Approximating the ARL of Changes in the Mean of a Seasonal Time Series Model with Exponential White Noise Running on a CUSUM Control Chart

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Abstract: - Control charts comprise an excellent statistical process control tool for monitoring industrial processes. Especially, the CUSUM control chart is very sensitive to small-to-moderate process parameter changes. The proposed approach utilizes the numerical integral equation (NIE) method to approximate the average run length (ARL) of changes in the mean of a seasonal time series model with underlying exponential white noise running on a CUSUM control chart. This was achieved by solving a system of linear equations and integration through partitioning and summation using the area under the curve of a function obtained by applying the Gauss-Legendre quadrature. A numerical study was conducted to compare the capabilities of the ARL derivations obtained using the NIE method and explicit formulas to detect changes in the mean of a long-memory $\mathrm{SARFIMA}(p,d)(P,D,Q)_s$ model with exponential white noise running on a CUSUM control chart. The results reveal that the performances of both were comparable in terms of the accuracy percentage, which was greater than 95%, meaning that the ARL values were highly consistent. Thus, the NIE method can be used to validate ARL results for this situation.

Key-Words: - CUSUM control chart, average run length (ARL), exponential white noise, $\mathrm{SARFIMA}(p,d)(P,D,Q)_s$ process, numerical integral equation (NIE).

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1 Introduction

Statistical process control (SPC) has been extensively employed for monitoring processes and services to avoid the occurrence of problems in industrial processes. One of the most commonly utilized SPC tools is the control chart, which is robust, reliable, and powerful for monitoring industrial processes. Moreover, they are easy to implement and interpretation of their output is unambiguous. Control charts are designed to detect changes in a process parameter from the in-control state to the out-of-control state. Information about a process characteristic is plotted against time in conjunction with so-called control limits. A signal is transmitted once the plotted statistic exceeds the predetermined control limit and indicates that the process is possibly out-of-control, [1]. There are two categories of control charts: memoryless (e.g., Shewharts) and memory-type (e.g., the cumulative sum (CUSUM) and exponentially weighted moving average (EWMA)). The CUSUM, [2], and EWMA, [3], control charts are extensively utilized to detect small-to-moderate changes in the parameter of a process while the Shewhart control chart is better at detecting large changes. Much research has been performed to evaluate the efficacy of the CUSUM control chart, [4], [5], [6]. Our research is mainly concentrated on the upper one-sided CUSUM control chart, [7].

An important control chart evaluation criterion is the average run length (ARL). It is the average number of observations taken until a control chart signals that the process is out of control. It comprises two parts: ARL_0 when the process is in control, which should be as large as possible, and ARL_1 when the process is out of control, which should be as small as possible. Various methods have been developed to derive the ARL for changes in the mean of a normal process running on a CUSUM control chart. For example, [8], derived the approximate

ARL for changes in the parameters of processes running on a CUSUM control chart with a zero head-start as the ratio of numerical solutions to two integral equations. Likewise, [9], provided a solution for the ARL under similar circumstances but with a non-zero head-start using the Markov chain and integral equation methods for normally distributed observations.

Autocorrelation can have a substantial impact on the effectiveness of a CUSUM control chart, [10]. Nevertheless, since it is often an inherent part of a process, it must be modeled and monitored appropriately. Econometric data has been used to develop some of these models since it often fits autoregressive (AR), moving-average (MA), ARMA, or AR fractionally integrated MA (ARFIMA) models. Measuring errors (the difference between the actual and estimated values) is crucial when creating a model: the lower the number of errors, the higher the efficiency of the model. A time series model with autocorrelated data often contains errors indicated as white noise, which in certain situations, is exponentially distributed, [11], [12], [13].

A time series has long-memory properties when the differencing parameter d in a $\mathrm{ARFIMA}(p,d,q)$ model lies within the range (0, ½), [14], [15]. This characteristic is exemplified either by the hyperbolic decline of the autocorrelation function or the lack of bounds for the spectral density function. In contrast, an ARMA model shows a geometric rate of reduction in the correlation between the observations. Our primary interest lies in long-memory $\mathrm{SARFIMA}(p,d,q)\ (P,D,Q)_S$ models with the added complication of seasonality, [16], [17], [18], which often appears empirically. The study, [17], adopted the Kalman filter methodology to deduce the values of parameters d and D for a $\mathrm{SARFIMA}(0,d,0)(0,D,0)_S$ process. In the present study, we explore this specific scenario under the presumption that the white noise follows an exponential distribution. Several control charts have been adapted to run processes with the fractional integration element, [18], [19]. The present research is centered on identifying shifts in the mean of a long-memory $\mathrm{SARFIMA}(p,d,q)(P,D,Q)_S$ process running on a CUSUM control chart.

The principal methods for calculating the ARL have been based on utilizing Monte Carlo simulation, the Markov chain technique, explicit formulas, and integral equations. The study, [20], employed the Markov chain methodology to determine the ARL of a process running on a CUSUM control chart while presupposing that the observations are independently and identically distributed (i.i.d.). The author in, [21], refined this approach through the incorporation of Richardson extrapolation for observations from a comprehensive array of distributions, including the Chi-squared distribution. Integral equations such as the Fredholm integral equation of the second kind have been utilized within the numerical integral equation (NIE) methodology to compute the ARL, [22], [23], [24], [25], [26], [27], [28]. Alternatively, the Gauss-Legendre quadrature has been employed in the integral equation approach, [22], in which it is noteworthy that the sample variance adheres to a right-skewed Chi-squared distribution restricted to the half-real line. The study, [29], used a piecewise collocation technique as an alternative to the Gauss-Legendre quadrature for ARL approximation. Numerical integration (or quadrature) is a commonly utilized method for approximating integrals. As well as the Gauss-Legendre quadrature, other examples include using the midpoint, composite trapezoidal, composite Simpson's, and Gaussian rules. In the present study, we obtained approximated the ARL for changes in the mean of a long-memory $\mathrm{SARFIMA}(p,d)(P,D,Q)_S$ process with underlying exponential white noise running on a CUSUM control chart by using the NIE method based on an integral equation using the Gauss-Legendre quadrature.

The remainder of the paper Is organized as follows. Section 2 provides a concise overview of the upper-sided CUSUM control chart and the generalized long-memory $\mathrm{SARFIMA}(p,d)(P,D,Q)_S$ model with underlying exponential white noise. Similarly, the design of the upper-sided CUSUM control chart is presented in Section 3. Approximating the ARL for changes in the mean of a long-memory $\mathrm{SARFIMA}(p,d)(P,D,Q)_S$ with underlying exponential white noise running on a CUSUM control chart by using the NIE method is covered in Section 4. The ARL numerical results obtained using the NIE method and the explicit formulas are compared in Section 5. Last, conclusions and future recommendations are set out in Section 7.

