Comparison between the two HUM and no-regret control methods

BOUAFI NADIA, MERABTI NESRINE LAMYA, REZZOUG IMAD
Department of Mathematics and Computer Science
University of Oum El Bouaghi,
Laboratory of Dynamics systems and control
ALGERIA

Abstract: We present in our work the steps of the Hilbert Uniqueness Method (HUM) and characterization of the low regret and no regret control. At the end of this article, we compare these methods.

Key-Words: Distributed system ; Incomplete data ; Low-regret control ; no-regret control ; HUM method ; Optimality system.

Received: January 2, 2023. Revised: September 4, 2023. Accepted: October 9, 2023. Published: November 8, 2023.

1 Introduction

Several domains are modeled by dynamic or stationary systems, [1], the sentinel theory, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13], is an important tool for the identification of some system data based on control theory, [14, 15, 16, 17], control plays an interesting role in resolving the different systems in the different domains.

Bellow we Present the organization of our memory.

In the first section, we present a description of the HUM method for solving the problem of the controllability system.

In the second section, we present the standard optimal control theory and we consecrated to study the notion of no-regret control and low-regret of distributed system, [18, 19, 20].

Finally, we conclude by comparison between the HUM method and the low regrets control method.

2 Hilbert Uniqueness Method

The construction of the Hilbertian spaces adapted to the building of the system according to the criteria of the specific uniqueness of the homogeneous system associated with it, and the method adopted for that is Hibert Uniqueness Method (HUM), the following algorithm describes the basics of applying the HUM method to solving the problem of exact system controllability.

The basic idea is the following :

Assuming that the system is exactly controllable, characterize the control that minimizes the associated cost function among the set of admissible controls by an optimality system.

2.1 Exact controllability and penalization

2.1.1 Orientation

Let be Ω a bounded domain of R^n, $n \geq 1$, at the border Γ of class C^2.

We consider the wave equation

$$y" - \Delta y = 0, \tag{1}$$

in $Q = \Omega \times [0, T]$ with $T > 0$ fixed.

We assume that we can act on the system through the intermediary of the control $v = v(x, t)$ on the edge $\Sigma = \Gamma \times [0, T]$, so that

$$y = v, \tag{2}$$

on Σ. Let the initial data be

$$y(x, 0) = y^0(x); y'(x, 0) = y^1(x), \tag{3}$$

on Ω. Let $x^0 \in R^n$, $m(x) = x - x^0$ and

$$R(x^0) = \max |m(x)|, x \in \bar{\Omega}.$$

Consider the usual partition of the boundary
$\Gamma(x^0) = x \in \Gamma / m(x).v(x) > 0$,
$\Gamma_*(x^0) = \Gamma \backslash \Gamma(x^0)$,
and
$\Sigma(x^0) = \Gamma(x^0) \times [0, T]$,
$\Sigma_*(x^0) = \Sigma / \Sigma(x^0)$.

Let be the exact controllability of the following equation.

If $T > T(x^0) = 2R(x^0)$ for each pair of initial data $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, there is a control $v \in L^2(\Sigma(x^0))$, such as the solution $y = y(v)$ in $(1-3)$ checked $y(T, v) = y^{'}(T, v) = 0$.

The fact that the control v is defended $\Sigma(x^0)$ must be interpreted as meaning $y = v$ in $\Sigma(x^0), y = 0$ in $\Sigma_*(x^0)$.

For each pair of initial data we have $\{y^0, y^1\} \in L^2(\Omega) \times H^{-1}(\Omega)$.

The set of admissible controls
$U_{ad} = \left\{ v \in L^2(\Sigma(x^0)); y(T,v) = y^{'}(T,v) = 0 \mid_{\Omega} \right\}$,
contains an infinity of elements.

We will now show that the control given by HUM is that realizes the minimum of the cost function $J(v) = \frac{1}{2}\int_{\Sigma(x^0)} |v|^2 \, d\Sigma$,

on all admissible controls U_{ad} we will next characterize the control v using the optimality system.

2.2 Characterization of control

Theorem 1 *For each pair of initial data $\{y^0; y^1\} \in L^2(\Omega) \times H^{-1}(\Omega)$, the control $v \in L^2(\Sigma(x^0))$ given by HUM is the one that minimizes the cost function $J(v)$ on all admissible controls U_{ad}.*

2.2.1 First step

We consider the minimization problem

$$\inf J(v), v \in U_{ad} \tag{4}$$

the (4) problem is an optimal control problem with constraint.

Theorem 2 *By a penalization method we define the function*
$J_\epsilon(v,z) = \frac{1}{2}\int_{\Sigma(x^0)} |v|^2 \, d\Sigma + \frac{1}{2\varepsilon}\int_Q (z'' - \Delta z)^2 dxdt$,
with $\varepsilon > 0$,$v \in L^2(\Sigma(x^0))$ and $z = z(x,t)$ a function such as
$z'' - \Delta z \in L^2(Q)$,
$z(0) = y^0, z'(0) = y^1$ *in* Ω,

$$z\chi_{\Sigma(x^0)} = v, \tag{5}$$

$z = 0$ *in* $\Sigma_*(x^0)$, $z(T) = z'(T) = 0$ *in* Ω,
recess for each $v \in U_{ad}$ the function $y = y(v)$ of $(1-3)$ verifies these condition.
The term $\frac{1}{2\varepsilon}\int_Q (z'' - \Delta z)^2 dxdt$ is a penalty term.

