Solution of Multi-Dimensional Non-linear Fractional Differential Equations of Higher Orders

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Abstract: - In our paper, we are used here two methods to solve non-linear differential equations from a higher order: the first-one is domain decomposition method is used to estimate the Maxi. Abso. Trunc. Error of Adomain series and the second-one proposed numerical (PNM), these types of equations are studied. When we use these methods, an exclusive solution will be provided, and the approximate analyses of this method applied to these types of equations will be overlooked, and the maximum error that has been informed to solve the ADOMIANS series will be classified. A digital example is prepared clarify the impact method provided and significant following of these equations in our paper is Bagley-Torvik equation.

Key-Words: - Non-linear fractional differential equation, higher order, adomian decomposition method (ADM), and proposed numerical method.

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1 Introduction

Fractional differential equations are practiced to sample expansive space of physical problems including non-linear vacillation of earth shakes, [1], fluid-dynamic passing (in 1999), [2], and hesitancy dependent on the waning behavior of many relativistic materials.

Fractional differential equations that contain only one fracture derivative are a good understandable tool, they are often emploied for the sporty depiction of plenty material procedures, but they are not eternity enough to reverse all suitable phenomena

The authors in, [3], investigated strategies for the numerical solution of the initial value problem with initial conditions where $0<\alpha_1<\alpha_2<\cdots<\alpha_v$. Here $y(\alpha_j)$ denotes the derivative of order $y(\alpha_j)>0$ (not necessarily $\alpha_j \in \mathbb{N}$) in the sense of Caputo.

The authors in, [4], returned and expanded multi-term homogeneous differential equations with caputo-type derivatives and fixed transactions through the necessary stability, instability conditions and stability and caffeine.

The authors in, [5], reviewed two methods of the most action groups of digital methods of fractional arrangement problems and discussed some major mathematical issues such as the effective treatment of the term continuous memory and the solution of nonlinear systems participating in implicit ways.

The use of fractional differential operators in mathematical models has become increasingly widespread in recent years. Several forms of fractional differential equations have been proposed in standard models, and there has been significant interest in developing numerical schemes for their solution the authors in, [6], show how the numerical approximation of the multi-wheel -term differential differential formula solution can be calculated by reducing the problem to a system of regular differential equations and fracture in each unit in most unit.

The author described in, [7], two sports ways to use equations of multi-term and multi-arranged systems have shown the relationships between these two concepts. Then examine its most important analytical characteristics. Finally, he is considerd the digital methods of its approach solution.

The author was collected in, [8], with the linear way to devise the Adams Fource Molton method for real, non-linear, non-linear equations with a firm delay or change of time, then use this method to estimate the late fractures- arrange differential equations.

The authors in, [9], introduced a new method of analytical and digital solution of a non-Dynican N-Range N-Range by the well-known vibration engineers, that is, the average consensual balance method.

The authors have suggested at, [10], the definition of Mittag-Lfler's stability and entered the direct Lyapunov method. The principle of broken comparison is presented and the Riemann -Liouville Fairville system is extended using the Order Caputo systems.

The authors were presented in, [11], and used the results of modern stability of broken equations and methods of analytical types include linear, non-linear and chronological delay differential equations. Some of the inferences of regularity are similar to the inferences of differential equations for the classic order.

The authors were obtained in, [12], for multi-term homogeneous differential equations with three Caputo derivatives and fixed laboratories through the necessary and sufficient stability of instability conditions.

In 2013, the authors in, [13], were informed of the theories of stable point, the presence and peerless of solutions for non -linear non -non-linear equations, and presented two examples to clarify the results.

The authors in, [14], built two new schemes to solve a numerical solution to the non-linear differential equation of fractional kinds in one-dimensions and two-dimension. The scheme-I one and two-dimension mythical (SLP) uses fundamental functions while the chart-II uses 1D and 2D (IBF) basis functions as main functions.

