	15.4581	24.6347	11.2509	***10.4136***	**10.6289**
		0.8	6.2173	6.1100	6.1472	6.0036	6.0262	6.1958	5.9416	***5.9207***	**5.9210**
	10	0.9	11.3093	10.9326	11.0674	10.5627	10.6454	11.2334	10.3522	***10.2809***	**10.2831**
		0.95	21.6263	20.2345	20.7609	18.8909	19.2192	21.3441	18.1645	***17.9139***	**17.9292**
		0.99	104.4463	76.6669	90.0797	53.2847	61.8325	98.5387	45.0034	***41.6544***	**42.5154**

Two estimators with Minimum MSE values are bolded in each row. The smaller of the two is italicized.

Table 4. Estimated MSE when k=0.6, d=0.2

N	σ^2	rho	OLS	RIDGE	LIU	K-L	MRT	TP	NTPKL	MTPKL1	MTPKL2
50	1	0.8	0.1363	0.1263	0.1234	0.1167	0.1244	0.1283	0.1224	***0.1149***	**0.1153**
		0.9	0.2462	0.2132	0.2046	0.1827	0.2074	0.2196	0.2009	***0.1774***	**0.1794**
		0.95	0.4682	0.3564	0.3333	0.2604	0.3391	0.3775	0.3173	***0.2461***	**0.2566**
		0.99	2.2417	0.7679	0.7223	**0.0801**	0.6563	0.9998	0.4381	***0.0644***	0.2020
	3	0.8	1.2270	1.1362	1.1098	1.0491	1.1193	1.1540	1.1012	***1.0330***	**1.0362**
		0.9	2.2158	1.9183	1.8412	1.6431	1.8663	1.9760	1.8072	***1.5955***	**1.6133**
		0.95	4.2136	3.2080	2.9999	2.3436	3.0516	3.3978	2.8556	***2.2146***	**2.3094**
		0.99	20.1751	6.9114	6.5009	**0.7210**	5.9070	8.9983	3.9425	***0.5794***	1.8174
	5	0.8	3.4082	3.1559	3.0825	2.9139	3.1090	3.2055	3.0587	***2.8690***	**2.8779**
		0.9	6.1551	5.3284	5.1142	4.5638	5.1839	5.4888	5.0199	***4.4316***	**4.4810**
		0.95	11.7045	8.9111	8.3330	6.5101	8.4768	9.4384	7.9322	***6.1518***	**6.4149**
		0.99	56.0420	19.1985	18.0581	**2.0029**	16.4084	24.9953	10.9513	***1.6093***	5.0484
	10	0.8	13.6328	12.6233	12.3294	11.6547	12.4356	12.8219	12.2341	***11.4752***	**11.5108**
		0.9	24.6204	21.3135	20.4565	18.2547	20.7355	21.9550	20.0791	***17.7258***	**17.9234**
		0.95	46.8180	35.6446	33.3323	26.0407	33.9073	37.7537	31.7291	***24.6072***	**25.6600**
		0.99	224.1680	76.7940	72.2327	**8.0117**	65.6337	99.9814	43.8053	***6.4375***	20.1937
100	1	0.8	0.0622	0.0601	0.0594	0.0580	0.0597	0.0605	0.0592	***0.0576***	**0.0576**
		0.9	0.1131	0.1058	0.1036	0.0987	0.1044	0.1072	0.1029	***0.0974***	**0.0976**
		0.95	0.2163	0.1898	0.1828	0.1651	0.1851	0.1949	0.1798	***0.1608***	**0.1622**
		0.99	1.0445	0.5881	0.5345	**0.2651**	0.5341	0.6687	0.4515	***0.2342***	0.2920
	3	0.8	0.5596	0.5405	0.5345	0.5218	0.5368	0.5443	0.5330	***0.5182***	**0.5185**
		0.9	1.0178	0.9518	0.9322	0.8881	0.9394	0.9648	0.9262	***0.8762***	**0.8782**
		0.95	1.9464	1.7080	1.6447	1.4859	1.6658	1.7544	1.6184	***1.4469***	**1.4600**
		0.99	9.4002	5.2929	4.8107	**2.3853**	4.8065	6.0184	4.0635	***2.1077***	2.6277
	5	0.8	1.5543	1.5014	1.4848	1.4495	1.4912	1.5119	1.4806	***1.4394***	**1.4403**
		0.9	2.8273	2.6439	2.5896	2.4669	2.6094	2.6800	2.5728	***2.4338***	**2.4395**
		0.95	5.4066	4.7444	4.5687	4.1273	4.6272	4.8733	4.4956	***4.0191***	**4.0555**
		0.99	26.1116	14.7025	13.3631	**6.6258**	13.3514	16.7177	11.2874	***5.8547***	7.2991
	10	0.8	6.2173	6.0056	5.9393	5.7978	5.9646	6.0476	5.9223	***5.7577***	**5.7613**
		0.9	11.3093	10.5755	10.3582	9.8677	10.4378	10.7202	10.2913	***9.7352***	**9.7580**
		0.95	21.6263	18.9777	18.2747	16.5093	18.5089	19.4930	17.9825	***16.0762***	**16.2220**
		0.99	104.4463	58.8099	53.4523	**26.5030**	53.4054	66.8709	45.1496	***23.4186***	29.1961

Two estimators with Minimum MSE values are bolded in each row. The smaller of the two is italicized.