We consider the optimal control problem

$$\inf J_\varepsilon(v,z), \tag{6}$$

for each $\varepsilon > 0$ there exist a unique solution $\{u_\varepsilon, z_\varepsilon\}$ of this problem, i.e. $J_\varepsilon(u_\varepsilon, z_\varepsilon) = \inf J_\varepsilon(v,z)$.

2.2.2 Second step

Note that the sequence $(u_\varepsilon)_{\varepsilon\rangle 0}$ is bounded in $L^2(\Sigma(x^0))$.

Let $v \in U_{ad}$ and $y = y(v)$ the solution of the problem $(1-3)$ associated. The couple $\{v, y(v)\}$ is admissible for the minimization problem (6) for every $\varepsilon > 0$ and so
$J_{_\varepsilon}(u_\varepsilon, z_\varepsilon) \leq J_\varepsilon(v, y(v))$.
But as $y(x)$ verifies
$y'' - \triangle y = 0$ in Q.
We see that
$J_\varepsilon(v, y(v)) = J(v)$,$\forall \epsilon > 0$,
so
$J_\varepsilon(u_\varepsilon, z_\varepsilon) \leq J(v), \forall \epsilon > 0$,
and this for each $v \in U_{ad}$, so we have
$J_\varepsilon(u_\varepsilon, z_\varepsilon) \leq \inf J(v), \forall \varepsilon > 0$.
Especially
$J(u_\varepsilon) \leq \inf J(v), \forall \epsilon > 0$.
And, if we put
$f_\epsilon = \frac{1}{\sqrt{\varepsilon}}(z''_\varepsilon - \triangle z_\varepsilon)$.
We have (f_ε) bounded in $L^2(Q)$.

2.2.3 Third step

Quite to extract subsequences we will have
$u_{\varepsilon\to 0}\ \widehat{v}$ in $L^2(\Sigma(x^0))$ weak.
We moreover
$\|z^{''}_\varepsilon - \triangle z_\varepsilon\|_{L^2(Q)} \leq C\sqrt{\varepsilon}, \forall \varepsilon > 0$.
We fined (z_ε) bounded in
$L^\infty(0,T,L^2(\Omega)) \cap W^{1.\infty}(0,T,H^{-1}(\Omega))$,
on particular

$$\|z_\varepsilon\|_{L^2}(Q) \leq C, \ \forall\varepsilon > 0, \tag{7}$$

and even it means extracting yet another sub-suite

$$z\varepsilon \to \widehat{y}, \varepsilon \to 0 \tag{8}$$

in $L^2(Q)$ weak.
According to (5) and (8) we have
$\widehat{y''} - \Delta\widehat{y} = 0$,
$\widehat{y} = \widehat{v}$ in $\Sigma(x^0), \widehat{y} = 0$ in $\Sigma_*(x^0)$,
$\widehat{y}(T) = \widehat{y'}(T) = 0$ in Ω,
$\widehat{y}(0) = y^0; \widehat{y'}(0) = y^1$ in Ω,
so we have
$J_\epsilon(u_\epsilon, z_\epsilon) \geq J(u_\epsilon), \widehat{v} \in U_{ad}$,
and after the week lower semi continuity of J we have
$J(\widehat{v}) \leq \liminf J(u_\varepsilon) \leq \liminf J_\varepsilon(u_\varepsilon, z_\varepsilon)$,
we conclude
$J(\widehat{v}) = \inf J(v)$.
We have also proved that
$\lim J(u_\varepsilon) = J(\widehat{v}), (\varepsilon \longrightarrow 0)$
which, combined with (7) gives
$u_\varepsilon \longrightarrow \widehat{v}$ in $L^2(\Sigma(x^0))$ (strong).

2.2.4 Fourth step

Consider the sequel
$P_\varepsilon = \frac{1}{\varepsilon}(z''_\varepsilon - \Delta z_\varepsilon), \forall\varepsilon > 0$,
obviously
$P_\varepsilon = \frac{1}{\varepsilon} f_\varepsilon, \forall\varepsilon > 0$,

we say that $(f_\varepsilon)_{\varepsilon>0}$ is bounded in $L^2(Q)$ but we do not yet have of estimate on $(P_\varepsilon)_{\varepsilon>0}$.

By writing the equation of Euler associated with the problem of minimization (6). We have

$$\int_{\Sigma(x^0)} u_\epsilon v d\Sigma - \int_Q p_\epsilon(\zeta'' - \Delta\zeta)dxdt = 0, \quad (9)$$

