Reducing the size of a waiting line optimally

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Abstract: The problem of reducing the number of customers waiting for service in a modified $M/G/k$ queueing model is considered. We assume that the optimizer can decide how many servers are working at any time instant. The optimization problem ends as soon as the objective has been achieved or a time limit has been reached. Cases when dynamic programming can be used to determine the optimal control even if the service time is not exponentially distributed are presented.

Key-Words: $M/G/k$ model, first-passage time, homing problem, dynamic programming.

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1 Introduction

Assume that customers arrive in a queueing system according to a Poisson process with rate λ and are served one at a time. Then, the time A between two customers has an exponential distribution with parameter λ. If there are k servers and if the service time S has an exponential distribution with parameter μ for any server, then the queueing system is denoted by $M/M/k$. It is also assumed that the service times are independent random variables, and are also independent of the times between successive customers. The system capacity c can be finite or infinite. Finally, the service policy is First in First out (FIFO).

When the queueing system is in equilibrium, or steady state, customers leave according to a Poisson process. As this is a *Markovian* stochastic process, the notation M is used to denote the model.

When the service time S has a general distribution, the notation becomes $M/G/k$. Important particular cases are when S is a constant, when it is uniformly distributed, and when it has a gamma distribution.

Next, suppose that at the initial time t_0 the total number $X(t_0)$ of customers in the system is equal to $k+l$, where $l \geq 1$. Moreover, they are all waiting for service. The time t_0 could be the opening time of a store.

In, [1], the following stochastic optimal control problem was considered: assuming that the optimizer can choose the number $n(t)$ of servers who are working at any time instant t, find how many servers must be used in order to minimize the expected value of the cost (or reward) function

$$J(t_0, l) := \int_{t_0}^{T(t_0,l)} \{q_0 n(t) - m[n(t)]\} \, \mathrm{d}t. \quad (1)$$

In the cost function, q_0 is a positive constant, $m[n(t)]$ is the money earned by the system, per unit time, when $n(t) \in \{1, \ldots, k\}$ servers are working, and $T(t_0, l)$ is a *first-passage time* defined by

$$T(t_0, l) = \inf\{t > t_0 : X(t) = k + r \text{ or } t = t_1\}, \quad (2)$$

where $0 \leq r < l$ and $t_1 > t_0$.

The above problem, which is an extended *homing problem*, was treated in, [1], in the case of the (modified) $M/M/k/c$ queueing system.

Homing problems, in which a stochastic process is controlled until a given event occurs, were studied for n-dimensional diffusion processes in, [2]. The author also considered the case when the cost criterion takes the risk-sensitivity of the optimizer into account in, [3].

The author has written many papers on homing problems; see, for instance, [4]. See also, [5], [6], and, [7].

In, [1], the authors used dynamic programming to prove the following proposition, in which the function $F(t_0, l)$ is defined by

$$F(t_0, l) := \inf_{\substack{n(t) \\ t \in [t_0, T)}} E[J(t_0, l)]. \quad (3)$$

The function $F(t_0, l)$ (called the *value function*) is such that it satisfies the condition $F(t_0, l) = 0$ if

$l \leq r$. Moreover, the authors assumed that a cost F_1 is incurred if the objective has not been achieved by time t_1:

$$F(t_0, l) = F_1 > 0 \quad \text{if}\, T(t_0, l) = t_1. \tag{4}$$

Proposition 1.1. *Let* $\eta = n(t_0)$. *The value function* $F(t_0, l)$ *satisfies the* dynamic programming equation

$$-\frac{\partial}{\partial t_0} F(t_0, l) = \inf_{\eta} \{q_0 \eta - m(\eta) + F(t_0, l+1)\,\lambda + F(t_0, l-1)\,\eta\mu - F(t_0, l)\,(\lambda + \eta\mu)\}\,. \tag{5}$$

In the current paper, we suppose that the service time S is *not* exponentially distributed. In, [8], the author assumed that the random variable S is actually a constant. Here, in particular, the case when S has a uniform distribution will be considered.

Optimal control of queuing systems has been the subject of numerous research papers. See, for example, [9], [10], and, [11], for very recent ones. Here, contrary to other articles found in the literature, the final time is a random variable.

2 Non-exponential service times

Let W be an exponentially distributed random variable with parameter α. We have

$$P[W \leq t] = 1 - e^{-\alpha t} = \alpha t + o(t). \tag{6}$$

Moreover, if $W_1, \ldots, W_j$ are independent random variables distributed as W, then, as is well known, $\min\{W_1, \ldots, W_j\}$ has an exponential distribution with parameter $j\,\alpha$.