Table 5. Estimated MSE when k=0.6, d=0.5

N	σ^2	rho	OLS	RIDGE	LIU	K-L	MRT	TP	NTPKL	MTPKL1	MTPKL2
50	1	0.8	0.1363	0.1263	0.1281	0.1167	0.1217	0.1312	0.1169	***0.1123***	**0.1126**
		0.9	0.2462	0.2132	0.2197	0.1827	0.1992	0.2294	0.1838	***0.1699***	**0.1714**
		0.95	0.4682	0.3564	0.3811	0.2604	0.3154	0.4103	0.2666	***0.2268***	**0.2343**
		0.99	2.2417	0.7679	1.1925	**0.0801**	0.5301	1.4066	0.1837	***0.0488***	0.1193
	3	0.8	1.2270	1.1362	1.1530	1.0491	1.0947	1.1811	1.0508	***1.0095***	**1.0120**
		0.9	2.2158	1.9183	1.9774	1.6431	1.7923	2.0643	1.6530	***1.5281***	**1.5414**
		0.95	4.2136	3.2080	3.4302	2.3436	2.8383	3.6931	2.3995	***2.0409***	**2.1086**
		0.99	20.1751	6.9114	10.7325	**0.7210**	4.7705	12.6591	1.6527	***0.4386***	1.0732
	5	0.8	3.4082	3.1559	3.2026	2.9139	3.0407	3.2808	2.9187	***2.8038***	**2.8106**
		0.9	6.1551	5.3284	5.4928	4.5638	4.9784	5.7340	4.5914	***4.2443***	**4.2812**
		0.95	11.7045	8.9111	9.5284	6.5101	7.8841	10.2588	6.6653	***5.6693***	**5.8573**
		0.99	56.0420	19.1985	29.8126	**2.0029**	13.2514	35.1642	4.5908	***1.2183***	2.9811
	10	0.8	13.6328	12.6233	12.8101	11.6547	12.1622	13.1229	11.6739	***11.2140***	**11.2411**
		0.9	24.6204	21.3135	21.9710	18.2547	19.9132	22.9359	18.3648	***16.9765***	**17.1238**
		0.95	46.8180	35.6446	38.1136	26.0407	31.5367	41.0351	26.6614	***22.6773***	**23.4292**
		0.99	224.1680	76.7940	119.2504	**8.0117**	53.0058	140.6570	18.3634	***4.8734***	11.9245
100	1	0.8	0.0622	0.0601	0.0604	0.0580	0.0591	0.0611	0.0580	***0.0570***	**0.0571**
		0.9	0.1131	0.1058	0.1071	0.0987	0.1024	0.1094	0.0988	***0.0955***	**0.0956**
		0.95	0.2163	0.1898	0.1950	0.1651	0.1784	0.2028	0.1659	***0.1546***	**0.1557**
		0.99	1.0445	0.5881	0.7055	0.2651	0.4662	0.7996	0.3035	***0.1974***	**0.2345**
	3	0.8	0.5596	0.5405	0.5438	0.5218	0.5314	0.5500	0.5220	***0.5129***	**0.5131**
		0.9	1.0178	0.9518	0.9639	0.8881	0.9213	0.9845	0.8892	***0.8588***	**0.8604**
		0.95	1.9464	1.7080	1.7548	1.4859	1.6055	1.8252	1.4931	***1.3914***	**1.4012**
		0.99	9.4002	5.2929	6.3497	2.3853	4.1956	7.1966	2.7312	***1.7767***	**2.1101**
	5	0.8	1.5543	1.5014	1.5107	1.4495	1.4760	1.5277	1.4499	***1.4246***	**1.4253**
		0.9	2.8273	2.6439	2.6774	2.4669	2.5591	2.7348	2.4700	***2.3854***	**2.3898**
		0.95	5.4066	4.7444	4.8743	4.1273	4.4597	5.0699	4.1475	***3.8648***	**3.8921**
		0.99	26.1116	14.7025	17.6381	6.6258	11.6543	19.9905	7.5867	***4.9351***	**5.8613**
	10	0.8	6.2173	6.0056	6.0427	5.7978	5.9041	6.1110	5.7997	***5.6983***	**5.7011**
		0.9	11.3093	10.5755	10.7097	9.8677	10.2365	10.9391	9.8800	***9.5416***	**9.5591**
		0.95	21.6263	18.9777	19.4972	16.5093	17.8388	20.2795	16.5901	***15.4592***	**15.5684**
		0.99	104.4463	58.8099	70.5524	26.5030	46.6173	79.9619	30.3465	***19.7401***	**23.4451**

Two estimators with Minimum MSE values are bolded in each row. The smaller of the two is italicized.

