Controllability of leader-following multi-agent systems

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Abstract: In this work, the controllability of a class of multi-agent linear systems that are interconnected via communication channels is studied. Condition for controllability have been presented and described in terms of the topology of the followers agents, in the case where the followers agents have the same linear dynamics.

Key–Words: Multiagent linear systems, graph, leaser-following, controllability.

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1 Introduction

A multi-agent system is a system made up of several agents that interact with each other where the dynamics of each agent and the leader is a linear system. Multi-agent systems can be used to solve problems that are difficult or impossible to solve in a single agent. Recently, the study of multi-agent systems has attracted the attention of many researchers because this class of systems appears in various areas of knowledge, such as the cooperative control of unmanned aerial vehicles, the consensus problem of communication networks, the training control of mobile robots, neural networks modeling the brain structure, Etc., [1], [2], [3], [4],[5], [6], [7].

An exciting topic is the study of a group of agents with a leader, where the leader is a special agent whose movement is affected by that of all the others, but it does influence the rest of the agents, which is why we speak of a leading agent who is followed by all the others, [8]. In this sense, [9] examines the stability of leader-follower multi-agent systems with general linear dynamics and switching network topologies.

It is known that the human brain can be interpreted mathematically as a multi-agent linear dynamic system that moves through various cognitive regions, promoting more or less complicated behaviors. The dynamics of the cerebral neuronal system play a considerable role in cognitive function and are, therefore, of interest in the attempt to understand the processes and evolution of possible disorders. The controllability of a system refers to the possibility of manipulating its components to drive the system along a desired trajectory: a set of states culminating in a target state chosen for its functional utility (leader). The study of system controllability could be a mechanism of cognitive control in critical locations within the anatomical system acting as drivers that move the system (brain) towards specific modes of action (cognitive functions), [3], [10].

In this paper, the leader-following multi-agent systems are considered. In [2], the author analyzes the case of multi-agent systems without a leader where the significant difference in the study is that in this case the graph considered is undirected and connected, assuming, therefore, the symmetry of the Laplacian matrix property that, in general, is not fulfilled in the case of leader-following multi-agent systems.

2 Preliminaries

Let us consider a group of k agents. The following linear dynamical systems give the dynamic of each agent

$$\left.\begin{array}{ll} \dot{x}^0(t) & = A_0x^0(t) + C_0v^0(t) \\ \dot{x}^1(t) & = A_1x^1(t) + B_1u^1(t) + C_1v^1(t) \\ \vdots & \\ \dot{x}^k(t) & = A_kx^k(t) + B_ku^k(t) + C_1v^k(t) \end{array}\right\} \quad (1)$$

$A_i \in M_n(\mathbb{C})$, $B_i \in M_{n\times m}(\mathbb{C})$, $C_i \in M_{n\times p}(\mathbb{C})$, $x^i(t) \in \mathbb{C}^n$, $u^i(t) = f^i(x^1(t),\dots,x^k(t)) \in \mathbb{C}^m$, $v^i(t) \in \mathbb{C}^p$, $1 \leq i \leq k$.

Sometimes, the considered internal controls u^i are given by means a communication topology defined by a graph with

i) Vertex set: $V = \{0, 1, \dots, k\}$

ii) Edge set: $E = \{i, j) \mid i, j \in V\} \subset V \times V$

iii) Neighbor of i: $\mathcal{N}_i = \{j \in V \mid (i,j) \in E\}$. (In the case where $j = i$ the edge is called self-loop).

defining the communication topology among agents.

The leader is represented by vertex 0, and the leader sends information to the agents located in the leader's neighbors. Then there are edges $(0, j)$ but not $(j, 0)$ for $j \neq 0$

Associate to the graph, we have the Laplacian matrix defined in the following manner.

$$\mathcal{L} = (l_{ij}) = \begin{cases} |\mathcal{N}_i| & \text{if } i = j \\ -1 & \text{if } j \in \mathcal{N}_i \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Example:
Consider the graph in figure 1.

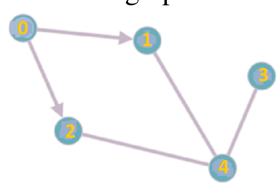

Figure 1: Graph with leader node

The Laplacian matrix is:

$$\mathcal{L} = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & -1 & -1 & -1 & 3 \end{pmatrix} \tag{3}$$

We observe that $\mathcal{L}$ is a block upper triangular matrix

$$\begin{pmatrix} \ell_0 & L \\ 0 & \mathcal{L}_1 \end{pmatrix}$$

where the first diagonal block is a 1×1 matrix. When the subgraph with vertices set $V' = \{1, \dots, k\}$ is undirected and connected, the second diagonal block is symmetric. This submatrix is the Laplacian matrix corresponding to the subgraph V'.