2 Preliminaries

Here, we overview the upper-sided CUSUM control chart and the generalized long-memory $\text{SARFIMA}(p,d)(P,D,Q)_S$ model with underlying exponential white noise.

2.1 The Upper-Sided CUSUM Control Chart

CUSUM statistic Z_t, at time t is defined as follows:

$$Z_t = \max\{Z_{t-1} + Y_t - k,\ 0\},\ \text{for } t = 1,2,\ldots,$$

where Y_t is the sequence of the generalized $\text{SARFIMA}(p,d)(P,D,Q)_S$ process with exponential white noise and k is a reference value. The starting value Z_0 is set to ψ in this study whereby $\psi \le h$, where h is either the decision parameter or upper control limit of the CUSUM control chart.

2.2 The Generalized Long-Memory SARFIMA(*p, d, q*)(*P, D, Q*)s Model With Underlying Exponential White Noise

A time series has long-term dependence or long memory if its autocorrelation coefficient does not decay. If the coefficient of autocorrelation of order k, ρ_k, satisfies the condition $\sum_{k=1}^{\infty}|\rho_k| = \infty$, then such a time series is called a long-memory process. As the latter has often been observed in many economic time series, several models for describing it have been developed. Analysis of long-term dependency on the volatility of exchange rates has often been performed using the ARFIMA model, [14], [15]. Nevertheless, the utilization of fractional differencing (or integration) alone does not cover the characteristics of seasonality. Consequently, the SARFIMA model, which is the ARFIMA model with a seasonality component, has been devised. The parameters of the $\text{SARFIMA}(p,d,q)(P,D,Q)_S$ model can be described in terms of a seasonal time series (Y_t) as follows:

$$\phi_p(B)\Phi_P(B^s)(1-B)^d(1-B^s)^D Y_t = \theta_q(B)\Theta_Q(B^s)\varepsilon_t, \tag{2}$$

where ε_t is a white noise process assumed to be exponentially distributed with $\varepsilon_t \sim Exp(\lambda)$ when shift parameter $\lambda > 0$.

$$\phi_p(B) = (1-\phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p) = 1 - \sum_{i=1}^{p}\phi_i B^i, \text{ and}$$

$$\theta_q(B) = (1-\theta_1 B - \theta_2 B^2 - \ldots - \theta_q B^q) = 1 - \sum_{j=1}^{q}\theta_j B^j,$$

are non-seasonal AR and MA polynomials in B of order p and q respectively;

$$\Phi_P(B^s) = (1-\Phi_1 B^s - \Phi_2 B^{2s} - \ldots \Phi_P B^{Ps}) = 1 - \sum_{k=1}^{P}\Phi_k B^{ks},$$

and (1)

$$\Theta_Q(B^s) = (1-\Theta_1 B^s - \Theta_2 B^{2s} - \ldots - \Theta_Q B^{Qs}) = 1 - \sum_{l=1}^{Q}\Theta_l B^{ls},$$

are seasonal AR and MA polynomials in B of order P and Q, respectively; B is the backshift operator satisfying $BY_t = Y_{t-1}$, and $B^s Y_t = Y_{t-s}$, ; d and D are the annual and seasonal fractionally differencing parameter, respectively, and s is the number of time periods utilized in a year (e.g., $s=12$ is a monthly time series). In particular, the $\text{SARFIMA}(p,d,q)(P,D,Q)_S$ process is when $p=q=P=Q=0.$ This process is a non-seasonal and seasonal fractionally integrated $(\text{SARFIMA}(0,d,0)\ (0,D,0)_S)$ model, which can be defined as

$$(1-B)^d(1-B^S)^D Y_t = \varepsilon_t,$$

where d and D are the non-seasonal and seasonal differencing parameters, respectively. All real values of $d, D > -1$, can be expressed in terms of their binomial expansion as follows:

$$(1-B^\kappa)^x = \sum_{v=0}^{\infty}\binom{\partial}{v}(-B^\kappa)^v = 1 - \partial B^\kappa + \frac{\partial(\partial-1)}{2}B^{2\kappa} - \ldots;$$

$$\kappa = 1, s, \quad \text{and } \partial = d, D, \tag{3}$$

where $\binom{\partial}{v} = \frac{\Gamma(\partial+1)}{\Gamma(v+1)\Gamma(\partial-v+1)}$, and $\Gamma(.)$ is a gamma function.

For $q=0$, the $\text{SARFIMA}(p,d,0)(P,D,Q)_S$ or $\text{SARFIMA}(p,d)(P,D,Q)_S$ model can be defined as

$$\phi_p(B)\Phi_P(B^s)(1-B)^d(1-B^s)^D Y_t = \Theta_Q(B^s)\varepsilon_t, \tag{4}$$

$\text{SARFIMA}(p,d)(P,D,Q)_S$ models are commonly used to model time series with long-memory behavior. They have the same characteristics as the corresponding ARFIMA model (i.e. stationarity and invertibility) when $|d+D| < 1/2$ and d and D are less than $1/2$, which indicates a long-memory process.

The equation (4) can be rearranged in favor of (Y_t) for the generalized $\mathrm{SARFIMA}(p,d)(P,D,Q)_S$ model on the CUSUM control chart as follows:

$$Y_t = \frac{\Theta_Q(B^s)\varepsilon_t}{\phi_p(B)\Phi_P(B^s)(1-B)^d(1-B^s)^D}, \text{ or}$$

$$Y_t = (1-\sum_{i=1}^{p}\phi_i B^i)^{-1}(1-\sum_{j=1}^{P}\Phi_j B^{is})^{-1}(1-B)^{-d}(1-B^s)^{-D} .(\varepsilon_t - \Theta_1\varepsilon_{t-s} - \Theta_2\varepsilon_{t-2s} - \ldots - \Theta_Q\varepsilon_{t-Qs}) \quad (5)$$

where $\varepsilon_t \sim Exp(\lambda)$. The initial value is normally the process mean (i.e., $\varepsilon_{t-s}, \varepsilon_{t-2s}, \ldots, \varepsilon_{t-Qs} = 1$.); the coefficient parameters are $\phi_{i,1\le i\le p}, \Phi_{j,1\le j\le P}, \Theta_{k,1\le k\le Q}$; and the initial value of the long-memory $\mathrm{SARFIMA}(p,d)(P,D,Q)_S$ process is 1.

3 The Design of Upper-Sided CUSUM Control Chart

Here, we discuss the design of the CUSUM control chart running a generalized long-memory $\mathrm{SARFIMA}(p,d)(P,D,Q)_S$ model with underlying exponential white noise.