Table 6. Estimated MSE when k=0.6, d=0.8

N	σ^2	rho	OLS	RIDGE	LIU	K-L	MRT	TP	NTPKL	MTPKL1	MTPKL2
50	1	0.8	0.1363	0.1263	0.1330	0.1167	0.1191	0.1343	0.1116	***0.1098***	**0.1100**
		0.9	0.2462	0.2132	0.2354	0.1827	0.1915	0.2394	0.1682	***0.1629***	**0.1636**
		0.95	0.4682	0.3564	0.4323	0.2604	0.2941	0.4446	0.2241	***0.2098***	**0.2131**
		0.99	2.2417	0.7679	1.7822	0.0801	0.4374	1.8841	0.0698	***0.0387***	**0.0601**
	3	0.8	1.2270	1.1362	1.1971	1.0491	1.0710	1.2085	1.0029	***0.9869***	**0.9881**
		0.9	2.2158	1.9183	2.1188	1.6431	1.7228	2.1545	1.5123	***1.4651***	**1.4712**
		0.95	4.2136	3.2080	3.8903	2.3436	2.6472	4.0012	2.0164	***1.8877***	**1.9174**
		0.99	20.1751	6.9114	16.0396	0.7210	3.9366	16.9565	0.6281	***0.3474***	**0.5407**
	5	0.8	3.4082	3.1559	3.3251	2.9139	2.9747	3.3569	2.7854	***2.7409***	**2.7440**
		0.9	6.1551	5.3284	5.8855	4.5638	4.7853	5.9848	4.2004	***4.0692***	**4.0861**
		0.95	11.7045	8.9111	10.8064	6.5101	7.3532	11.1144	5.6012	***5.2435***	**5.3261**
		0.99	56.0420	19.1985	44.5544	2.0029	10.9351	47.1015	1.7448	***0.9649***	**1.5019**
	10	0.8	13.6328	12.6233	13.3005	11.6547	11.8982	13.4276	11.1406	***10.9622***	**10.9749**
		0.9	24.6204	21.3135	23.5419	18.2547	19.1408	23.9392	16.8005	***16.2759***	**16.3434**
		0.95	46.8180	35.6446	43.2258	26.0407	29.4132	44.4577	22.4050	***20.9741***	**16.3434**
		0.99	224.1680	76.7940	178.2178	8.0117	43.7407	188.4059	**6.9793**	***3.8599***	16.3434
100	1	0.8	0.0622	0.0601	0.0615	0.0580	0.0585	0.0617	0.0568	***0.0565***	**0.0565**
		0.9	0.1131	0.1058	0.1107	0.0987	0.1004	0.1116	0.0949	***0.0936***	**0.0937**
		0.95	0.2163	0.1898	0.2076	0.1651	0.1721	0.2108	0.1531	***0.1488***	**0.1493**
		0.99	1.0445	0.5881	0.9008	0.2651	0.4107	0.9425	0.2025	***0.1689***	**0.1836**
	3	0.8	0.5596	0.5405	0.5532	0.5218	0.5260	0.5557	0.5112	***0.5076***	**0.5077**
		0.9	1.0178	0.9518	0.9961	0.8881	0.9037	1.0044	0.8538	***0.8419***	**0.8427**
		0.95	1.9464	1.7080	1.8685	1.4859	1.5485	1.8974	1.3778	***1.3391***	**1.3436**
		0.99	9.4002	5.2929	8.1072	2.3853	3.6959	8.4827	1.8218	***1.5201***	**1.6520**
	5	0.8	1.5543	1.5014	1.5368	1.4495	1.4611	1.5437	1.4200	***1.4100***	**1.4103**
		0.9	2.8273	2.6439	2.7668	2.4669	2.5103	2.7901	2.3715	***2.3386***	**2.3406**
		0.95	5.4066	4.7444	5.1902	4.1273	4.3015	5.2705	3.8271	***3.7197***	**3.7322**
		0.99	26.1116	14.7025	22.5199	6.6258	10.2664	23.5632	5.0604	***4.2223***	**4.5888**
	10	0.8	6.2173	6.0056	6.1472	5.7978	5.8445	6.1747	5.6799	***5.6399***	**5.6412**
		0.9	11.3093	10.5755	11.0674	9.8677	10.0413	11.1605	9.4859	***9.3542***	**9.3624**
		0.95	21.6263	18.9777	20.7609	16.5093	17.2058	21.0822	15.3082	***14.8786***	**14.9288**
		0.99	104.4463	58.8099	90.0797	26.5030	41.0653	94.2527	20.2414	***16.8890***	**18.3550**

Two estimators with Minimum MSE values are bolded in each row. The smaller of the two is italicized.