We use the following control law for agent i:

$$u^i = -K_i \sum_{j \in \mathcal{N}_i} (x^j - x^i), \quad i = 0, 1, \dots, k,$$

where $K_i \in M_{m \times n}(\mathbb{R})$ are feedback matrices to be designed. (We are interested in the case where $K_i = K$ for $i = 0, \dots, k$.

Writing $\mathbb{X} = (x^0, x^1, \dots, x^k)^t$, $\mathbb{U} = (0, u^i, \dots, u^k)^t$ and $\mathbb{V} = (v^0, v^1, \dots, v^k)^t$,

$$\begin{aligned} \mathbb{A} &= \operatorname{diag}(A_0, A_1, \dots, A_k) \\ &\in M_n(\mathbb{C}) \times \overset{k}{\dots} \times M_n(\mathbb{C}) \\ \mathbb{B} &= \operatorname{diag}(0, B_1, \dots, B_k) \\ &\in M_{n \times m}(\mathbb{C}) \times \overset{k}{\dots} \times M_{n \times m}(\mathbb{C}) \\ \mathbb{C} &= \operatorname{diag}(C_1, \dots, C_k) \\ &\in M_{n \times p}(\mathbb{C}) \times \overset{k}{\dots} \times M_{n \times p}(\mathbb{C}) \end{aligned}$$

and $\mathbb{X} = (x^0, x^1, \dots, x^k)^t$, $\mathbb{U} = (\mathcal{L} \otimes I_n)\mathbb{X} = (0, u^1, \dots, u^k)^t$ and $\mathbb{V} = (v^0, v^1, \dots, v^k)^t$, and in the case where the communication topology for internal control is considered, the control is written as $\mathbb{U} = (\mathcal{L} \otimes I_n)\mathbb{X}$, and $\mathbb{K} = \operatorname{diag}(K_0, K_1, \dots, K_k)$ the system 1 can be described as

$$\dot{\mathbb{X}}(t) = (\mathbb{A} + \mathbb{B}\mathbb{K}(\mathcal{L} \otimes I_n))\mathbb{X}(t) + \mathbb{C}\mathbb{V}(t). \tag{4}$$

Remember that $A = (a_{ij}) \in M_{n \times m}(\mathbb{C})$ and $B = (b_{ij}) \in M_{p \times q}(\mathbb{C})$ the Kronecker product is defined as follows:

Definition 1 *Let $A = (a^i_j) \in M_{n \times m}(\mathbb{C})$ and $B \in M_{p \times q}(\mathbb{C})$ be two matrices, the Kronecker product of A and B, write $A \otimes B$, is the matrix*

$$A \otimes B = \begin{pmatrix} a^1_1 B & a^1_2 B & \dots & a^1_m B \\ a^2_1 B & a^2_2 B & \dots & a^2_m B \\ \vdots & \vdots & & \vdots \\ a^n_1 B & a^n_2 B & \dots & a^n_m B \end{pmatrix} \in M_{np \times mq}(\mathbb{C})$$

(See, [11] for Kronecker product properties).

3 Controllability

The importance of the qualitative property of dynamic systems in the control theory, known as controllability, is well known.

The controllability concept involves taking the system from any initial state to any final state in finite time, regardless of the path or input. Let us consider the multi-agent system 1.

In our particular setup, the objective of leader-following multi-agent control is to make the state of each following agent consistent with that of the leader. For every agent $1 \leq i \leq k$, a external control $v^i(t)$ is required to realize

$$\lim_{t \to \infty} \|x^i(t) - x^0(t)\| = 0, \; 1 \leq i \leq k$$

It is important to emphasize that various definitions of controllability are derived, depending to a large extent on the class of dynamic systems and the form of allowable controls, [12].

In our particular setup, the controllability character can be described as

$$\operatorname{rank} \begin{pmatrix} \mathbb{A} - \lambda I_{n \times k} & \mathbb{B} & \mathbb{C} \end{pmatrix} = nk$$

If a fixed $\mathbb{B}$-feedback $\mathbb{K}$ and a fixed topology communication is considered, the controllability character is described as

$$\text{rank}\begin{pmatrix} \mathbb{A}+\mathbb{B}\mathbb{K}(\mathcal{L}\otimes I_n) & \mathbb{C}\end{pmatrix} = nk$$

The controllability of the system can be analyzed by computing the rank of the controllability matrix:

$$\begin{matrix}(\mathbb{C} & (\mathbb{A}+\mathbb{B}\mathbb{K})(\mathcal{L}\otimes I_n))\mathbb{C}) \\ \dots & (\mathbb{A}+\mathbb{B}\mathbb{K})(\mathcal{L}\otimes I_n))^{nk-1}\mathbb{C})\end{matrix}$$

The rank of this matrix is invariant under feedback, that is to say

Proposition 2 *The matrix controllability of the system 1 is invariant under external feedback*