It follows that in the case of the modified $M/M/k/c$ model considered in, [1], we can write that

$$P[A \leq \Delta t] = \lambda \Delta t + o(\Delta t) \tag{7}$$

and

$$P[D \leq \Delta t] = \eta\mu\Delta t + o(\Delta t), \tag{8}$$

where A is the time taken for a new customer to arrive in the system after time t_0 and D is the time taken for a customer to leave the system if there are η servers working at time t_0.

Both Eq. (7) and Eq. (8) are needed to derive the dynamic programming equation (5). Unfortunately, in the case of an $M/G/k$ queueing system, the equation

$$P[D \leq \Delta t] = \kappa \Delta t + o(\Delta t), \tag{9}$$

where κ is a positive constant, is generally not valid.

Assuming that the service time is exponentially distributed is a *simplifying assumption*. Indeed, this hypothesis is often made because then it is not necessary to consider what happened since the initial time. One only needs to observe the state of the system at a given time instant to determine the future.

In practice, the service time S cannot be exactly exponentially distributed. More realistic cases include the ones when S is deterministic, or uniformly distributed, etc.

In this section, we will present three distributions for which Eq. (9) *does* hold. Let $S_1, \ldots, S_j$ be independent random variables distributed as S and $M := \min\{S_1, \ldots, S_j\}$.

Case I. Assume first that the service time S is uniformly distributed on the interval $(0, L)$. We have

$$P[M > t] = \left(\frac{L - t}{L}\right)^j \quad \text{for } t \in (0, L). \tag{10}$$

If follows that

$$\begin{aligned} P[M \leq \Delta t] &= 1 - \left(\frac{L - \Delta t}{L}\right)^j \\ &= 1 - \frac{1}{L^j}\,(L - \Delta t)^j \\ &= 1 - \frac{1}{L^j}\,\left[L^j - j\,L^{j-1}\,\Delta t + o(\Delta t)\right] \\ &= \frac{1}{L}\,j\,\Delta t + o(\Delta t), \end{aligned} \tag{11}$$

as required.

Case II. Assume next that the probability density function of the service time is given by

$$f_S(s) = \frac{1}{\ln(1+L)}\,\frac{1}{1+s} \quad \text{for } s \in (0, L). \tag{12}$$

We calculate

$$F_S(s) = \frac{\ln(1+s)}{\ln(1+L)} \quad \text{for } s \in (0, L). \tag{13}$$

Hence, making use of the formula

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots \quad \text{for } |x| < 1, \tag{14}$$

we can write that

$$\begin{aligned} P[M \leq \Delta t] &= 1 - \left[1 - \frac{\ln(1+\Delta t)}{\ln(1+L)}\right]^j \\ &= 1 - \left[1 - \frac{1}{\ln(1+L)}\,[\Delta t + o(\Delta t)]\right]^j \\ &= 1 - \left[1 - \frac{j}{\ln(1+L)}\,\Delta t + o(\Delta t)\right] \\ &= \frac{1}{\ln(1+L)}\,j\,\Delta t + o(\Delta t). \end{aligned} \tag{15}$$

Case III. Finally, we suppose that the probability density function of S is

$$f_S(s) = \frac{\cos(s)}{\sin(L)} \quad \text{for } s \in (0, L), \tag{16}$$

where $L \leq \pi/2$. This time, we use the formula

$$\sin(x) = x - \frac{x^3}{3!} + \dots \quad \text{for } x \in \mathbb{R} \tag{17}$$

to write that

$$\begin{aligned} P[M \leq \Delta t] &= 1 - \left[1 - \frac{\sin(\Delta t)}{\sin(L)}\right]^j \\ &= 1 - \left[1 - \frac{1}{\sin(L)}\left[\Delta t + o(\Delta t)\right]\right]^j \\ &= 1 - \left[1 - \frac{j}{\sin(L)}\,\Delta t + o(\Delta t)\right] \\ &= \frac{1}{\sin(L)}\, j\,\Delta t + o(\Delta t). \end{aligned} \tag{18}$$

In the next section, we will present a case when Eq. (9) does not hold, but the problem can be transformed into the one considered in, [1].

3 Gamma distributed inter-arrival and service times

As mentioned in the previous section, in practice, the service time S cannot be exactly exponentially distributed. Similarly, the time A between arrivals cannot exactly follow an exponential distribution either. A distribution that generalizes the exponential distribution and is more more likely to be a good (approximate) model for S is the gamma distribution. Indeed, because the gamma distribution has a *shape parameter*, there are many real-life situations for which S could be well approximated by a gamma distribution.