Table 7. Estimated MSE when k=0.7, d=0.2

N	σ^2	rho	OLS	RIDGE	LIU	K-L	MRT	TP	NTPKL	MTPKL1	MTPKL2
50	1	0.8	0.1363	0.1247	0.1234	0.1137	0.1226	0.1270	0.1203	***0.1117***	**0.1122**
		0.9	0.2462	0.2083	0.2046	0.1738	0.2019	0.2157	0.1944	***0.1680***	**0.1706**
		0.95	0.4682	0.3419	0.3333	0.2361	0.3230	0.3655	0.2986	***0.2211***	**0.2340**
		0.99	2.2417	0.6731	0.7223	**0.0378**	0.5677	0.9122	0.3503	***0.0304***	0.1489
	3	0.8	1.2270	1.1221	1.1098	1.0222	1.1028	1.1427	1.0819	***1.0040***	**1.0082**
		0.9	2.2158	1.8748	1.8412	1.5634	1.8164	1.9406	1.7489	***1.5109***	**1.5339**
		0.95	4.2136	3.0769	2.9999	2.1241	2.9068	3.2895	2.6869	***1.9893***	**2.1058**
		0.99	20.1751	6.0583	6.5009	**0.3396**	5.1093	8.2098	3.1526	***0.2729***	1.3398
	5	0.8	3.4082	3.1167	3.0825	2.8391	3.0632	3.1739	3.0051	***2.7883***	**2.8000**
		0.9	6.1551	5.2076	5.1142	4.3425	5.0455	5.3905	4.8579	***4.1967***	**4.2603**
		0.95	11.7045	8.5469	8.3330	5.9004	8.0744	9.1375	7.4636	***5.5258***	**5.8494**
		0.99	56.0420	16.8287	18.0581	**0.9434**	14.1924	22.8051	8.7573	***0.7582***	3.7218
	10	0.8	13.6328	12.4665	12.3294	11.3553	12.2522	12.6954	12.0197	***11.1520***	**11.1990**
		0.9	24.6204	20.8301	20.4565	17.3694	20.1815	21.5618	19.4312	***16.7859***	**17.0405**
		0.95	46.8180	34.1876	33.3323	23.6018	32.2976	36.5501	29.8547	***22.1034***	**23.3978**
		0.99	224.1680	67.3149	72.2327	**3.7737**	56.7697	91.2205	35.0295	***3.0328***	14.8873
100	1	0.8	0.0622	0.0597	0.0594	0.0573	0.0593	0.0602	0.0588	***0.0569***	**0.0569**
		0.9	0.1131	0.1046	0.1036	0.0965	0.1030	0.1063	0.1014	***0.0950***	**0.0953**
		0.95	0.2163	0.1859	0.1828	0.1579	0.1806	0.1918	0.1746	***0.1531***	**0.1550**
		0.99	1.0445	0.5425	0.5345	**0.2074**	0.4873	0.6296	0.3989	***0.1800***	0.2436
	3	0.8	0.5596	0.5374	0.5345	0.5158	0.5332	0.5418	0.5288	***0.5117***	**0.5121**
		0.9	1.0178	0.9415	0.9322	0.8682	0.9273	0.9565	0.9120	***0.8547***	**0.8574**
		0.95	1.9464	1.6727	1.6447	1.4207	1.6252	1.7257	1.5710	***1.3774***	**1.3944**
		0.99	9.4002	4.8827	4.8107	**1.8662**	4.3853	5.6661	3.5896	***1.6196***	2.1920
	5	0.8	1.5543	1.4929	1.4848	1.4327	1.4810	1.5050	1.4687	***1.4212***	**1.4224**
		0.9	2.8273	2.6151	2.5896	2.4117	2.5757	2.6569	2.5334	***2.3741***	**2.3816**
		0.95	5.4066	4.6464	4.5687	3.9463	4.5145	4.7937	4.3637	***3.8262***	**3.8734**
		0.99	26.1116	13.5630	13.3631	**5.1838**	12.1814	15.7392	9.9711	***4.4987***	6.0889
	10	0.8	6.2173	5.9714	5.9393	5.7309	5.9241	6.0202	5.8750	***5.6847***	**5.6896**
		0.9	11.3093	10.4605	10.3582	9.6467	10.3029	10.6275	10.1335	***9.4961***	**9.5264**
		0.95	21.6263	18.5858	18.2747	15.7850	18.0579	19.1747	17.4549	***15.3045***	**15.4934**
		0.99	104.4463	54.2520	53.4523	**20.7350**	48.7256	62.9567	39.8844	***17.9946***	24.3556

Two estimators with Minimum MSE values are bolded in each row. The smaller of the two is italicized.

Table 8. Estimated MSE when k=0.7, d=0.5

N	σ^2	rho	OLS	RIDGE	LIU	K-L	MRT	TP	NTPKL	MTPKL1	MTPKL2
50	1	0.8	0.1363	0.1247	0.1281	0.1137	0.1195	0.1304	0.1140	***0.1088***	**0.1092**
		0.9	0.2462	0.2083	0.2197	0.1738	0.1927	0.2269	0.1753	***0.1599***	**0.1617**
		0.95	0.4682	0.3419	0.3811	0.2361	0.2975	0.4025	0.2438	***0.2013***	**0.2104**
		0.99	2.2417	0.6731	1.1925	**0.0378**	0.4511	1.3408	0.1257	***0.0233***	0.0790
	3	0.8	1.2270	1.1221	1.1530	1.0222	1.0749	1.1739	1.0245	***0.9775***	**0.9807**
		0.9	2.2158	1.8748	1.9774	1.5634	1.7341	2.0416	1.5763	***1.4373***	**1.4542**
		0.95	4.2136	3.0769	3.4302	2.1241	2.6776	3.6222	2.1941	***1.8108***	**1.8928**
		0.99	20.1751	6.0583	10.7325	**0.3396**	4.0594	12.0670	1.1309	***0.2088***	0.7102
	5	0.8	3.4082	3.1167	3.2026	2.8391	2.9856	3.2607	2.8455	***2.7148***	**2.7237**
		0.9	6.1551	5.2076	5.4928	4.3425	4.8167	5.6710	4.3784	***3.9920***	**4.0391**
		0.95	11.7045	8.5469	9.5284	5.9004	7.4378	10.0618	6.0946	***5.0299***	**5.2578**
		0.99	56.0420	16.8287	29.8126	**0.9434**	11.2761	33.5193	3.1414	*0.5799*	1.9726
	10	0.8	13.6328	12.4665	12.8101	11.3553	11.9416	13.0428	11.3809	***10.8577***	**10.8934**
		0.9	24.6204	20.8301	21.9710	17.3694	19.2663	22.6840	17.5127	***15.9669***	**16.1552**
		0.95	46.8180	34.1876	38.1136	23.6018	29.7514	40.2472	24.3787	***20.1200***	**21.0313**
		0.99	224.1680	67.3149	119.2504	**3.7737**	45.1045	134.0774	12.5656	***2.3198***	7.8906
100	1	0.8	0.0622	0.0597	0.0604	0.0573	0.0586	0.0609	0.0574	***0.0562***	**0.0563**
		0.9	0.1131	0.1046	0.1071	0.0965	0.1008	0.1088	0.0967	***0.0928***	**0.0931**
		0.95	0.2163	0.1859	0.1950	0.1579	0.1731	0.2008	0.1589	***0.1463***	**0.1477**
		0.99	1.0445	0.5425	0.7055	0.2074	0.4192	0.7726	0.2509	***0.1484***	**0.1879**
	3	0.8	0.5596	0.5374	0.5438	0.5158	0.5269	0.5484	0.5160	***0.5055***	**0.5059**
		0.9	1.0178	0.9415	0.9639	0.8682	0.9066	0.9792	0.8697	***0.8350***	**0.8371**
		0.95	1.9464	1.6727	1.7548	1.4207	1.5578	1.8068	1.4302	***1.3164***	**1.3290**
		0.99	9.4002	4.8827	6.3497	1.8662	3.7726	6.9538	2.2581	***1.3349***	**1.6907**
	5	0.8	1.5543	1.4929	1.5107	1.4327	1.4636	1.5234	1.4334	***1.4041***	**1.4051**
		0.9	2.8273	2.6151	2.6774	2.4117	2.5184	2.7201	2.4158	***2.3193***	**2.3251**
		0.95	5.4066	4.6464	4.8743	3.9463	4.3272	5.0190	3.9727	***3.6565***	**3.6916**
		0.99	26.1116	13.5630	17.6381	5.1838	10.4795	19.3161	6.2725	***3.7079***	**4.6962**
	10	0.8	6.2173	5.9714	6.0427	5.7309	5.8543	6.0937	5.7335	***5.6165***	**5.6203**
		0.9	11.3093	10.4605	10.7097	9.6467	10.0734	10.8805	9.6631	***9.2770***	**9.3001**
		0.95	21.6263	18.5858	19.4972	15.7850	17.3088	20.0761	15.8906	***14.6258***	**14.7662**
		0.99	104.4463	54.2520	70.5524	20.7350	41.9180	77.2645	25.0898	***14.8312***	**18.7847**