for all solution of
$\zeta'' - \Delta\zeta \in L^2(Q)$,
$\zeta(0) = \zeta'(0) = \zeta(T) = \zeta'(T) = 0$
$\zeta = v$ in $\Sigma(x^0), \zeta = 0$ in $\Sigma_*(x^0)$,
with $v \in L^2(\Sigma x^0))$.
By means of the Green formula we obtain
$p''_\epsilon - \Delta p_\epsilon = 0$ in Q,
$p_\epsilon = 0$ on Σ,
$\frac{\partial p_\epsilon}{\partial \nu} = u_\epsilon$ on $\Sigma(x^0)$,
in effect
$\int_Q p_\varepsilon(\zeta'' - \triangle\zeta)\, dxdt =$
$\int_Q (p''_\varepsilon - p_\varepsilon)\zeta dxdt$
$- \int_\Sigma p_\varepsilon \frac{\partial\zeta}{\partial v} d\Sigma + \int_{\Sigma(x^0)} \frac{\partial p_\varepsilon}{\partial v} v d\Sigma$,
and so, after (9)
$\int_{\Sigma(x^0)} u_\varepsilon v\, d\Sigma =$
$\int_Q (p''_\varepsilon - p_\varepsilon)\zeta dxdt$
$- \int_\Sigma p_\varepsilon \frac{\partial\zeta}{\partial v} d\Sigma + \int_{\Sigma(x^0)} \frac{\partial p_\varepsilon}{\partial v} v d\Sigma$,
which is equivalent.

2.2.5 Fifth step

According to the inverse inequality we obtain
$0.5 \times (T - 2R(x^0)) \left\{ |\nabla p_\varepsilon(0)|^2 + |p'_\varepsilon(0)|^2 \right\} \leq$
$0.5 \times R(x^0) \int_{\Sigma(x^0)} \left| \frac{\partial p_\varepsilon}{\partial v} \right|^2 d\Sigma = \frac{R(x^0)}{2} \int_{\Sigma(x^0)} |u_\varepsilon|^2 d\Sigma$,
the sequence $(u_\varepsilon)_{\varepsilon>0}$ being bounded in $L^2(\Sigma(x^0))$, we see that
$|\nabla p_\varepsilon(0)| + |p'_\varepsilon(0)| \leq 0, \forall \varepsilon > 0$,
and according to the law of conservation of energy ,we have
$p_\varepsilon \to p$ in $L^\infty((0, T, H^1_0(\Omega)) \cap W^{1.\infty}(0, T, L^2(\Omega))$,
$p_\varepsilon \to p$ on $L^\infty((0, T, H^1_0(\Omega))$ weak,
$p'_\varepsilon \to p'$ on $L^\infty((0, T, L^2(\Omega))$ weak,
$\left\{ p_\epsilon(0), p'_\varepsilon(0) \right\} \to \left\{ p(0), p'(0) \right\}$ on $H^1_0(\Omega) \times L^2(\Omega)$ weak.
So the function $p = p(x, t)$ solution of
$p'' - \triangle p = 0$ in Q,
$p = 0$ on Σ,
$\frac{\partial p}{\partial v} = \overrightarrow{v}$ on $\Sigma(x^0)$,
$p(0) = p\, ; p'(0) = p^1$ in Ω.

2.2.6 Sixth step

We pose $\Phi = p$; $\Phi^0 = p^0$; $\Phi^1 = p^1$ and $\psi = \widehat{y}$.
According to to (9) we have
$\Phi'' - \Delta\Phi = 0$ in Q,
$\Phi = 0$ on Σ,
$\Phi(0) = \Phi^0, \Phi'(0) = \Phi^1$ in Ω.
$\psi'' - \Delta\psi = 0$ in Q.
$\psi = \frac{\partial\psi}{\partial v}$ on $\Sigma(x^0), \psi = 0$ on $\Sigma_*(x^0)$.
On the other hand as
$\psi(0) = y^0$; $\psi'(0) = y^1$,
we have
$\Lambda \left\{ \Phi^0, \Phi^1 \right\} = \left\{ y^1, -y^0 \right\}$.
With Λ is the isomorphism between $H^1_0(\Omega) \times L^2(\Omega)$ and $H^{-1}(\Omega) \times L^2(\Omega)$ introduced in the application of HUM.
We thus see that the control $\overrightarrow{v}$ which by construction minimizes $J'(v)$ on U_{ad} is the control given by HUM since $\widehat{v} = \frac{\partial p}{\partial v} = \frac{\partial\Phi}{\partial v}$.

3 Standard optimal control of distributed system

In this chapter, we will study the optimal control of linear PDE's ,(the dimension of space of solution is infinite) We start by the presentation of the classical theory of the optimal control when we prove the existence, uniqueness and characterization of the optimum and we give some examples Then we study the optimal control for a linear system with incomplete data by present the notion of no-regret control [21], and associated with low-regret control which converges to the no-regret control, then we characterize them and we give example.

3.1 Position of problem

Let $\mathcal{Y}, \mathcal{U}$ and $\mathcal{Z}$ be infinite dimensional Hilbert spaces of states, controls and observation resp. $\mathcal{U}_{ad} \subset \mathcal{U}$ is a subset of admissible controls supposed non empty, closed and convex.

f is a source function in y . Consider the (10) well-posed abstract linear partial differential equation :

$$\mathcal{A}y = f + \mathcal{B}v. \quad (10)$$

Where $\mathcal{A} \in \mathcal{L}(\mathcal{Y})$ is a linear partial differential operator stationary or evolutionary (elliptic, parabolic and hyperbolic) makes an isomorphism on $\mathcal{Y}'$ identified to$\mathcal{Y}$, $\mathcal{B} \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ is the control operator.