Let $\varepsilon_t, t = 1,2,\ldots,$ represent a sequence of continuous i.i.d. random variables from an exponential distribution with parameter λ. The process is considered in-control when $\lambda = \lambda_0$, whereas it is out-of-control when $\lambda = \lambda_1$. The following are the change points for ε_t:

$$\varepsilon_t = \begin{cases} Exp(\lambda_0), & t = 1, 2, \ldots, m-1 \quad \text{(no change)} \\ Exp(\lambda_1 > \lambda_0), & t = m, m+1, \ldots \quad \text{(change)}, \end{cases} \quad (6)$$

The ARFIMA process in Equation (5) can be substituted into Equation (1), so the CUSUM statistic becomes

$$Z_t = Z_{t-1} + Y_t - k$$

$$Z_1 = \psi + (1-\sum_{i=1}^{p}\phi_i B^i)^{-1}(1-\sum_{j=1}^{P}\Phi_j B^{is})^{-1}(1-B)^{-d}(1-B^s)^{-D} .(\varepsilon_t - \Theta_1\varepsilon_{t-s} - \Theta_2\varepsilon_{t-2s} - \ldots - \Theta_Q\varepsilon_{t-Qs}) - k \quad (7)$$

where $Z_{t-1} = \psi$. Hence, the CUSUM stopping time (τ_h) can be written as

$$\tau_h = \inf\{t > 0;\, Z_t > h\}, \text{for } \psi \le h, \quad (8)$$

Note that $0 < Z_t < h,$ indicates that the process is in control whereas $Z_t > h$ indicates that the process is out of control.

In the context of the in-control process, modifying the CUSUM statistic by reorganizing the error term (ε_t) is possible, resulting in ε_t being between 0 and h. Subsequently, ε_t can be rearranged as follows:

$$k - \psi + (1-\sum_{i=1}^{p}\phi_i B^i)^{-1}(1-\sum_{j=1}^{P}\Phi_j B^{is})^{-1}(1-B)^{-d}(1-B^s)^{-D} .(-\Theta_1\varepsilon_{1-s} - \Theta_2\varepsilon_{1-2s} - \ldots - \Theta_Q\varepsilon_{1-Qs})$$
$$< \varepsilon_1 < h + k - \psi + (1-\sum_{i=1}^{p}\phi_i B^i)^{-1}(1-\sum_{j=1}^{P}\Phi_j B^{is})^{-1}(1-B)^{-d}(1-B^s)^{-D} .(-\Theta_1\varepsilon_{1-s} - \Theta_2\varepsilon_{1-2s} - \ldots - \Theta_Q\varepsilon_{1-Qs}),$$

4 Approximating the ARL for Changes in the Mean of a Process Running on a CUSUM Control Chart Via the NIE Method

The ARL can be approximated by utilizing the NIE method based on Fredholm's integral equation of the second kind, [23]. Subsequently, the NIE method was used in conjunction with the CUSUM statistic to approximate the ARL. In this section, the application of the Gauss-Legendre rule technique for the numerical calculation of the integral equations of the NIE method is proposed.

To evaluate the performance of the CUSUM control chart, it is necessary to determine the stochastic properties of the corresponding stopping time (τ_h). Assuming there is a change point in Equation (6), it is possible to establish a rigorous definition of the ARL using $E_m(.)$ under the assumption that the change point occurs at time m. Consequently,

$$\mathrm{ARL} = \begin{cases} E_\infty(\tau_h), & \lambda = \lambda_0; \text{ in-control state } (\mathrm{ARL}_0) \\ E_1(\tau_h), & \lambda \ne \lambda_0; \text{ out-of-control state} (\mathrm{ARL}_1). \end{cases}$$

Let $\mathrm{L}(\psi)$ be the ARL of a change in the mean of a long-memory $\mathrm{SARFIMA}(p,d,q)(P,D,Q)_S$ model conditioned on the initial value of the CUSUM statistic $Z_0 = \psi;\quad 0 < \psi < h.$ This structure utilizes the stopping time (τ_h) for the process such that the ARL is given as

$$\mathrm{L}(\psi) = \mathrm{E}_\infty(\tau_h) < \infty,$$

where $\mathrm{E}_\infty(\tau_h)$ is the expectation under the density function $f(\varepsilon_t, \lambda)$.

The solution of the integral equation becomes

$$\mathrm{L}(\psi) = 1 + \mathrm{P}_z\{Z_1 = 0\}\mathrm{L}(0) + \mathrm{E}_z[I\{0 < Z_1 < h\}\mathrm{L}(Z_1)], \quad (9)$$

where Z_1 is the first observation and $I\{0 < Z_1 < h\}$ represents the indicator function.

The integral equation utilized to determine the ARL of a change in the process parameter running on a one-sided CUSUM control chart is obtained by employing Equation (9) and applying a Fredholm integral equation of the second kind. This allows us to rewrite for $\mathrm{L}(\psi)$ as follows:

$$\begin{aligned}\mathrm{L}(\psi) = 1 + \mathrm{L}(0)F(k - \psi - (1 - \sum_{i=1}^{p}\phi_i B^i)^{-1}(1 - \sum_{j=1}^{P}\Phi_j B^{is})^{-1} \\ (1-B)^{-d}(1-B^s)^{-D}(\varepsilon_t - \Theta_1\varepsilon_{t-s} - \ldots - \Theta_Q\varepsilon_{t-Qs})) \\ + \int_0^h \mathrm{L}(u) f(u + k - \psi - (1 - \sum_{i=1}^{p}\phi_i B^i)^{-1}(1 - \sum_{j=1}^{P}\Phi_j B^{is})^{-1} \\ (1-B)^{-d}(1-B^s)^{-D}(\varepsilon_t - \Theta_1\varepsilon_{t-s} - \ldots - \Theta_Q\varepsilon_{t-Qs}))du,\end{aligned} \quad (10)$$

where $F(\psi) = 1 - e^{-\lambda\psi}$ and $f(\psi) = \lambda e^{-\lambda\psi}$. When applying the final term in Equation (1 0) to the quadrature rule, the integral can be estimated by summation of the rectangles.

The Gauss-Legendre quadrature rule can be utilized in numerical solutions based on integral equations in the final term in Equation (10). Clearly, integral $\int_0^h f(u)du$ can be approximated by summing the areas of the rectangles with bases h/m and heights maintained as the values of f at the zero-based midpoints of the intervals of length h/m. Interval $[0, h]$ is partitioned into $0 \le a_1 \le \ldots \le a_m \le h$ partitions while $w_j = h/m \ge 0$ is a set of weights. Thus, the integral can be approximated in summation form as

$$\int_0^h W(u) f(u) du \approx \sum_{j=1}^{m} w_j f(a_j)$$

where $W(u)$ is a weight function, $a_j = \frac{h}{m}\left(j - \frac{1}{2}\right)$,

and $w_j = h/m; j = 1, 2, \ldots, m.$

Solving the system of m linear equations in mm unknowns can be used to approximate the solution for $\mathrm{L}(\psi)$ on interval $[0, h]$ by substituting ψ with a_i in Equation (10) as follows:

$$\begin{aligned}\hat{\mathrm{L}}(a_i) = 1 + \hat{\mathrm{L}}(a_1)F(k - \psi - (1 - \sum_{i=1}^{p}\phi_i B^i)^{-1}(1 - \sum_{j=1}^{P}\Phi_j B^{is})^{-1} \\ .(1-B)^{-d}(1-B^s)^{-D}(-\Theta_1\varepsilon_{1-s} - \ldots - \Theta_Q\varepsilon_{1-Qs})) \\ + \sum_{j=1}^{m} w_j \hat{\mathrm{L}}(a_j) f(a_j + k - a_i - (1 - \sum_{i=1}^{p}\phi_i B^i)^{-1}(1 - \sum_{j=1}^{P}\Phi_j B^{is})^{-1} \\ .(1-B)^{-d}(1-B^s)^{-D}(-\Theta_1\varepsilon_{1-s} - \ldots - \Theta_Q\varepsilon_{1-Qs})),\ i = 1, 2, \ldots, m.\end{aligned}$$