The authors in, [15], presented a simple and effective analytical algorithm of two steps from two steps to solve multidimensional times.

2 Formulation of Issue with the Solution Algorithm

2.1 First method: Adomian decomposition method

Let

$$\xi_k \in (k-1,k), \mathrm{k}=1,\dots,\mathrm{n}, 0<\xi_1<\xi_2<\cdots<\xi_m < n,k,m$$

Let nonlinear fractional differential equation:

$$\frac{d^n x(t)}{dt_n} = \mathrm{f}\left(\mathrm{t},\mathrm{x},\frac{d^{\xi_1}x(t_1)}{dt_{\xi_1}},\frac{d^{\xi_2}x(t_2)}{dt_{\xi_2}},\dots,\frac{d^{\xi_m}x(t_m)}{dt_{\xi_m}}\right),\quad x^{(j)}(0)=c,\quad j=0,1,2,\dots,n-1 \tag{1}$$

Where $x=x(t), t\in J=[0,T], T\in R^+, x\in C(J)$ and the fractional derivative is,

$$\frac{d^{\xi}x(t)}{dt_{\xi}} = I^{n-\xi}\frac{d^n x(t)}{\partial dt_n}, n-1<\xi\leq n$$

$$I^{\xi}x(t) = \frac{1}{\Gamma(\xi)}\int_0^t (t-\tau)x(\tau)d\tau.$$

Let f confirmed Lipschitz condition with constant L such as

$$|\mathrm{f}(\mathrm{t},\mathrm{y}_0,\mathrm{y}_2,\dots,\mathrm{y}_\mathrm{m}) - \mathrm{f}(\mathrm{t},\mathrm{z}_0,\mathrm{z}_1,\dots,\mathrm{z}_\mathrm{m})| \leq \mathrm{L}\sum_{\mathrm{i}=0}^{m}|\mathrm{y}_\mathrm{i}-\mathrm{z}_\mathrm{i}|. \tag{2}$$

which implies that,

$$\left|\mathrm{f}\left(\mathrm{t},\sum_{j=0}^{n-1}c_j\frac{t^j}{j!}+I^n y(t),\dots,I^{n-\xi_m}y(t)\right) - \mathrm{f}\left(\mathrm{t},\sum_{j=0}^{n-1}c_j\frac{t^j}{j!} + I^n z(t),\dots,I^{n-\xi_m}z(t)\right)\right| \leq \mathrm{L}\sum_{\mathrm{i}=0}^{\mathrm{m}}\left|I^{n-\xi_i}y(t) - I^{n-\xi_i}z(t)\right|. \tag{3}$$

The solution algorithm by using the domain decomposition method is:

$$y_0(t)=p(t), \tag{4}$$

$$y_j(t)=A_{j-1}(t), j\geq 1. \tag{5}$$

Where, Aj are Adomian polynomials of non-linear $f\left(t, \sum_{j=0}^{n-1} c_j \frac{t^j}{j!} + I^n y(t), \dots, I^{n-\xi_m} y(t)\right)$ which take the form,

$$A_j = \frac{1}{j!}\left[\frac{d}{d\lambda^j} f\left(t, \sum_{j=0}^{n-1} c_j \frac{t^j}{j!} + \sum_{i=0}^{\infty} \lambda^i I^n y_i \dots, \sum_{i=0}^{\infty} \lambda^i I^{n-\xi_m} y_i\right)\right]_{\lambda=0} \quad (6)$$

And the solution of the equations (1) and (2) will be,

$$y(t) = \sum_{i=0}^{\infty} y_i(t) \quad (7)$$

Finally,

$$x(t) = \sum_{j=0}^{n-1} c_j \frac{t^j}{j!} + X(t) = \sum_{j=0}^{n-1} c_j \frac{t^j}{j!} + I^n y(t) \quad (8)$$