Our optimal control problem consists in looking for a control function $u \in \mathcal{U}_{ad}$ which minimizes the following cost function

$$J(v) = \|\mathcal{C}y(v) - y_d\|^2_{\mathcal{Z}} + N \|v\|^2_{\mathcal{U}} \quad \forall v \in \mathcal{U}_{ad}, \quad (11)$$

J is convex function from $\mathcal{U}_{ad} \subset \mathcal{U}$ to $R\cup\{+\infty\}$, $\mathcal{C} \in \mathcal{L}(\mathcal{Y},\mathcal{Z})$: the observation operator and N is a symmetric definite positive operator bounded in $\mathcal{U}$.

y_d is the fixed observation in $\mathcal{Z}$.

we search u solution of : find

$$u \in \mathcal{U}_{ad} \tag{12}$$

such that $J(u) = \inf J(v), v \in \mathcal{U}_{ad}$

Theorem 3 *"Existence and uniqueness of optimal control" Let $\mathcal{U}_{ad} \subset \mathcal{U}$ closed and nonempty, J is lower semi continuous, bounded from below and coercive on $\mathcal{U}_{ad}$. Then there exists a minimize for J on $\mathcal{U}_{ad}$. Moreover, if J is strictly convex the minimize is unique.*

3.2 Optimal systems (Optimal control characterization)

We have by a first order optimality condition

$J'(u)(v-u) \geq 0\ \forall v \in \mathcal{U}_{ad}$,

J is Gateaux-differentiable function

$J'(u)(v-u) = \lim t^{-1}(J(u+t(v-u)) - J(u))$ for every $v \in \mathcal{U}_{ad}, t \longrightarrow 0$.

with a calculatation we fined

$J(u+t(v-u)) = J(u) + t^2 \|\mathcal{C}y(v-u)\|_z^2$
$+2t(\mathcal{C}y(u) - y_d, \mathcal{C}y(v-u))_{\mathcal{Z}}$
$+t^2N\|v-u\|_{\mathcal{U}}^2 + 2tN(u,v-u)_{\mathcal{U}}$,

which gives

$t^{-1}(J(u+t(v-u)) - J(u))$
$= t\|\mathcal{C}y(v-u)\|_{\mathcal{Z}}^2$
$+2(\mathcal{C}y(u) - y_d, \mathcal{C}y(v-u))_{\mathcal{Z}} + tN\|v-u\|_{\mathcal{U}}^2 + 2N(u,v-u)_{\mathcal{U}}$,

when $t \to 0$ we find

$J'(u)(v-u) = 2(\mathcal{C}^*(\mathcal{C}y(u) - y_d), y(v-u))_{\mathcal{Y}}$
$+2N(u,v-u)_{\mathcal{U}} \geq 0,\ \forall v \in \mathcal{U}_{ad}$.

Remark 4 *A condition of the (12) from $J'(u)(v-u)$ is called the variationel inequality.*

$\mathcal{C}^* \in \mathcal{L}(\mathcal{Z},\mathcal{Y})$ is the adjoint of $\mathcal{C}$,$\mathcal{A}^*$ is the adjoint operator of $\mathcal{A}$ and introduce the adjoint state $p = p(u)$ given by

$\mathcal{A}^*p(u) = \mathcal{C}^*(\mathcal{C}y(u) - y_d)$,

then

$(\mathcal{C}^*(\mathcal{C}y(u) - y_d), \delta y(v-u))_{\mathcal{Y}}$
$= (\mathcal{A}^*p(u), \delta y(v-u))_{\mathcal{Y}}$
$= (\mathcal{B}^*p(u), (v-u))_{\mathcal{U}}$.

Hence,

$J'(u)(v-u) = (\mathcal{B}^*p(u) + Nu, v-u)_{\mathcal{U}} \geq 0,\ \forall v \in \mathcal{U}_{ad}$.

The optimal control problem (10, 11, 12) has a unique solution u characterized by the following optimality system

$$\left\{\begin{array}{l} \mathcal{A}y(u) = f + \mathcal{B}u, \\ \mathcal{A}^*p(u) = \mathcal{C}^*(\mathcal{C}y(u) - y_d), \\ (\mathcal{B}^*p + Nu, v-u)_{\mathcal{U}} \geq 0, \forall v \in \mathcal{U}_{ad}. \end{array}\right. \tag{13}$$

The equation 1 and 2 from (13) must be associated to some appropriate boundary and initial condition.

We called the pair $(u, p(u))$ by the optimal pair.

Remark 5 *We have no constraints on control, by space structure of $\mathcal{U}$ (if $\mathcal{U}_{ad} = \mathcal{U}$) we deduce that we also have $J'(u)(v-u) \leq 0\ \ \forall v \in \mathcal{U}_{ad}$,*

and with the previous condition we get

$J'(u)(v-u) = 0\ \ \forall v \in \mathcal{U}$,

therefore the optimality system become as following

$\mathcal{A}y(u) = f + \mathcal{B}u$,
$\mathcal{A}^*p(u) = \mathcal{C}^*(\mathcal{C}y(u) - y_d)$,
$(\mathcal{B}^*p(u) + Nu, v-u)_{\mathcal{U}} = 0, \forall v \in \mathcal{U}$.

3.3 No-regret control and low-regret control to solve distributed system with missing data

In this section, we make an initiation to the theory of the optimal control of problems with incomplete data, where we introduce this leads to define the notion of no- regret control, low regret control, [22]. Moreover, we give existence, uniqueness , and prove that it converges to the no-regret control, then we characterize them via optimality systems and we give example.