Let X be a random variable having a gamma distribution with shape parameter α and inverse scale parameter β, which will be denoted by $\mathrm{G}(\alpha, \beta)$. Assume that $\alpha = 2$ and $\beta = 1$, so that

$$f_X(x) = x\,e^{-x} \quad \text{for } x > 0. \tag{19}$$

We have

$$F_X(x) = 1 - (x+1)\,e^{-x} \quad \text{for } x > 0. \tag{20}$$

Therefore,

$$\begin{aligned} P[X \leq \Delta x] &= 1 - (\Delta x + 1)\left[1 - \Delta x + o(\Delta x)\right] \\ &= o(\Delta x). \end{aligned} \tag{21}$$

Remark. The above result is also valid for a random variable Y having a $\mathrm{G}(2, \mu)$ distribution, for which

$$f_Y(y) = \mu^2\, y\, e^{-\mu y} \quad \text{for } y > 0. \tag{22}$$

Moreover, if $M := \min\{X_1, \dots, X_j\}$, where $X_n \sim \mathrm{G}(2, \mu)$ for $n = 1, \dots, j$ and $X_1, \dots, X_j$ are independent, then we find that

$$P[M \leq \Delta t] = o(\Delta t). \tag{23}$$

Assume next, for the sake of simplicity, that both the inter-arrival time A and the service time S have a $\mathrm{G}(2, 1)$ distribution. Let us define $Z = X^2$. We calculate

$$f_Z(z) = \frac{1}{2}\, e^{-\sqrt{z}} \quad \text{for } z > 0. \tag{24}$$

It follows that

$$\begin{aligned} P[Z \leq \Delta z] &= 1 - \left(\sqrt{\Delta z} + 1\right) e^{-\sqrt{\Delta z}} \\ &= \frac{1}{2}\Delta z + o(\Delta z). \end{aligned} \tag{25}$$

Similarly, proceeding as in the previous section, we find that

$$P[\min\{Z_1, \dots, Z_j\} \leq \Delta t] = \frac{j}{2}\Delta t + o(\Delta t), \tag{26}$$

where the random variables $Z_1, \dots, Z_j$ are independent and distributed as Z.

Now, in order to obtain the dynamic programming equation (5), we must also have

$$\int_{t_0}^{t_0+\Delta t} \{q_0\, n(t) - m[n(t)]\}\, \mathrm{d}t = \gamma\, \Delta t + o(\Delta t), \tag{27}$$

where γ is a constant.

Suppose that the cost function $J(t_0, l)$ defined in Eq. (1) is replaced by

$$C(t_0, l) := \int_{t_0}^{T(t_0,l)} \{q_0\, n(t) - m[n(t)]\}\, t\, \mathrm{d}t. \tag{28}$$

If we make the change of variable $z = t^2$, we obtain that

$$\begin{aligned} &\int_{t_0}^{t_0+\Delta t} \{q_0\, n(t) - m[n(t)]\}\, t\, \mathrm{d}t \\ &= \int_{t_0^2}^{(t_0+\Delta t)^2} \left\{q_0\, n(\sqrt{z}) - m[n(\sqrt{z})]\right\} \frac{1}{2}\, \mathrm{d}z \\ &\approx \{q_0\, n(t_0) - m[n(t_0)]\}\, \frac{1}{2}\left[(\Delta t)^2 + 2 t_0\, \Delta t\right] \\ &= \{q_0\, n(t_0) - m[n(t_0)]\}\, t_0\, \Delta t + o(\Delta t). \end{aligned} \tag{29}$$

From what precedes, we can state that if A and S have a $\mathrm{G}(2, 1)$ distribution and the cost function is the one defined in Eq. (28), then by making the change of variable $z = t^2$, the optimal control problem considered in this note (with $t_0 > 0$) is transformed into a problem for which the dynamic programming equation that corresponds to Eq. (5) can be derived.

4 Conclusion

In this note, the problem of reducing the size of a waiting line as quickly as possible, taking into account control costs, has been extended to the case where the inter-arrival and service times are not exponentially distributed.

In Section 2, we gave three probability density functions for which Eq. (9) is valid. This equation is necessary for the derivation of the dynamic programming equation in Proposition 1.1.

Then, in Section 3, we presented a problem that, although Eqs. (7) and (9) do not hold, we were able to transform into a problem for which it is possible to use dynamic programming to obtain the optimal solution.

Once the dynamic programming equation has been derived, we need to solve difference equations in order to determine the optimal control.

Finally, it would be interesting to try to solve problems for which only one of Eqs. (7) and (9 holds. As we saw in Section 3, we must also define the cost function appropriately. Moreover, in general one cannot use dynamic programming to obtain the optimal control.

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