Two estimators with Minimum MSE values are bolded in each row. The smaller of the two is italicized.

Table 9. Estimated MSE when k=0.7, d=0.8

N	σ^2	rho	OLS	RIDGE	LIU	K-L	MRT	TP	NTPKL	MTPKL1	MTPKL2
50	1	0.8	0.1363	0.1247	0.1330	0.1137	0.1166	0.1340	0.1080	***0.1061***	**0.1062**
		0.9	0.2462	0.2083	0.2354	0.1738	0.1843	0.2384	0.1581	***0.1523***	**0.1532**
		0.95	0.4682	0.3419	0.4323	0.2361	0.2751	0.4413	0.1991	***0.1841***	**0.1881**
		0.99	2.2417	0.6731	1.7822	0.0378	0.3674	1.8533	0.0389	***0.0188***	**0.0335**
	3	0.8	1.2270	1.1221	1.1971	1.0222	1.0481	1.2056	0.9703	***0.9522***	**0.9537**
		0.9	2.2158	1.8748	2.1188	1.5634	1.6574	2.1453	1.4212	***1.3691***	**1.3768**
		0.95	4.2136	3.0769	3.8903	2.1241	2.4752	3.9715	1.7915	***1.6562***	**1.6918**
		0.99	20.1751	6.0583	16.0396	0.3396	3.3064	16.6799	0.3493	***0.1685***	**0.3004**
	5	0.8	3.4082	3.1167	3.3251	2.8391	2.9110	3.3488	2.6947	***2.6443***	**2.6484**
		0.9	6.1551	5.2076	5.8855	4.3425	4.6037	5.9590	3.9473	***3.8026***	**3.8240**
		0.95	11.7045	8.5469	10.8064	5.9004	6.8757	11.0321	4.9764	***4.6005***	**4.6993**
		0.99	56.0420	16.8287	44.5544	0.9434	9.1844	46.3331	0.9702	***0.4681***	**0.8345**
	10	0.8	13.6328	12.4665	13.3005	11.3553	11.6432	13.3951	10.7776	***10.5756***	**10.5922**
		0.9	24.6204	20.8301	23.5419	17.3694	18.4143	23.8360	15.7880	***15.2092***	**15.2949**
		0.95	46.8180	34.1876	43.2258	23.6018	27.5030	44.1283	19.9058	***18.4022***	**18.7972**
		0.99	224.1680	67.3149	178.2178	3.7737	36.7379	185.3324	3.8809	***1.8724***	**3.3380**
100	1	0.8	0.0622	0.0597	0.0615	0.0573	0.0579	0.0617	0.0560	***0.0556***	**0.0556**
		0.9	0.1131	0.1046	0.1107	0.0965	0.0985	0.1114	0.0922	***0.0907***	**0.0908**
		0.95	0.2163	0.1859	0.2076	0.1579	0.1661	0.2100	0.1448	***0.1400***	**0.1406**
		0.99	1.0445	0.5425	0.9008	0.2074	0.3647	0.9307	0.1559	***0.1247***	**0.1398**
	3	0.8	0.5596	0.5374	0.5532	0.5158	0.5207	0.5551	0.5037	***0.4995***	**0.4997**
		0.9	1.0178	0.9415	0.9961	0.8682	0.8867	1.0023	0.8295	***0.8160***	**0.8170**
		0.95	1.9464	1.6727	1.8685	1.4207	1.4947	1.8899	1.3023	***1.2595***	**1.2652**
		0.99	9.4002	4.8827	8.1072	1.8662	3.2818	8.3766	1.4023	***1.1214***	**1.2578**
	5	0.8	1.5543	1.4929	1.5368	1.4327	1.4465	1.5419	1.3989	***1.3874***	**1.3879**
		0.9	2.8273	2.6151	2.7668	2.4117	2.4630	2.7842	2.3040	***2.2665***	**2.2692**
		0.95	5.4066	4.6464	5.1902	3.9463	4.1518	5.2497	3.6175	***3.4984***	**3.5144**
		0.99	26.1116	13.5630	22.5199	5.1838	9.1161	23.2683	3.8951	***3.1149***	**3.4939**
	10	0.8	6.2173	5.9714	6.1472	5.7309	5.7859	6.1677	5.5957	***5.5496***	**5.5514**
		0.9	11.3093	10.4605	11.0674	9.6467	9.8519	11.1367	9.2156	***9.0659***	**9.0767**
		0.95	21.6263	18.5858	20.7609	15.7850	16.6072	20.9990	14.4698	***13.9932***	**14.0574**
		0.99	104.4463	54.2520	90.0797	20.7350	36.4642	93.0733	15.5802	***12.4593***	**13.9752**