3.3.1 Position of problem

We keep the same theorical framework as mentioned in the last paragraphed, the difference here is the presence of missing data. For this reason, we define a new operator $\beta \in \mathcal{L}(F,\mathcal{Y})$ where

F is a Hilbert space of uncertainties (missing data), G is a non-empty closed subspace of F.

For $f \in \mathcal{Y}$ the abstract equation related to the control $v \in \mathcal{U}_{ad}$ and the uncertainty $g \in G$ is given by

$$\mathcal{A}y(v,g) = f + \mathcal{B}v + \beta g. \tag{14}$$

The equation (14) is well posed in $\mathcal{Y}$ and has a unique solution $y(v,g)$, which associate to her the following cost function :

$$J(v,g) = \| \mathcal{C}y(v,g) - y_d \|_{\mathcal{Z}}^2 + N \| v \|_{\mathcal{U}}^2, \forall v \in \mathcal{U}_{ad}, \forall g \in G. \tag{15}$$

as usual, we are concerned with the optimal control of (14) and (15) is to search u solution of

$$\inf J(v,g), \forall g \in G, v \in \mathcal{U}_{ad}$$

when G is an infinite dimensional space the problem (14) has no sene, this problem is solved in, [23], they using many notion like no-regret control and pareto control, [24], there equivalents is proved in, [25]. We take

$\inf(g \in G\sup J(v,g)), v \in \mathcal{U}_{ad}$, but G is an infinite dimensional space we can get $g \in G \sup J(v,g) = +\infty$ and by the way the problem has no sense. So, to avoid this difficulty, we introduce the concept of "No-regret control".

Remark 6 *If $G = \{0\}$ then $J(v,g) = J(v,0)$. Therefore, the problem (14) becomes a standard optimal control problem :*
find $u \in \mathcal{U}_{ad}$ such that, $J(u) = \inf J(v), v \in \mathcal{U}_{ad}$.

To avoid difficulties arise when we get $\sup(v,g) = +\infty, g \in G$, we take only controls such that $\forall v \in \mathcal{U}_{ad}$:
$J(v,g) \leq J(0,g), \forall g \in G$
$J(v,g) - J(0,g) \leq 0, \forall g \in G$.
Thus, we can say that $\sup(J(v,g) - J(0,g)), g \in G$ exists.

3.4 No-regret control

Definition 7 *We say that $u \in \mathcal{U}_{ad}$ is a no-regret control for (14) and (15) if u solves*

$$\inf(\sup(J(v,g) - J(0,g))), v \in \mathcal{U}_{ad}, g \in G. \tag{16}$$

Remark 8 *of course , the next problem is defied only for controls such that*
$\sup(J(v,g) - J(0,g)) < \infty, g \in G$.

Lemma 9 *For every $u \in \mathcal{U}_{ad}$ and $g \in G$ we have :*

$$J(v,g) - J(0,g) = J(v.0) - J(0,0) + 2(S(v), g)_{G',G}, \tag{17}$$

where $S(v) = \beta^* \xi(v)$ and $\xi(v)$ defined for $v \in \mathcal{U}_{ad}$ by
$\mathcal{A}^* \xi(v) = \mathcal{C}^* \mathcal{C}(y(v,0) - y(0,0))$.

$\mathcal{A}$ is a linear operator in $\mathcal{Y}$, so :
$y(v.g) = y(v,0) + y(0,g) - y(0,0)$,
$y(0,g) = y(0,0) + y(0,g) - y(0,0)$,
with $y(v,0)$ and $y(0,g)$ are a solution of (14) when $g = 0$ and $v = 0$ resp.
By the definition of $J(v,g)$ one obtain
$J(v,g) = J(v,0) + \|\mathcal{C}(y(0,g) - y(0,0)\|^2_{\mathcal{Z}}$
$+2(\mathcal{C}y(v,0) - y_d, \mathcal{C}(y(0,g) - y(0,0)))_{\mathcal{Z}}$,
and
$J(0,g) = J(0,0) + \|\mathcal{C}(y(0,g) - y(0,0))\|^2_{\mathcal{Z}}$
$+2(\mathcal{C}(y(0,0) - y_d, \mathcal{C}(y(0,g) - y(0,0)))_{\mathcal{Z}}$,
then
$J(v,g) - J(0,g) = J(v,0) - J(0,0)$
$+2(\mathcal{C}^*\mathcal{C}(y(v,0) - y(0,0)), y(0,g) - y(0,0))_{\mathcal{Y}}$.
Introduce an adjoint state $\xi(v)$ given by $\mathcal{A}^*\xi(v) = \mathcal{C}^*\mathcal{C}(y(v,0) - y(0,0))$ to write

$$J(v,g) - J(0,g) = J(v,0) - J(0,0) + 2(S(v), g)_{G',G} \tag{18}$$

where $S(v) = \beta^*\xi(v)$, the last equation leads to (18).

Remark 10 *1. By (18) you can see that condition (17) holds iff $v \in k$, where*
$K = \{v \in \mathcal{U}_{ad}, (S(v), g) = 0 \forall g \in G\}$,
is a closed subspace of $\mathcal{U}$. Then, u is a no-regret control iff $u \in k$.
2. The notion of no-regret control could be generalized to no-regret control related to any a fixed control $u_0 \in U_{ad}$, i.e , we want controls v s.t
$J(v,g) \leq J(u_{0,}g) \quad \forall \; g \in G$

Definition 11 *we say that $u \in U_{ad}$ is a no-regret control related to $u \in U_{ad}$ for (14)-(15) if u solve*
$\inf \sup(J(v,g) - J(u_{0,}g)$.