Let $\hat{\mathrm{L}}(\psi)$ denote the approximated ARL using the NIE method when applying the Gauss-Legendre rule on interval $[0, h]$. Therefore, the integral equation in Equation (10) comprises the set $\hat{\mathrm{L}}(\psi) = \hat{\mathrm{L}}(a_1), \hat{\mathrm{L}}(a_2), \ldots, \hat{\mathrm{L}}(a_m)$, which can be approximated as

$$\begin{aligned}\hat{\mathrm{L}}(a_1) = 1 + \hat{\mathrm{L}}(a_1)\begin{bmatrix} F(k - a_1 - (1 - \sum_{i=1}^{p}\phi_i B^i)^{-1}(1 - \sum_{j=1}^{P}\Phi_j B^{is})^{-1}(1-B)^{-d} \\ .(1-B^s)^{-D}(-\Theta_1\varepsilon_{1-s} - \ldots - \Theta_Q\varepsilon_{1-Qs})_t) \\ + w_1 f(k - (1 - \sum_{i=1}^{p}\phi_i B^i)^{-1}(1 - \sum_{j=1}^{P}\Phi_j B^{is})^{-1} \\ .(1-B)^{-d}(1-B^s)^{-D}(-\Theta_1\varepsilon_{1-s} - \ldots - \Theta_Q\varepsilon_{1-Qs})_t) \end{bmatrix} \\ + \sum_{j=2}^{m} w_j \hat{\mathrm{L}}(a_j) f(a_j + k - a_1 - (1 - \sum_{i=1}^{p}\phi_i B^i)^{-1}(1 - \sum_{j=1}^{P}\Phi_j B^{is})^{-1} \\ .(1-B)^{-d}(1-B^s)^{-D}(-\Theta_1\varepsilon_{t-s} - \ldots - \Theta_Q\varepsilon_{t-Qs})),\end{aligned}$$

$$\begin{aligned}\hat{\mathrm{L}}(a_2) = 1 + \hat{\mathrm{L}}(a_1)\begin{bmatrix} F(k - a_2 - (1 - \sum_{i=1}^{p}\phi_i B^i)^{-1}(1 - \sum_{j=1}^{P}\Phi_j B^{is})^{-1} \\ .(1-B)^{-d}(1-B^s)^{-D}(\varepsilon_t - \Theta_1\varepsilon_{1-s} - \ldots - \Theta_Q\varepsilon_{1-Qs})) \\ + w_1 f(a_1 + k - a_2 - (1 - \sum_{i=1}^{p}\phi_i B^i)^{-1}(1 - \sum_{j=1}^{P}\Phi_j B^{is})^{-1} \\ .(1-B)^{-d}(1-B^s)^{-D}(-\Theta_1\varepsilon_{1-s} - \ldots - \Theta_Q\varepsilon_{1-Qs})) \end{bmatrix} \\ + \sum_{j=2}^{m} w_j \hat{\mathrm{L}}(a_j) f(a_j + k - a_2 - (1 - \sum_{i=1}^{p}\phi_i B^i)^{-1}(1 - \sum_{j=1}^{P}\Phi_j B^{is})^{-1} \\ .(1-B)^{-d}(1-B^s)^{-D}(-\Theta_1\varepsilon_{1-s} - \ldots - \Theta_Q\varepsilon_{1-Qs})),\end{aligned}$$

$$\vdots$$

$$\hat{\mathrm{L}}(a_m) = 1 + \hat{\mathrm{L}}(a_1)\left[\begin{array}{l} F(k - a_m - (1-\sum_{i=1}^{p}\phi_i B^i)^{-1}(1-\sum_{j=1}^{P}\Phi_j B^{is})^{-1} \\ .(1-B)^{-d}(1-B^s)^{-D}(\varepsilon_t - \Theta_1\varepsilon_{1-s} - \ldots - \Theta_Q\varepsilon_{1-Qs})_t) \\ +w_1 f(a_1 + k - a_m - (1-\sum_{i=1}^{p}\phi_i B^i)^{-1}(1-\sum_{j=1}^{P}\Phi_j B^{is})^{-1} \\ .(1-B)^{-d}(1-B^s)^{-D}(\varepsilon_t - \Theta_1\varepsilon_{1-s} - \ldots - \Theta_Q\varepsilon_{1-Qs})) \end{array}\right]$$
$$+\sum_{j=2}^{m} w_j \hat{\mathrm{L}}(a_j) f(a_j + k - a_m - (1-\sum_{i=1}^{p}\phi_i B^i)^{-1}(1-\sum_{j=1}^{P}\Phi_j B^{is})^{-1}$$
$$.(1-B)^{-d}(1-B^s)^{-D}(\varepsilon_t - \Theta_1\varepsilon_{1-s} - \ldots - \Theta_Q\varepsilon_{1-Qs})),$$

The above set of m equations in m unknowns can be expressed in matrix form:

$$\mathbf{L}_{m\times 1} = \mathbf{1}_{m\times 1} + \mathbf{C}_{m\times m}\mathbf{L}_{m\times 1}$$

which is equivalent to $(\mathbf{I}_m - \mathbf{C}_{m\times m})\mathbf{L}_{m\times 1} = \mathbf{1}_{m\times 1}$.

where $\mathbf{L}_{m\times 1} = \left[\hat{\mathrm{L}}(a_1), \hat{\mathrm{L}}(a_2), \ldots, \hat{\mathrm{L}}(a_m)\right]'$, $\mathbf{1}_{m\times 1} = \left[1,1,\ldots,1\right]'$ is a column vector of ones, and $\mathbf{I}_m = diag(1,1,\ldots,1)$ is the unit matrix order m. Matrix $\mathbf{C}$ with dimensions $m\times m$ becomes

$$\mathbf{C}_{m\times m} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1m} \\ c_{21} & c_{22} & \cdots & c_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mm} \end{bmatrix},$$

where $c_{ij};\ i,j = 1,2,\ldots,m$ can be expanded as

$$c_{ij} = F(k - a_i - (1-\sum_{i=1}^{p}\phi_i B^i)^{-1}(1-\sum_{j=1}^{P}\Phi_j B^{is})^{-1}$$
$$.(1-B)^{-d}(1-B^s)^{-D}(-\Theta_1\varepsilon_{1-s} - \ldots - \Theta_Q\varepsilon_{1-Qs})_t)$$
$$+w_j f(a_j + k - a_i - (1-\sum_{i=1}^{p}\phi_i B^i)^{-1}(1-\sum_{j=1}^{P}\Phi_j B^{is})^{-1}$$
$$.(1-B)^{-d}(1-B^s)^{-D}(-\Theta_1\varepsilon_{1-s} - \ldots - \Theta_Q\varepsilon_{1-Qs}))$$