Two estimators with Minimum MSE values are bolded in each row. The smaller of the two is italicized.

Table 10. Estimated MSE when k=0.9, d=0.2

N	σ^2	rho	OLS	RIDGE	LIU	K-L	MRT	TP	NTPKL	MTPKL1	MTPKL2
		0.8	0.1363	0.1217	0.1234	0.1081	0.1191	0.1246	0.1162	***0.1056***	**0.1064**
	1	0.9	0.2462	0.1992	0.2046	0.1575	0.1915	0.2082	0.1823	***0.1508***	**0.1546**
		0.95	0.4682	0.3154	0.3333	**0.1937**	0.2941	0.3434	0.2651	***0.1783***	0.1958
		0.99	2.2417	0.5301	0.7223	**0.0227**	0.4374	0.7761	0.2297	***0.0168***	0.0854
		0.8	1.2270	1.0947	1.1098	0.9706	1.0710	1.1205	1.0447	***0.9484***	**0.9550**
	3	0.9	2.2158	1.7923	1.8412	1.4156	1.7228	1.8733	1.6395	***1.3553***	**1.3892**
		0.95	4.2136	2.8383	2.9999	**1.7424**	2.6472	3.0910	2.3858	***1.6035***	1.7613
50		0.99	20.1751	4.7705	6.5009	**0.2033**	3.9366	6.9844	2.0674	***0.1504***	0.7679
		0.8	3.4082	3.0407	3.0825	2.6954	2.9747	3.1124	2.9016	***2.6339***	**2.6522**
		0.9	6.1551	4.9784	5.1142	3.9317	4.7853	5.2034	4.5538	***3.7640***	**3.8584**
		0.95	11.7045	7.8841	8.3330	**4.8400**	7.3532	8.5861	6.6273	***4.4542***	4.8924
	5	0.99	56.0420	13.2514	18.0581	**0.5647**	10.9351	19.4012	5.7429	***0.4178***	2.1329
		0.8	13.6328	12.1622	12.3294	10.7802	11.8982	12.4492	11.6056	***10.5338***	**10.6070**
		0.9	24.6204	19.9132	20.4565	15.7255	19.1408	20.8131	18.2145	***15.0550***	**15.4324**
		0.95	46.8180	31.5367	33.3323	**19.3600**	29.4132	34.3446	26.5095	*17.8169*	19.5697
	10	0.99	224.1680	53.0058	72.2327	**2.2591**	43.7407	77.6050	22.9718	*1.6713*	8.5317
		0.8	0.0622	0.0591	0.0594	0.0561	0.0585	0.0597	0.0578	***0.0555***	**0.0556**
	1	0.9	0.1131	0.1024	0.1036	0.0923	0.1004	0.1045	0.0983	***0.0904***	**0.0909**
		0.95	0.2163	0.1784	0.1828	0.1444	0.1721	0.1857	0.1646	***0.1388***	**0.1416**
		0.99	1.0445	0.4662	0.5345	**0.1233**	0.4107	0.5630	0.3144	***0.1036***	0.1730
		0.8	0.5596	0.5314	0.5345	0.5040	0.5260	0.5369	0.5204	***0.4988***	**0.4995**
	3	0.9	1.0178	0.9213	0.9322	0.8299	0.9037	0.9402	0.8845	***0.8133***	**0.8176**
		0.95	1.9464	1.6055	1.6447	1.2988	1.5485	1.6709	1.4813	***1.2486***	**1.2740**
100		0.99	9.4002	4.1956	4.8107	**1.1091**	3.6959	5.0670	2.8297	***0.9318***	1.5563
		0.8	1.5543	1.4760	1.4848	1.3999	1.4611	1.4915	1.4455	***1.3855***	**1.3874**
	5	0.9	2.8273	2.5591	2.5896	2.3051	2.5103	2.6116	2.4569	***2.2591***	**2.2710**
		0.95	5.4066	4.4597	4.5687	3.6077	4.3015	4.6415	4.1146	***3.4681***	**3.5388**
		0.99	26.1116	11.6543	13.3631	**3.0808**	10.2664	14.0751	7.8603	***2.5881***	4.3229
		0.8	6.2173	5.9041	5.9393	5.5996	5.8445	5.9660	5.7819	***5.5417***	**5.5496**
	10	0.9	11.3093	10.2365	10.3582	9.2202	10.0413	10.4465	9.8275	***9.0362***	**9.0838**
		0.95	21.6263	17.8388	18.2747	14.4306	17.2058	18.5659	16.4584	***13.8722***	**14.1550**
		0.99	104.4463	46.6173	53.4523	**12.3231**	41.0653	56.3003	31.4412	***10.3522***	17.2912

Two estimators with Minimum MSE values are bolded in each row. The smaller of the two is italicized.