3 Convergence Analysis

Consider F is mapping with Banach space E,

$$(C(J), \|.\|).$$

All continual functions on J with

$$\| y \| = \max_{t \in J} e^{-Nt} \left| y(t) \right|, N > 0.$$

Theorem 3.1.(**Existence and uniqueness**): Let f satisfies the Lipschitz condition

$$|f(t, y_0, y_2, \dots, y_m) - f(t, z_0, z_1, \dots, z_m)| \le L \sum_{i=0}^{m} |y_i - z_i|. \quad (9)$$

Then, the nonlinear fractional differential equation has a unique solution

$$y \in C(J).$$

Proof:

Let F: E →E is defined as

$$Fy = f\left(t, \sum_{j=0}^{n-1} c_j \frac{t^j}{j!} + I^n y(t), \dots, I^{n-\xi_m} y(t)\right). \quad (10)$$

Let

$$y, z \in E$$

then,

$$F_y - F_z = f\left(t, \sum_{j=0}^{n-1} c_j \frac{t^j}{j!} + I^n y(t), \dots, I^{n-\xi_m} y(t)\right) - f\left(t, \sum_{j=0}^{n-1} c_j \frac{t^j}{j!} + I^n z(t), \dots, I^{n-\xi_m} z(t)\right) \quad (11)$$

This implies that:

$$\left|F_y - F_z\right| = \left| f\left(t, \sum_{j=0}^{n-1} c_j \frac{t^j}{j!} + I^n y(t), \dots, I^{n-\xi_m} y(t)\right) - f\left(t, \sum_{j=0}^{n-1} c_j \frac{t^j}{j!} + I^n z(t), \dots, I^{n-\xi_m} z(t)\right) \right| \quad (12)$$

$$\le L \sum_{i=0}^{m} \left| I^{n-\xi_i} y - I^{n-\xi_i} z \right|$$

$$\le L \sum_{i=0}^{m} \left| \frac{1}{\Gamma(n-\xi_i)} \int_0^t (t-\tau)^{n-\xi_i-1} (y-z) d\tau. \right|$$

$$e^{-Nt}\left|F_y - F_z\right| \le L \sum_{i=0}^{m} \left| \frac{1}{\Gamma(n-\xi_i)} \int_0^t e^{-N(t-\tau)} e^{-N\tau} (t-\tau)^{n-\xi_i-1} (y-z) d\tau. \right| \quad (13)$$

$$\max_{t\in J} e^{-Nt}\left|F_y - F_z\right| \le L\sum_{i=0}^{m}\left|\frac{1}{\Gamma(n-\xi_i)}\max_{t\in J}\int_0^t e^{-N(t-\tau)}e^{-N\tau}(t-\tau)^{n-\xi_i-1}(y-z)d\tau.\right| \tag{14}$$

$$\left\|F_y - F_z\right\| \le L\sum_{i=0}^{m}\frac{1}{\Gamma(n-\xi_i)}\|y-z\|\int_0^t e^{-Ns}s^{n-\xi_i-1}ds.$$

$$\le L\sum_{i=0}^{m}\frac{1}{\Gamma(n-\xi_i)}\|y-z\|\int_0^\infty e^{-Ns}s^{n-\xi_i-1}ds.$$

$$\le L\sum_{i=0}^{m}\frac{1}{N^{n-\xi_i}}\|y-z\| \tag{15}$$

Now, we choosen N large enough s.t

$L\sum_{i=0}^{m}\frac{1}{N^{n-\xi_i}} < 1,$

then, we get:

$$\left\|F_y - F_z\right\| \le \|y-z\| \tag{16}$$

Therefore, the mapping F is constriction.

Theorem 3.2. (Proof of convergence):
the series solution

$$\mathrm{y(t)} = \sum_{\mathrm{i=0}}^{\infty}\mathrm{y_i(t)}$$

Then, by using Adomain decomposition method converges if $|y_1(t)| < c$, where c is positive no.