Unfortunately, the main difficulty with no-regret control arises when we want to characterize the set k, for this reason we shall approximate the no-regret control by a sequence of controls called low regret controls

3.4.1 Characterization of the no-regret control

The optimality system of no-regret control is given by :
$\mathcal{A}y = f + Bu$,
$\mathcal{A}^*\zeta = \mathcal{C}^*\mathcal{C}y(u,0) - y_d$,
$\mathcal{A}\rho = \beta\lambda, \lambda \in G$,
$\mathcal{A}^*p = \mathcal{C}^*(\mathcal{C}y(u,0) - y_d) + \mathcal{C}^*\mathcal{C}\rho$,
$(\mathcal{B}^*p + Nu, v - u)_{\mathcal{U}} \geq 0 \;\; \forall v \in \mathcal{U}_{ad}$.
where $y(u,0) = y, \xi(u) = \xi$.

3.5 The low-regret control

One through to relax (16) by making some quadratic perturbation on $J(0,g)$, in other words, we search controls v such that
$J(v,g) \leq J(0,g) + \gamma \|g\|^2_G, \gamma > 0, g \in G$.

Definition 12 *We say that $u_\gamma \in U_{ad}$ is a low -regret control for (14)-(15) if u solves*
$\inf \sup(J(v,g) - J(0,g) - \gamma \|g\|^2_G), \gamma > 0, v \in \mathcal{U}_{ad}, g \in G$.

So we have the equivalence
$\inf(J(v,0) - J(0,0) +$
$\sup(2(S(v),g)_G - \gamma \|g\|^2_G))$,
$v \in \mathcal{U}_{ad}, g \in G$.
Legendre transform, for
$\sup(2(S(v),g)_G - \gamma \|g\|^2_G) = \frac{1}{\gamma}\|S(v)\|^2_G, g \in G$.
then
$\inf J'(v), v \in \mathcal{U}_{ad}$
where
$J'(v) = J(v,0) - J(0,0) + \frac{1}{\gamma}\|S(v)\|^2_G$.
Now, we can define the low-regret by

Definition 13 *We say that $u_\gamma \in \mathcal{U}_{ad}$ is a low-regret control for (14) and (15) if u solves*
$\inf \sup(J(v,g) - J(0,g) - \gamma \|g\|^2_G, \gamma > 0)$,
$v \in \mathcal{U}_{ad}, g \in G$.

Theorem 14 *"Low-regret control: existence and uniqueness" The problem* (14) *and* (17) *with* (18) *has a unique solution* u_γ.

Theorem 15 *The unique low-regret control* u_γ *is converge weakly when* $\gamma \to 0$ *to the unique no-regret control* u *in* $\mathcal{U}_{ad}$.

Let u_γ be a low-regret control in $\mathcal{U}_{ad}$ then for all $v \in \mathcal{U}_{ad}$
$J'(u_\gamma) \leq J^\gamma(v)$,
$J(u_\gamma, 0) - J(0,0) + \frac{1}{\gamma}\|\beta^*\zeta(u_\gamma)\|_G^2$
$\leq J(v,0) - J(0,0) + \frac{1}{\gamma}\|\beta^*\zeta(v)\|_G^2, \forall v \in \mathcal{U}_{ad}$,
by implies
$J(u_\gamma, 0) + \frac{1}{\gamma}\|\beta^*\zeta(u_\gamma)\|_G^2$
$\leq J(v,0) + \frac{1}{\gamma}\|\beta^*\zeta(v)\|_G^2, \forall v \in \mathcal{U}_{ad}$,
we choose $v = 0$ to find
$J(u_\gamma, 0) + \frac{1}{\gamma}\|\beta^*\zeta(u_\gamma)\|_G^2 = \text{constant}$,
then
$\|u_\gamma\|_{\mathcal{U}} \leq C$,
$\|\mathcal{C}y(u_\gamma, 0)\|_{\mathcal{Z}} \leq C$,
$\|\beta^*\zeta(u_\gamma)\|_G \leq \sqrt{\gamma}C$,
where C is a constant independent of γ.

(u_γ) is bounded in $\mathcal{U}_{ad}$ then we can extract a subsequence still be denoting (u_γ) converges weakly to $u \in \mathcal{U}_{ad}$.

It's clear that for every $v \in \mathcal{U}_{ad}$
$J(v,g) - J(0,g) - \gamma\|g\|_G^2 \leq$
$J(v,g) - J(0,g), \forall g \in G$,
i.e,
$J(v,g) - J(0,g) - \gamma\|g\|_G^2 \leq$
$\sup(J(v,g) - J(0,g)), \forall g \in G$,
from another side we have
$J(u_\gamma, g) - J(0,g) - \gamma\|g\|_G^2$
$\leq J(v,g) - J(0,g) - \gamma\|g\|_G^2$,
so
$J(u_\gamma, g) - J(0,g) - \gamma\|g\|_G^2$
$\leq \sup(J(v,g) - J(0,g)), \forall g \in G$,
when γ tend to 0 we obtain
$J(u,g) - J(0,g) \leq$
$\sup(J(v,g) - J(0,g)), \forall g \in G$,
which means that
$\sup(J(u,g) - J(0,g))$
$= \inf\{\sup(J(v,g) - J(0,g))\}$.
In conclusion, u is a no-regret control.