If $(\mathbf{I}_m - \mathbf{C}_{m\times m})^{-1}$ is invertible and exists, then the ARL approximation for the NIE can be reformed into a system of linear equations in matrix form as follows:

$$\mathbf{L}_{m\times 1} = (\mathbf{I}_m - \mathbf{C}_{m\times m})^{-1}\mathbf{1}_{m\times 1}, \tag{12}$$

After the computation, $\hat{\mathrm{L}}(a_1), \hat{\mathrm{L}}(a_2), \ldots, \hat{\mathrm{L}}(a_m)$ is substituted for a_i by ψ as follows:

$$\hat{\mathrm{L}}(\psi) = 1 + \hat{\mathrm{L}}(a_1)F(k - \psi - (1-\sum_{i=1}^{p}\phi_i B^i)^{-1}(1-\sum_{j=1}^{P}\Phi_j B^{is})^{-1}$$
$$.(1-B)^{-d}(1-B^s)^{-D}(-\Theta_1\varepsilon_{1-s} - \ldots - \Theta_Q\varepsilon_{1-Qs}))$$
$$+\sum_{j=1}^{m} w_j \hat{\mathrm{L}}(a_j) f(a_j + k - \psi - (1-\sum_{i=1}^{p}\phi_i B^i)^{-1}(1-\sum_{j=1}^{P}\Phi_j B^{is})^{-1}$$
$$.(1-B)^{-d}(1-B^s)^{-D}(-\Theta_1\varepsilon_{1-s} - \ldots - \Theta_Q\varepsilon_{1-Qs})), \tag{13}$$

with $w_j = h/m$, and $a_j = h/m\left(j - \frac{1}{2}\right);\ j = 1,2,\ldots,m.$

This is the proposed approximation of the ARL for changes in the mean of a long-memory $\mathrm{SARFIMA}(p,d,0)(P,D,Q)_S$ process with underlying exponential white noise on a CUSUM control chart using the NIE method. The Gauss-Legendre quadrature rule technique can be used to approximate the ARL quite accurately, as shown in the next section.

5 The Performance Evaluation Results of the Numerical Study

Here, we present the numerical results of a comparative analysis conducted to evaluate the performances of the proposed NIE method utilizing explicit formulas.

5.1 Derivation of the ARL using Explicit Formulas

To evaluate the performance of the proposed ARL for a long-memory $\mathrm{SARFIMA}(p,d)(P,D,Q)_S$ model running on a CUSUM control chart, we used the ARL derived by using explicit formulas denoted as $\mathrm{L}(\psi)$, which can be written in the form

$$\mathrm{L}(\psi) = \exp\{\lambda h\}\left(1 - \lambda h + \exp\left\{\lambda(k - (1-\sum_{i=1}^{p}\phi_i B^i)^{-1}(1-\sum_{j=1}^{P}\Phi_j B^{is})^{-1}\right.\right.$$
$$\left.\left..(1-B)^{-d}(1-B^s)^{-D}(-\Theta_1\varepsilon_{1-s} - \ldots - \Theta_Q\varepsilon_{1-Qs}))\right\}\right) - \exp\{\lambda\psi\} \tag{14}$$

where λ is replaced by λ_0 for the in-control ARL ($\mathrm{ARL_0}$) process and λ is replaced by λ_1 for the out-of-control ARL ($\mathrm{ARL_1}$).

5.2 The Standard Deviation of the Run Length (SDRL) Performance Measure

As well as the ARL, we also computed the SDRL used as a performance measure for the CUSUM control chart for the situation described earlier. The in-control SDRL ($\mathrm{SDRL_0}$) is defined as

$$\mathrm{SDRL_0} = \sqrt{\frac{1-\alpha}{\alpha^2}} \tag{15}$$

where $\alpha = 1 - P(0 < Z_t < h|\,\lambda)$ is a false alarm (or type I error); the probability at which a false alarm occurs is known as the false alarm rate (FAR). Thus, the probability of a type I error or FAR is 0.0027. Hence, $\mathrm{ARL}_0 = \frac{1}{\alpha} \approx 370,$ and $\mathrm{SDRL}_0 \approx 370.$

On the contrary, the out-of-control SDRL (SDRL_1) can be characterized as

$$\mathrm{SDRL}_1 = \sqrt{\frac{\beta}{(1-\beta)^2}}, \tag{16}$$

where $\beta = P(0 < Z_t < h \mid \lambda_1)$ is the probability of a type II error occurring. $\mathrm{ARL}_1 = \frac{1}{(1-\beta)} \approx 1$ corresponds to ARL_0 = 370 for large changes in the process mean.

5.3 The Percentage Accuracy Performance Measure

Let $\hat{\mathrm{L}}(\psi)$ and $\mathrm{L}(\psi)$ be the ARL values obtained by using the NIE and explicit formula methods, respectively. Thus,

$$\%\,\mathrm{Accuracy} = 100 - \left| \frac{\mathrm{L}(\psi) - \hat{\mathrm{L}}(\psi)}{\mathrm{L}(\psi)} \right| \times 100\%, \tag{17}$$

An accuracy percentage of greater than 95% implies that the ARL results using both methods are close to each other.

This work aims to numerically approximate the average run lengths (ARLs) for long memory with $\mathrm{SARFIMA}(p,d)(P,D,Q)_S$ a model underly exponential white noise when implemented on a CUSUM control chart using the NIE method and explicit formulas. The white noise in the long-memory $\mathrm{SARFIMA}(p,d)(P,D,Q)_S$ process was distributed exponentially where the mean parameter of the exponential is λ in the study situation. In addition, the value of $\lambda = \lambda_0$ is equal to 1 for the in-control process, whereas $\lambda_1 = (1+\delta)\lambda_0$ represents the value for the out-of-control process, where δ is the magnitude of shift size; δ = 0.025, 0.05, 0.10, 0.25, 0.50, 0.75, 1.0, 1.5, 2.0, 2.5, 3.0, 4.0, or 5.0 respectively. In the long-memory process with $\mathrm{SARFIMA}(1, 0.1)(1, 0.1, 1)_4$ the model, coefficient parameters ϕ_1 = 0.1, 0.3, 0.5, or 0.7, Φ_1 = 0.10, and Θ_1 = 0.10 are employed and compared. The NIE method employs 800 division points, denoted as m, to solve systems of linear equations in calculations. The ARL results using the NIE method were compared to the ARL results derived from explicit formulas. The results showed comparable performance of ARL obtained from both methods in detecting changes in the mean process.