Proof:

$$\begin{array}{l}\text{rank}\begin{pmatrix}(\mathbb{A}+\mathbb{B}\mathbb{K})(\mathcal{L}\otimes I_n)+\mathbb{C}\mathbb{F} & \mathbb{C}\end{pmatrix} = \\ \text{rank}\begin{pmatrix}(\mathbb{A}+\mathbb{B}\mathbb{K})(\mathcal{L}\otimes I_n) & \mathbb{C}\end{pmatrix}\begin{pmatrix} I & \\ \mathbb{F} & I\end{pmatrix}\end{array}$$

□

We are going to carry out the study for a particular case in which all the systems have the same dynamics, that is, $A_i = A$, $B_i = B$, $C_i = C$ and $K_i = K$ for all $1 \leq i \leq k$; and the graph defining the topology relating to the systems is undirected and connected. Being un undirected graph the matrix $\mathcal{L}_1$ is symmetric, then there exist an orthogonal matrix P such that $P\mathcal{L}_1P^t = \mathcal{D}$, and the connection ensures that 0 is a simple eigenvalue of $\mathcal{L}_1$.

Proposition 3 *Under these conditions, the system can be described as*

$$\begin{array}{rl}\dot{\mathbb{X}}(t) & = \\ & (I_k\otimes A)+(\bar{I}_k\otimes BK)(\mathcal{L}\otimes I_n))\mathbb{X}(t)+ \\ & (I_n\otimes C)\mathbb{V}(t)\end{array} \tag{5}$$

where $\bar{I}_k = \begin{pmatrix}0 & \\ & I_{k-1}\end{pmatrix}$.

In our particular setup, we have that there exists $Q = \begin{pmatrix}1 & 0\\ 0 & P\end{pmatrix} \in Gl(k,\mathbb{R})$ with P orthogonal such that $P\mathcal{L}P^t = \mathcal{D} = \text{diag}\,(\lambda_1,\dots,\lambda_k)$,
$(\lambda_1 \geq \dots \geq \lambda_{k-1} > \lambda_k = 0)$.
that is

$$\begin{pmatrix}1 & 0\\ 0 & P\end{pmatrix}\begin{pmatrix}\ell_0 & L\\ 0 & \mathcal{L}_1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & P^t\end{pmatrix} = \begin{pmatrix}\ell_0 & LP^t\\ 0 & \mathcal{D}\end{pmatrix} = \mathcal{T}$$

For the matrix $\mathcal{L}_1$ given in 3 the matrix $\mathcal{D}$ is

$$\mathcal{D} = \begin{pmatrix}0.0000 & 0 & 0 & 0\\ 0 & 1.0000 & 0 & 0\\ 0 & 0 & 1.0000 & 0\\ 0 & 0 & 0 & 4.0000\end{pmatrix}$$

and P is

$$P = \begin{pmatrix}0.5000 & -0.4082 & 0.7071 & -0.2887\\ 0.5000 & -0.4082 & -0.7071 & -0.2887\\ 0.5000 & 0.8165 & -0.0000 & -0.2887\\ 0.5000 & 0 & 0 & 0.8660\end{pmatrix}$$

Corollary 4 *The system can be described in terms of the matrices A, B, C the feedback K and the eigenvalues of $\mathcal{L}$.*

Proof:

Following the properties of Kronecker product, we have that.

$$(Q\otimes I_n)(I_k\otimes A)(^t\otimes I_n) = (I_k\otimes A)$$

$$\begin{array}{l}(Q\otimes I_n)(\bar{I}_k\otimes BK)(Q^t\otimes I_n) = \\ (\bar{I}_k\otimes BK)\end{array}$$

$$(P\otimes I_n)(I_k\otimes C)(P^t\otimes I_k) = (I_k\otimes C)$$

$$(Q\otimes I_n)(\mathcal{L}\otimes I_n)(Q^t\otimes I_n) = (\mathcal{T}\otimes I_n)$$

and calling $\widehat{\mathbb{X}} = (Q\otimes I_n)\mathbb{X}$, and $\widehat{\mathbb{V}} = (P\otimes I_k)\mathbb{V}$ we have

$$\dot{\widehat{\mathbb{X}}} = ((I_k\otimes A)+(I_k\otimes BK)(\mathcal{D}\otimes I_n))\widehat{\mathbb{X}}+(I_k\otimes C)\widehat{\mathbb{V}}.$$