3.5.1 Characterization of the low-regret control

By a first order optimality condition we have
$J'(u_\gamma)(v - u_\gamma) \geq 0, \forall v \in \mathcal{U}_{ad}$,
where
$J'(u_\gamma)(v - u_\gamma) =$
$\lim h^{-1}(J(u_\gamma + h(v - u_\gamma)) - J(u_\gamma)), \forall v \in \mathcal{U}_{ad}$,
we have
$h^{-1}(J(u_\gamma + t(v - u_\gamma)) - J(u_\gamma)) =$
$h\|\mathcal{C}y(v - u_\gamma, 0)\|_{\mathcal{Z}}^2 + hN\|v - u_\gamma\|_{\mathcal{U}}^2$
$+\frac{h}{\gamma}\|S(v - u_\gamma)\|_G^2 + 2(\mathcal{C}y(u_\gamma, 0) - y_d, \mathcal{C}y(v - u_\gamma, 0))_{\mathcal{Z}}$
$+2N(u_\gamma, v - u_\gamma)_{\mathcal{U}} + \frac{2}{\gamma}(S(u_\gamma), S(v - u_\gamma))_G$,
when $h \to 0$ we find
$J'(u_\gamma)(v - u_\gamma) = 2(\mathcal{C}y(u_\gamma, 0) - y_d, \mathcal{C}y(v - u_\gamma, 0))_{\mathcal{Z}}$
$+2N(u_\gamma, v - u_\gamma)_{\mathcal{U}} + \frac{2}{\gamma}(S(u_\gamma), S(v - u_\gamma))_G$.
By linearity of the operator C in Z we have
$J^{\gamma\prime}(u_\gamma)(v - u_\gamma) =$
$2(\mathcal{C}^*(\mathcal{C}y(u_\gamma, 0) - y_d), y(v,0) - y(u_\gamma, 0))_{\mathcal{Y}}$
$+2N(u_\gamma, v - u_\gamma)_{\mathcal{U}} + \frac{2}{\gamma}(S(u_\gamma), S(v - u_\gamma))_G$,
$y(v,0) - y(u_\gamma, 0) = y(v - u_\gamma, 0) - y(0,0)$,
then
$J'(u_\gamma)(v - u_\gamma) = 2(\mathcal{C}^*(\mathcal{C}y(u_\gamma, 0) - y_d), y(v - u_\gamma, 0) - y(0,0))_{\mathcal{Y}}$
$+2N(u_\gamma, v - u_\gamma)_{\mathcal{U}} + \frac{2}{\gamma}(S(u_\gamma), S(v - u_\gamma))_G$.
The adjoint state
$\mathcal{A}^*\xi(u_\gamma) = \mathcal{C}^*\mathcal{C}(y(u_\gamma, 0) - y(0,0))$, then
$(S(u_\gamma), S(v - u_\gamma))_G = (\beta\beta^*\xi(u_\gamma), \xi(v - u_\gamma))_{\mathcal{Y}}$.
Introduce the state $\rho_\gamma = \rho(u_\gamma)$ by
$\mathcal{A}\rho_\gamma = \frac{1}{\gamma}\beta\beta^*\xi(u_\gamma)$,
this leads to the following equality
$(\mathcal{A}\rho_\gamma, \xi(v - u_\gamma))_{\mathcal{Y}} = (\mathcal{C}^*\mathcal{C}\rho_\gamma, y(v - u_\gamma, 0) - y(0,0))_{\mathcal{Y}}$,
introducing the new adjoint state $p_\gamma = p(u_\gamma)$ by
$\mathcal{A}^*p_\gamma = \mathcal{C}^*(\mathcal{C}y_\gamma - y_d) + \mathcal{C}^*\mathcal{C}\rho_\gamma$,
to find
$(\mathcal{A}^*p_\gamma, y(v - u_\gamma, 0) - y(0,0))_{\mathcal{Y}} = (\mathcal{B}^*p_\gamma, v - u_\gamma)_{\mathcal{U}}$.
Hence, the optimality condition is given by
$J'(u_\gamma)(v - u_\gamma) = (\mathcal{B}^*p_\gamma + Nu_\gamma, v - u_\gamma)_{\mathcal{U}} \geq 0, \forall v \in \mathcal{U}_{ad}$.

Finally, the low-regret control is characterized by the following optimality system :
$\mathcal{A}y_\gamma = f + Bu_\gamma$,
$\mathcal{A}^*\xi_\gamma = \mathcal{C}^*\mathcal{C}(y_\gamma - y(0,0))$,
$\mathcal{A}\rho_\gamma = \frac{1}{\gamma}\beta\beta^*\xi_\gamma$,
$\mathcal{A}^*p_\gamma = \mathcal{C}^*(\mathcal{C}y_\gamma - y_d) + \mathcal{C}^*\mathcal{C}\rho_\gamma$,
$(\mathcal{B}^*p_\gamma + Nu_\gamma, v - u_\gamma)_{\mathcal{U}} \geq 0, \forall v \in \mathcal{U}_{ad}$,
where
$y(u_\gamma, 0) = y_\gamma, \xi(u_\gamma) = \xi_\gamma$.