Table 1 presents the chief findings of our proposed CUSUM control chart. We have used the sensitivity parameter of the control chart k = 2.5, 3.0, and 5.0, which are the optimal choices for calculating the upper control limit (h) from (13) such that the ARL_0 is fixed at 370.

Table 1. Values of CUSUM control limit with corresponding values of k for the desired ARL_0 = 370 of long-memory $\mathrm{SARFIMA}(1, 0.1)(1, 0.1, 1)_4$ models.

SARFIMA (1, 0.1) (0, 0.1, 1)$_4$			**In-control ARL_0 = 370**		
ϕ_1	Φ_1	Θ_1	k = 2.5	k = 3.0	k = 3.5
0.1	0.1	0.1	4.020943	3.303497	1.149340
0.3	0.1	0.1	3.937120	3.240665	1.098567
0.5	0.1	0.1	3.856962	3.178866	1.047840
0.7	0.1	0.1	3.779928	3.11,8004	0.997154

Table 1 contains reports on the upper control limits (h) and k for every scenario in the control process $(\delta = 0)$. The study revealed the value of h decreased as k was systematically increased for every coefficient parameter combination in each $\mathrm{SARFIMA}(1, 0.1)(1, 0.1, 1)_4$ model. Moreover, if we consider the coefficient parameters, it is found that as ϕ_1 increases, the value of h decreases for each value k as ϕ_1 changes between 0.1 and 0.7. Consequently, we propose a design structure based on the CUSUM statistic to detect changes in the process mean.

Numerical results of ARL_1 were obtained using the NIE method and explicit formulas for out-of-control ARL_1 $(\lambda_1 > \lambda_0)$, which can be calculated through the Wolfram Mathematica. Both methods of detecting changes in the process mean are reported in Table 2, Table 3, Table 4, and Table 5. We also proposed SDRL to compare the performance of ARL in this scenario. According to these findings, the ARL_1 results calculated using the NIE method in (13) and explicit formulas in (14) tended to decrease sensitively as the shift size increased for small to moderate shift magnitude shifts in the process. The ARL_1 is more effective at detecting small $(0 < \delta \le 0.5)$ to moderate $(0.5 < \delta \le 1.0)$ shifts than large $(1.0 < \delta \le 5.0)$ ones. In case of a small shift in the process mean, NIE methods for k = 2.5 (see Figure 1(a)) produced lower ARL_1 values compared to k =

3.0 and 5.0, respectively (see Figure 1(b)-(c)). In case k = 2.5, ARFIMA processes with small AR coefficient values $(\phi_1) = 0.1$ (see Figure 1(a)) were more sensitive to process mean shift detection by both methods than those with large AR coefficients of ϕ_1 = 0.3, 0.5, and 0.7, respectively (see Figure 1(b) – (c)). However, both methods produced the same ARL_1 value for all values of k for a large shift. Similar to the ARL_1 results, the $SDRL_1$ results also demonstrate a decreasing pattern as the shift size increased for all scenarios (see Table 2, Table 3, Table 4, and Table 5).

Furthermore, percentage change results calculated at various magnitudes of process mean shifts for the long-memory processes were greater than 95%, indicating that the proposed method is accurate and highly consistent with the explicit formulas.

The summary of the solution of the Integral equation can be approximated ARL using the Gauss-Legendre quadrature rule. The results demonstrate that the NIE method is a simpler alternative to ARL calculations, which approximate the accuracy of the ARL, [27], [28]. However, the Gauss-Legendre rule yielded the most straightforward ARL calculation and the highest accuracy for the given number of nodes. Lasty, the graphical displays of approximating the ARL_1 accurately using NIE method on the CUSUM control is presented in Figure 2.

Table 2. Comparison of ARL_1 between NIE method and explicit formulas and $SDRL_1$ of CUSUM chart for a long-memory $SARFIMA(1, 0.1)(1, 0.1, 1)_4$ model where $\phi_1 = 0.1$ at ARL_0 = 370

δ	k = 2.5		$SDRL_1$	Acc%	k = 3.0		$SDRL_1$	Acc%	k = 5.0		$SDRL_1$	Acc%
	NIE	Explicit			NIE	Explicit			NIE	Explicit		
0.025	313.089	313.759	313.259	99.79	316.375	316.968	316.468	99.81	319.751	319.973	319.473	99.93
0.05	267.614	268.156	267.656	99.80	273.019	273.516	273.016	99.82	278.423	278.612	278.112	99.93
0.10	1,99.797	200.173	199.672	99.81	207.401	207.755	207.254	99.83	215.082	215.221	214.720	99.94
0.25	96.020	96.165	95.664	99.85	103.693	103.842	103.341	99.86	112.081	112.144	111.643	99.94
0.50	39.889	39.931	39.428	99.89	44.571	44.621	44.118	99.89	50.366	50.389	49.886	99.95
0.75	21.765	21.782	21.276	99.92	24.551	24.572	24.067	99.91	28.370	28.381	27.877	99.96
1.0	14.060	14.068	13.559	99.94	15.806	15.817	15.309	99.93	18.419	18.425	17.918	99.97
1.5	7.863	7.866	7.349	99.96	8.663	8.667	8.152	99.95	10.046	10.048	9.535	99.98
2.0	5.451	5.453	4.928	99.96	5.877	5.879	5.356	99.97	6.707	6.708	6.188	99.99
2.5	4.237	4.238	3.704	99.98	4.487	4.488	3.957	99.98	5.031	5.032	4.504	99.98
3.0	3.523	3.524	2.982	99.97	3.680	3.681	3.141	99.97	4.060	4.061	3.526	99.98
4.0	2.736	2.737	2.180	99.96	2.806	2.806	2.251	100.00	3.016	3.016	2.466	100.00
5.0	2.317	2.317	1.747	100.00	2.351	2.351	1.782	100.00	2.480	2.480	1.916	100.00
h	4.020943				3.303497				1.149340			

Table 3. Comparison of ARL_1 between NIE method and explicit formulas and $SDRL_1$ of CUSUM chart for a long-memory $SARFIMA(1, 0.1)(1, 0.1, 1)_4$ model where $\phi_1 = 0.3$ at ARL_0 = 370