Table 11. Estimated MSE when n=100, k=0.9, d=0.5

N	σ^2	rho	OLS	RIDGE	LIU	K-L	MRT	TP	NTPKL	MTPKL1	MTPKL2
50	1	0.8	0.1363	0.1217	0.1281	0.1081	0.1153	0.1289	0.1085	***0.1022***	**0.1027**
		0.9	0.2462	0.1992	0.2197	0.1575	0.1808	0.2220	0.1596	***0.1416***	**0.1443**
		0.95	0.4682	0.3154	0.3811	0.1937	0.2662	0.3879	0.2046	***0.1585***	**0.1705**
		0.99	2.2417	0.5301	1.1925	**0.0227**	0.3386	1.2353	0.0595	***0.0124***	0.0362
	3	0.8	1.2270	1.0947	1.1530	0.9706	1.0369	1.1598	0.9742	***0.9168***	**0.9217**
		0.9	2.2158	1.7923	1.9774	1.4156	1.6262	1.9982	1.4351	***1.2721***	**1.2969**
		0.95	4.2136	2.8383	3.4302	1.7424	2.3957	3.4910	1.8403	***1.4257***	**1.5336**
		0.99	20.1751	4.7705	10.7325	**0.2033**	3.0474	11.1181	0.5349	***0.1103***	0.3246
	5	0.8	3.4082	3.0407	3.2026	2.6954	2.8800	3.2217	2.7055	***2.5457***	**2.5594**
		0.9	6.1551	4.9784	5.4928	3.9317	4.5169	5.5505	3.9859	***3.5330***	**3.6018**
		0.95	11.7045	7.8841	9.5284	4.8400	6.6547	9.6973	5.1120	***3.9601***	**4.2600**
		0.99	56.0420	13.2514	29.8126	**0.5647**	8.4651	30.8837	1.4857	***0.3064***	0.9014
	10	0.8	13.6328	12.1622	12.8101	10.7802	11.5189	12.8864	10.8204	***10.1808***	**10.2359**
		0.9	24.6204	19.9132	21.9710	15.7255	18.0671	22.2019	15.9424	***14.1306***	**14.4057**
		0.95	46.8180	31.5367	38.1136	19.3600	26.6191	38.7893	20.4483	***15.8405***	**17.0400**
		0.99	224.1680	53.0058	119.2504	**2.2591**	33.8606	123.5349	5.9429	***1.2255***	3.6058
100	1	0.8	0.0622	0.0591	0.0604	0.0561	0.0576	0.0606	0.0561	***0.0547***	**0.0547**
		0.9	0.1131	0.1024	0.1071	0.0923	0.0976	0.1077	0.0925	***0.0878***	**0.0882**
		0.95	0.2163	0.1784	0.1950	0.1444	0.1632	0.1969	0.1460	***0.1310***	**0.1331**
		0.99	1.0445	0.4662	0.7055	0.1233	0.3446	0.7259	0.1731	***0.0822***	**0.1227**
	3	0.8	0.5596	0.5314	0.5438	0.5040	0.5181	0.5454	0.5044	***0.4912***	**0.4917**
		0.9	1.0178	0.9213	0.9639	0.8299	0.8784	0.9689	0.8322	***0.7895***	**0.7927**
		0.95	1.9464	1.6055	1.7548	1.2988	1.4688	1.7717	1.3133	***1.1787***	**1.1974**
		0.99	9.4002	4.1956	6.3497	1.1091	3.1008	6.5331	1.5575	***0.7389***	**1.1035**
	5	0.8	1.5543	1.4760	1.5107	1.3999	1.4392	1.5149	1.4010	***1.3643***	**1.3658**
		0.9	2.8273	2.5591	2.6774	2.3051	2.4399	2.6915	2.3116	***2.1928***	**2.2018**
		0.95	5.4066	4.4597	4.8743	3.6077	4.0800	4.9213	3.6480	***3.2741***	**3.3260**
		0.99	26.1116	11.6543	17.6381	3.0808	8.6133	18.1475	4.3263	***2.0524***	**3.0650**
	10	0.8	6.2173	5.9041	6.0427	5.5996	5.7569	6.0596	5.6038	***5.4568***	**5.4628**
		0.9	11.3093	10.2365	10.7097	9.2202	9.7593	10.7658	9.2462	***8.7711***	**8.8071**
		0.95	21.6263	17.8388	19.4972	14.4306	16.3200	19.6851	14.5917	***13.0962***	**13.3036**
		0.99	104.4463	46.6173	70.5524	12.3231	34.4532	72.5899	17.3050	***8.2091***	**12.2596**

Two estimators with Minimum MSE values are bolded in each row. The smaller of the two is italicized.