4 Comparison between the controls calculated through HUM and the low regrets method

After the comprehensive and in-depth study of the two methods, we can draw the following comparison between the HUM method and the low-regret method. The HUM method has advantages represented in If

$U_{ad} = H$ the control is identifiable with the conjoint state of systems for systems satisfying the Mizohata hypotheses and the disadvantages represented in firstly if U_{ad} is empty this method does not work, secondly if U_{ad} is not empty (Slater) the method gives a duality between the control and the conjoint state of the systems. The advantages of the low-regret method ensure control existence even in the empty U_{ad} case and it gives characterization equations for singular systems, and the disadvantages that are not constrictive.

5 Conclusion

Generally, we conclude that the HUM method is used for the regulars systems, and the no-regret method is used for the singulars systems.

When $U_{ad} = H$ or interior non-empty (slater), we can use the HUM method.

Acknowledgements

The authors thank the referees for their careful reading and their precious comments. Their help is much appreciated.

References:

[1] E.H. Zerrik, A. Afifi, A. EL Jai. Systèmes dynamiques, Analyse régionale des systèmes linéaires distribués. Tome 2, Presses Universitaires de Perpignoom, 2008.

[2] F. Achab and I. Rezzoug. Average no regret sentinel. Innovative Journal of Mathematics. v. 1 no. 3, 1-15, (2022). doi:10.55059/ijm.2022.1.3/25

[3] A. Berhail and I. Rezzoug. Identification of the source term in Navier-Stokes system with incomplete data. AIMS Mathematics, 4(3): 516–526. DOI:10.3934/math.2019.3.516. 2019.

[4] T. Laib. Instantaneous sentinel for the identification of the pollution terme in Navier Stockes systems. Proyecciones. V40. N6. PP. 1489-1505. 2021.

[5] S.M. Kallab Debbih. Approximate solution for the Heat equation with delay time and missing data. 2021.

[6] N.L. Merabti and I. Rezzoug. Sentinel method and distributed systems with missing data. Innovative Journal of Mathematics. v. 1 no. 2, 44-62, (2022). doi:10.55059/ijm.2022.1.2/26

[7] I. Rezzoug and T.E. Oussaeif. Approximate Controllability. Wseas Transcations on Systems. Volume 19, (2020). DOI: 10.37394/23202.2020.19.3

[8] I. Rezzoug et al. Pollution detection for the singular linear parabolic equation. J. Appl. Math.& Informatics Vol 41(3), 647-656. 2023.

[9] I. Rezzoug et al. Regularization of a class of ill-posed Cauchy problems associated with generators of analytie semi-groups. PJM. V11, N1. PP. 485-495. 2022

[10] I. Rezzoug and T.E. Oussaeif. Solvability of a solution and controllability of partial fractional differential systems. JIM. V24. N5. PP 1175-1200. 2021.

[11] I. Rezzoug. Étude théorique et numérique des problèmes d'identification des systèmes gouvernes par des équations aux dérivées partielles. Thèse de Doctorat. Université oum el Boughi. 2014.

[12] I. Rezzoug. Identification d'une partie de la frontière inconnue d'une membrane. Thèse de Magister. Université oum el Boughi. 2009.

[13] M.B. Iqbal et I. Rezzoug. Pollition detection for the singular linear parabolic equation. J. Appl. Math. & Informatics Vol. 41, No. 3, pp. 647 - 656. 2023. https://doi.org/10.14317/jami.2023.647

[14] A. Benbrahim et I. Rezzoug. A revision of M. Asch notion of Discrete control. Wseas Transcations on Applied and Theorical Mechanics. Volume 17, (2022). DOI: 10.37394/232011.2022.17.29

[15] J. P. Raymond. Optimal control of partial differential equations. Université Paul Sabatier, 2013.

[16] L. J. Savage. The foundations of statistics. Courier Corporation. 1972.

[17] E. Trélat. Contrôle optimal : Théorie et applications. Université Pierre et Marie Curie (paris 6). 2013

[18] A. Ayadi and A. Hafdallah. Averaged optimal control for some evolution problems with missing parameter. Technical Report. DOI:10.13140/RG.2.1.3002.8563. 2016.

[19] J.L. Lions. Optimal control of systems governed by partial differential equations problèmes aux limites. 1971.

[20] J. Lohéac and E. Zuazua. Averaged controllability of parameter dependent conservative semigroups. Journal of Differential Equations, 262(3), 1540-1574. 2017.

[21] E. Zuazua. Averaged control. Automatica, 50(12), pp 3077-3087. 2014.

[22] J.L. Lions. Contrôle à moindres regrets des systèmes distribués. Comptes rendus de l'Académie des sciences. Série 1, Mathématique, 315(12), 1253-1257. 1992.

[23] J.L. Lions, Controllability, Penalty and stiffness, Anal, Nor, Serie 4, Tome 25, N3-4. PP. 547-610, 1997.

[24] J.L. Lions. Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Vol. 1 et 2, Masson, RMA, paris, 1988.

[25] O. Nakoulima and A. Omrane and J. Velin. On the pareto control and no-regret control for distributed systems with incomplete data. SIAM journal on control and optimization, 42(4), 1167-1184. 2003.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The authors have no conflicts of interest to declare.