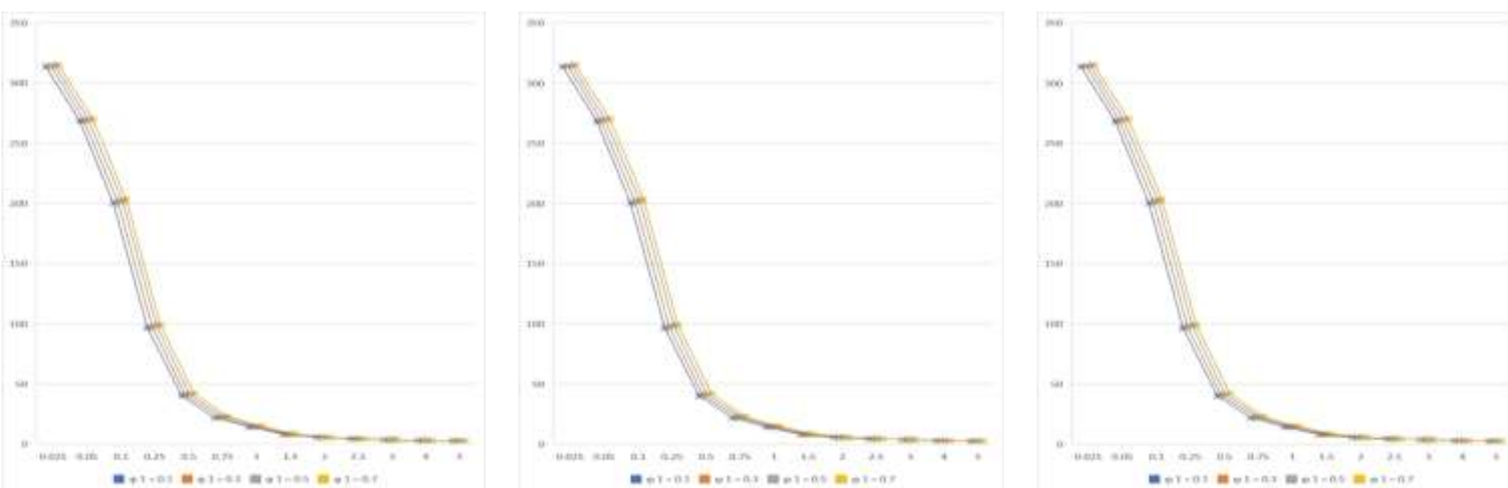
δ	k = 2.5		$SDRL_1$	Acc%	k = 3.0		$SDRL_1$	Acc%	k = 5.0		$SDRL_1$	Acc%
	NIE	Explicit			NIE	Explicit			NIE	Explicit		
0.025	313.570	314.225	313.725	99.79	316.387	317.171	316.671	99.75	319.774	319.987	319.487	99.93
0.05	268.404	268.944	268.444	99.80	273.367	273.857	273.357	99.82	278.456	278.636	278.136	99.94
0.10	200.899	201.275	200.774	99.81	207.894	208.244	207.743	99.83	215.125	215.258	214.757	99.94
0.25	97.103	97.249	96.748	99.85	104.209	104.357	103.856	99.86	112.131	112.191	111.690	99.95
0.50	40.523	40.567	40.064	99.89	44.903	44.952	44.449	99.89	50.405	50.427	49.924	99.96
0.75	22.131	22.148	21.642	99.92	24.756	24.777	24.272	99.92	28.398	28.408	27.904	99.96
1.0	14.283	14.291	13.782	99.94	15.940	15.951	15.443	99.93	18.439	18.445	17.938	99.97
1.5	7.960	7.963	7.446	99.96	8.728	8.732	8.217	99.95	10.058	10.061	9.548	99.97
2.0	5.500	5.502	4.977	99.96	5.913	5.915	5.392	99.97	6.715	6.716	6.196	99.99
2.5	4.264	4.265	3.732	99.98	4.509	4.511	3.980	99.96	5.037	5.038	4.510	99.98
3.0	3.540	3.540	2.999	100.00	3.695	3.696	3.157	99.97	4.064	4.065	3.530	99.98
4.0	2.743	2.743	2.187	100.00	2.814	2.814	2.259	100.00	3.018	3.019	2.469	99.97
5.0	2.320	2.320	1.750	100.00	2.355	2.355	1.786	100.00	2.482	2.482	1.918	100.00
h	3.937120				3.240665				1.098567			

Table 4. Comparison of ARL_1 between NIE method and explicit formulas and $SDRL_1$ of CUSUM chart for a long-memory with $\text{SARFIMA}(1, 0.1)(1, 0.1, 1)_4$ where $\phi_1 = 0.5$ at $ARL_0 = 370$

δ	$k = 2.5$		$SDRL_1$	Acc%	$k = 3.0$		$SDRL_1$	Acc%	$k = 5.0$		$SDRL_1$	Acc%
	NIE	Explicit			NIE	Explicit			NIE	Explicit		
0.025	314.001	314.651	314.151	99.79	316.784	317.360	316.860	99.82	319.798	320.000	319.500	99.94
0.05	269.114	269.651	269.151	99.80	273.692	274.174	273.674	99.82	278.487	278.659	278.159	99.94
0.10	201.894	202.269	201.768	99.81	208.356	208.702	208.201	99.83	215.167	215.293	214.792	99.94
0.25	98.087	98.236	97.735	99.85	104.695	104.842	104.341	99.86	112.177	112.234	111.733	99.95
0.50	41.108	41.153	40.650	99.89	45.217	45.266	44.763	99.89	50.440	50.462	49.959	99.96
0.75	22.470	22.489	21.983	99.92	24.952	24.973	24.468	99.92	28.424	28.434	27.930	99.96
1.0	14.491	14.501	13.992	99.93	16.068	16.079	15.571	99.93	18.459	18.464	17.957	99.97
1.5	8.052	8.056	7.539	99.95	8.791	8.795	8.280	99.95	10.070	10.072	9.559	99.98
2.0	5.548	5.549	5.024	99.98	5.949	5.951	5.428	99.97	6.724	6.724	6.204	100.00
2.5	4.291	4.292	3.759	99.98	4.531	4.533	4.002	99.96	5.042	5.043	4.515	99.98
3.0	3.556	3.557	3.016	99.97	3.710	3.711	3.172	99.97	4.068	4.069	3.534	99.98
4.0	2.749	2.749	2.193	100.00	2.821	2.821	2.267	100.00	3.021	3.021	2.471	100.00
5.0	2.322	2.322	1.752	100.00	2.359	2.359	1.790	100.00	2.484	2.484	1.920	100.00
h	**3.856962**				**3.178866**				**1.047840**			

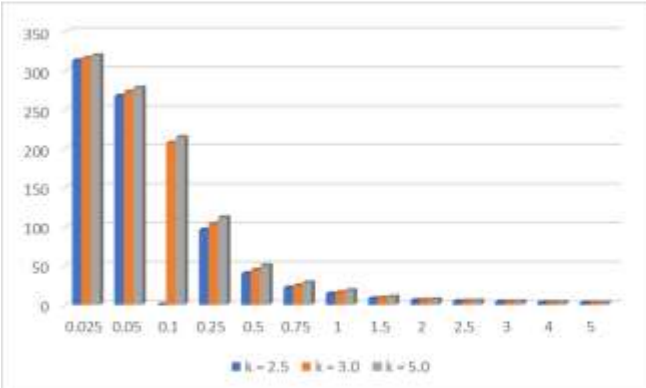
Table 5. Comparison of ARL_1 between NIE method and explicit formulas and $SDRL_1$ of CUSUM chart for a long-memory $\text{SARFIMA}(1, 0.1)(1, 0.1, 1)_4$ model where $\phi_1 = 0.7$ at $ARL_0 = 370$