Calling $LP^t = \begin{pmatrix}\ell_1 & \dots & \ell_k\end{pmatrix}$ the about equation is written in the following form

$$\begin{array}{l}\dot{\widehat{\mathbb{X}}} = \\ \left(\begin{pmatrix}A & & & \\ & A & & \\ & & \ddots & \\ & & & A\end{pmatrix}+\begin{pmatrix}0 & & & \\ & B & & \\ & & \ddots & \\ & & & BK\end{pmatrix}\begin{pmatrix}\ell_0 I_n & \ell_1 I_n & \dots & \ell_k I_n\\ & \lambda_1 I_n & & \\ & & \ddots & \\ & & & \lambda_k I_n\end{pmatrix}\right)\widehat{\mathbb{X}}+ \\ \begin{pmatrix}C & & & \\ & C & & \\ & & \ddots & \\ & & & C\end{pmatrix}\widehat{\mathbb{V}}\end{array}$$

That is to say

$$\dot{\widehat{\mathbb{X}}} = \begin{pmatrix}A & & & \\ & A+\lambda_1 BK & & \\ & & \ddots & \\ & & & A+\lambda_k I_n\end{pmatrix}\widehat{\mathbb{X}}+\begin{pmatrix}C & & & \\ & C & & \\ & & \ddots & \\ & & & C\end{pmatrix}\widehat{\mathbb{V}}$$

□

Using this description, the analysis of controllability is easier.

Proposition 5 *The system* (5) *is controllable if and only if the systems* (A, C) *and* $(A + \lambda_i BK, C)$ *are controllable, for each* $1 \leq i \leq k$.

If each system (A, C) $(A + \lambda_i BK, C)$ are controllable for $1 \leq i \leq k$ there exist external feedbacks F_i in such a way that each system $A + CF_0)$ and $A + \lambda_i BK + CF_i$ arrives to a final state preset, and that in this case is the same for each system.

When the multi-agent system is not controllable, one can try to change the proportionality of the interaction between the agents, that is, change the matrix K, looking for one that makes the final system controllable.

4 Conclusion

This work examines the controllability of leader-following multi-agent systems communicated by a graph, playing an essential role in describing the interaction topologies. A necessary and sufficient condition for controllability has been presented and described in terms of the eigenvalues of the subgraph defined by the topology of the follower's agents having the same linear dynamics. Based on these results, the author proposes considering the multi-agent linear system containing perturbation terms as future work. Furthermore, we want to take advantage of the theoretical results to investigate how the structural characteristics of a brain network determine the temporal characteristics of cognitive dynamics.

References:

[1] M.I. Garcia-Planas. Control properties of multi-agent dynamical systems modelling brain neural networks. In 2020 International Conference on Mathematics and Computers in Science and Engineering (MACISE) (pp. 106-113). IEEE. (2020, January).

[2] M.I. Garcıa-Planas. Obtaining Consensus of Multi-agent Linear Dynamic Systems. Advances in Applied and Pure Mathematics, pp. 91-95, (2014).

[3] M.I. Garcia-Planas, M.V. Garcia-Camba. Controllability of brain neural networks in learning disorders—a geometric approach. Mathematics. **10**, (3), pp. 331:1-331:13, (2022). doi.org/10.3390/math10030331

[4] J. Jiang, & Y. Jiang. (2020). Leader-following consensus of linear time-varying multi-agent systems under fixed and switching topologies. Automatica, 113, 108804.

[5] J. Wang, D. Cheng, X. Hu. (2008). Consensus of multi-agent linear dynamic systems. Asian Journal of Control, **10**, (2), pp. 144-155, (2008).

[6] Wei, Q., Wang, X., Zhong, X., & Wu, N. (2021). Consensus control of leader-following multi-agent systems in directed topology with heterogeneous disturbances. IEEE/CAA Journal of Automatica Sinica, 8(2), 423-431.

[7] G. Xie, L. Wang. Consensus control for a class of networks of dynamic agents: switching topology, Proc.2006 Amer.Contro.Conf.,pp.13821387,(2007).

[8] Ni, W., & Cheng, D. (2010). Leader-following consensus of multi-agent systems under fixed and switching topologies. Systems & control letters, **59**(3-4), 209-217.

[9] S.E. Li, Z. Wang, Y. Zheng, D. Yang, & K. You. (2020). Stability of general linear dynamic multi-agent systems under switching topologies with positive real eigenvalues. *Engineering*, **6**(6), pp. 688-694.

[10] S. Gu, F. Pasqualetti, M. Cieslak, Q.K. Telesford, A.B. Yu, A.E. Kahn, J.D. Medaglia, J.M. Vettel, M.B. Miller, S.T. Grafton & D.S. Bassett. Controllability of structural brain networks. Nat Commun **6**, 8414 (2015). https://doi.org/10.1038/ncomms9414

[11] P. Lancaster, M. Tismenetsky, *The Thoery of Matrices*. Academic–Press. San Diego, 1985.

[12] J. Klamka. Controllability of dynamical systems. A survey. Bulletin of the Polish Academy of Sciences: Technical Sciences, **61**, (2), pp. 335–342, (2013).

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The author has no conflicts of interest to declare that are relevant to the content of this article.