Table 12. Estimated MSE when n=100, k=0.9, d=0.8

N	σ^2	rho	OLS	RIDGE	LIU	K-L	MRT	TP	NTPKL	MTPKL1	MTPKL2
50	1	0.8	0.1363	0.1217	0.1330	0.1081	0.1118	0.1333	0.1013	***0.0989***	**0.0992**
		0.9	0.2462	0.1992	0.2354	0.1575	0.1710	0.2364	0.1399	***0.1333***	**0.1345**
		0.95	0.4682	0.3154	0.4323	0.1937	0.2422	0.4351	0.1578	***0.1421***	**0.1471**
		0.99	2.2417	0.5301	1.7822	0.0227	0.2703	1.8030	0.0131	***0.0101***	**0.0117**
	3	0.8	1.2270	1.0947	1.1971	0.9706	1.0046	1.1999	0.9087	***0.8867***	**0.8890**
		0.9	2.2158	1.7923	2.1188	1.4156	1.5378	2.1274	1.2567	***1.1968***	**1.2079**
		0.95	4.2136	2.8383	3.8903	1.7424	2.1794	3.9162	1.4186	***1.2771***	**1.3226**
		0.99	20.1751	4.7705	16.0396	0.2033	2.4324	16.2272	0.1167	***0.0898***	**0.1042**
	5	0.8	3.4082	3.0407	3.3251	2.6954	2.7899	3.3329	2.5232	***2.4621***	**2.4685**
		0.9	6.1551	4.9784	5.8855	3.9317	4.2714	5.9094	3.4902	***3.3237***	**3.3545**
		0.95	11.7045	7.8841	10.8064	4.8400	6.0539	10.8783	3.9406	***3.5473***	**3.6738**
		0.99	56.0420	13.2514	44.5544	0.5647	6.7567	45.0756	0.3240	***0.2492***	**0.2892**
	10	0.8	13.6328	12.1622	13.3005	10.7802	11.1585	13.3316	10.0908	***9.8463***	**9.8718**
		0.9	24.6204	19.9132	23.5419	15.7255	17.0847	23.6374	13.9592	***13.2930***	**13.4163**
		0.95	46.8180	31.5367	43.2258	19.3600	24.2157	43.5134	15.7624	***14.1894***	**14.6951**
		0.99	224.1680	53.0058	178.2178	2.2591	27.0269	180.3023	1.2961	***0.9966***	**1.1570**
100	1	0.8	0.0622	0.0591	0.0615	0.0561	0.0568	0.0615	0.0544	***0.0539***	**0.0539**
		0.9	0.1131	0.1024	0.1107	0.0923	0.0949	0.1109	0.0871	***0.0853***	**0.0855**
		0.95	0.2163	0.1784	0.2076	0.1444	0.1551	0.2084	0.1295	***0.1240***	**0.1249**
		0.99	1.0445	0.4662	0.9008	0.1233	0.2934	0.9100	0.0926	***0.0670***	**0.0816**
	3	0.8	0.5596	0.5314	0.5532	0.5040	0.5105	0.5539	0.4889	***0.4838***	**0.4840**
		0.9	1.0178	0.9213	0.9961	0.8299	0.8541	0.9981	0.7832	***0.7667***	**0.7682**
		0.95	1.9464	1.6055	1.8685	1.2988	1.3954	1.8755	1.1648	***1.1149***	**1.1233**
		0.99	9.4002	4.1956	8.1072	1.1091	2.6407	8.1898	0.8325	***0.6026***	**0.7334**
	5	0.8	1.5543	1.4760	1.5368	1.3999	1.4179	1.5385	1.3579	***1.3436***	**1.3443**
		0.9	2.827326 0	2.5591	2.7668	2.3051	2.3725	2.7726	2.1753	***2.1296***	**2.1338**
		0.95	5.4066	4.4597	5.1902	3.6077	3.8759	5.2096	3.2353	***3.0967***	**3.1201**
		0.99	26.1116	11.6543	22.5199	3.0808	7.3351	22.7494	2.3123	***1.6738***	**2.0371**
	10	0.8	6.2173	5.9041	6.1472	5.5996	5.6715	6.1540	5.4315	***5.3740***	**5.3768**
		0.9	11.3093	10.2365	11.0674	9.2202	9.4898	11.0902	8.7008	***8.5182***	**8.5349**
		0.95	21.6263	17.8388	20.7609	14.4306	15.5034	20.8384	12.9409	***12.3866***	**12.4801**
		0.99	104.4463	46.6173	90.0797	12.3231	29.3403	90.9977	9.2487	***6.6946***	**8.1480**

Two estimators with Minimum MSE values are bolded in each row. The smaller of the two is italicized.

Table 13. The results of regression coefficients and the corresponding MSE values.

	$\hat{\alpha}$	$\hat{\alpha}_{RE}$	$\hat{\alpha}_{LE}$	$\hat{\alpha}_{KL}$	$\hat{\alpha}_{NTP}$	$\hat{\alpha}_{MRT}$	$\hat{\alpha}_{NTPKL}$	$\hat{\alpha}_{MTPKL1}$	$\hat{\alpha}_{MTPKL2}$
$\hat{\alpha}_0$	-52.9936	-49.3576	-52.9936	-45.7216	41.6554	41.6554	-24.8522	-49.3576	-52.9936
$\hat{\alpha}_1$	0.0711	0.0703	0.0711	0.0696	47.4077	47.40767	47.42099	0.070317	0.071073
$\hat{\alpha}_2$	-0.4142	-0.4079	-0.4142	-0.4015	13.7557	13.75574	13.64773	-0.40785	-0.4142
$\hat{\alpha}_3$	-0.4235	-0.4323	-0.4235	-0.4411	0.8644	0.864403	1.018985	-0.43227	-0.42347
$\hat{\alpha}_4$	-0.5726	-0.5746	-0.5726	-0.5767	0.3746	0.374561	0.415282	-0.57463	-0.57257
$\hat{\alpha}_5$	48.4179	48.0144	48.4179	47.6109	-4.7292	-4.72919	2.27232	48.01439	48.41786
MSE	17095.15	15027.25	17095.14	13122.26	17095.14	13316.98	*11510.3*	***2915.66***	**5150.228**