δ	$k = 2.5$		$SDRL_1$	Acc%	$k = 3.0$		$SDRL_1$	Acc%	$k = 5.0$		$SDRL_1$	Acc%
	NIE	Explicit			NIE	Explicit			NIE	Explicit		
0.025	314.394	315.038	314.538	99.80	316.970	317.537	317.037	99.82	319.820	320.013	319.513	99.94
0.05	269.760	270.293	269.793	99.80	273.994	274.472	273.972	99.83	278.516	278.680	278.180	99.94
0.10	202.799	203.172	202.671	99.82	208.789	209.131	208.630	99.84	215.205	215.325	214.824	99.94
0.25	98.990	99.139	98.638	99.85	105.153	105.298	104.797	99.86	112.219	112.274	111.773	99.95
0.50	41.650	41.696	41.193	99.89	45.516	45.565	45.062	99.89	50.474	50.494	49.991	99.96
0.75	22.788	22.807	22.301	99.92	25.139	25.161	24.656	99.91	28.448	28.458	27.954	99.96
1.0	14.688	14.698	14.189	99.93	16.190	16.202	15.694	99.93	18.477	18.482	17.975	99.97
1.5	8.141	8.144	7.628	99.96	8.851	8.856	8.341	99.94	10.081	10.083	9.570	99.98
2.0	5.594	5.595	5.070	99.98	5.983	5.985	5.462	99.97	6.730	6.731	6.211	99.99
2.5	4.318	4.319	3.786	99.98	4.553	4.554	4.023	99.98	5.047	5.048	4.520	99.98
3.0	3.572	3.573	3.032	99.97	3.725	3.725	3.186	100.00	4.072	4.073	3.538	99.98
4.0	2.756	2.756	2.200	100.00	2.828	2.828	2.274	100.00	3.023	3.023	2.473	100.00
5.0	2.325	2.325	1.755	100.00	2.363	2.363	1.795	100.00	2.485	2.485	1.921	100.00
h	**3.779928**				**3.11,8004**				**0.997154**			

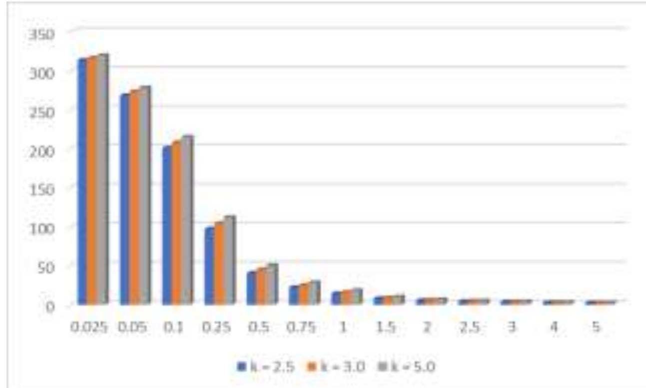


(a) $k = 2.5$ (b) $k = 3.0$ (c) $k = 5.0$

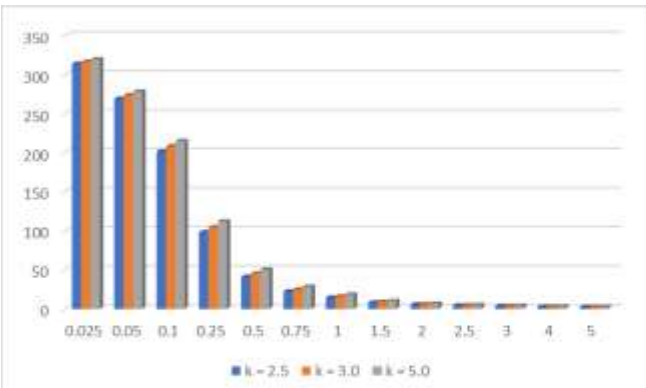
Fig. 1: Graphical displays of approximating the ARL_1 accurately using the NIE method on the CUSUM control chart for a long-memory $\text{ARFIMA}(1, 0.1)(0, 0.1, 1)_4$ model with coefficient value $\phi_1 = 0.1$: (a) $k = 2.5$, (b) $k = 3.0$ and (c) $k = 5.0$



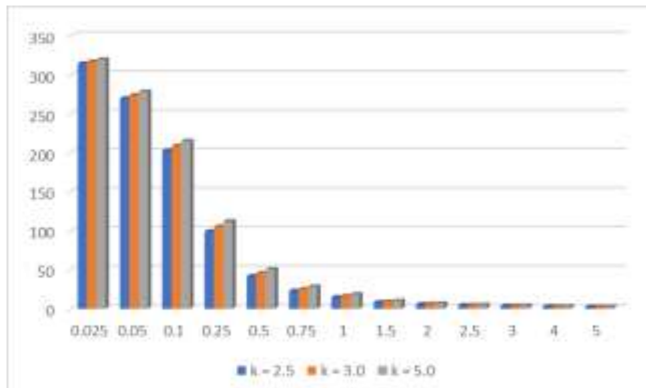
(a) $\phi_1 = 0.1$



(b) $\phi_1 = 0.3$



(c) $\phi_1 = 0.5$



(d) $\phi_1 = 0.7$

Fig. 2: Graphical displays of approximating the ARL_1 accurately using NIE method on the CUSUM control chart:(a) $\phi_1 = 0.1$, (b) $\phi_1 = 0.3$, (c) $\phi_1 = 0.5$ and (d) $\phi_1 = 0.7$

6 Conclusions and Recommendations

The ARL can be derived by using the NIE approach initially introduced by [2]. The ARL can be used to compare the performances of different control charts. The CUSUM control chart performs well at detecting small-to-moderate changes in the process mean. In this study, we applied the Gauss-Legendre quadrature to solve the integral equations for the NIE approach used to derive the ARL for changes in the mean of a long-memory $\text{ARFIMA}(p,d)(P,D,Q)_s$ $\text{ARFIMA}(p,d)(P,D,Q)_s$ model with underlying exponential white noise running on a CUSUM control chart. In addition to calculating the ARL using both the NIE and explicit formula approaches, we also calculated the SDRL. It was found that the ARL_1 and $SDRL_1$ values decreased rapidly and in the same direction.

The results indicate that the proposed NIE method is a good candidate for ARL determination in future research in this scenario. The method could be adapted for new memory-type control charts. In addition, the NIE method could be applied to real-life applications involving time series models.

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Conflict of Interest
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