Proof:
Let seq. $\{S_p\}$, s.t. $S_p = \sum_{\mathrm{i=0}}^{\mathrm{p}}\mathrm{y_i(t)}$ of partial sums from the series $\sum_{\mathrm{i=0}}^{\infty}\mathrm{y_i(t)}$ since,

$$f\left(t,\sum_{j=0}^{n-1}c_j\frac{t^j}{j!}+I^n y(t),\dots,I^{n-\xi_m}y(t)\right) = S_p$$

So, we can write

$$f\left(t,\sum_{j=0}^{n-1}c_j\frac{t^j}{j!}+I^n S_p,\dots,I^{n-\xi_m}S_p\right) = \sum_{i=0}^{p}A_i(t)$$

From equations (4) and (5), we have:

$$\sum_{i=0}^{\infty}y_i(t) = \mathrm{p(t)} + \sum_{i=0}^{\infty}A_{i-1}$$

Let Sp and Sq be partial sums and p greater than q, one can have:

$$S_p = \sum_{i=0}^{p}y_i(t) = \mathrm{p(t)} + \sum_{i=0}^{p}A_{i-1}$$

And

$$S_q = \sum_{i=0}^{q}y_i(t) = \mathrm{p(t)} + \sum_{i=0}^{q}A_{i-1}$$

Now, the Cauchy sequence $\{S_p\}$ in this E,

$$S_p - S_q = \sum_{i=0}^{p}A_{i-1} - \sum_{i=q+1}^{p}A_{i-1} = \sum_{i=q}^{p-1}A_i$$

$$= f\left(t,\sum_{j=0}^{n-1}c_j\frac{t^j}{j!}+I^n S_{p-1},\dots,I^{n-\xi_m}S_{p-1}\right) - f\left(t,\sum_{j=0}^{n-1}c_j\frac{t^j}{j!}+I^n S_{q-1},\dots,I^{n-\xi_m}S_{q-1}\right)$$

$$\left|S_p - S_q\right| = \left|f\left(t,\sum_{j=0}^{n-1}c_j\frac{t^j}{j!}+I^n S_{p-1},\dots,I^{n-\xi_m}S_{p-1}\right) - f\left(t,\sum_{j=0}^{n-1}c_j\frac{t^j}{j!}+I^n S_{q-1},\dots,I^{n-\xi_m}S_{q-1}\right)\right|$$

$$\le L\sum_{i=0}^{m}\left|I^{n-\xi_1}S_{p-1} - I^{n-\xi_1}S_{q-1}\right|$$

$$\le L\frac{1}{\Gamma(n-\xi_i)}\int_0^t (t-\tau)^{n-\xi_i-1}\left|S_{p-1}-S_{q-1}\right|d\tau.$$

$$e^{-Nt}\left|S_p - S_q\right| \le L\sum_{i=0}^{m}\frac{1}{\Gamma(n-\xi_i)}\int_0^t e^{-N(t-\tau)}(t-\tau)^{n-\xi_i-1}e^{-N\tau}\left|(S_{p-1}-S_{q-1}\right|d\tau$$

$$\left\|S_p - S_q\right\| \le L\sum_{i=0}^{m}\frac{1}{N^{n-\xi_i}}\left\|S_{p-1}-S_{q-1}\right\| \le \beta\left\|S_{p-1}-S_{q-1}\right\|$$

Let p=q+1 then,

$$\|S_{q+1}-S_q\| \le \beta\|S_q-S_{q-1}\| \le \beta^2\|S_{q-1}-S_{q-2}\| \le \cdots \le \beta^q\|S_1-S_0\|$$

$$\|S_p-S_q\| \le \|S_{q+1}-S_q\| \le \|S_{q+2}-S_{q+1}\| + \cdots + \|S_p-S_{p-1}\|$$

$$\le \beta^q+\beta^{q+1}+\cdots+\beta^{p-1}\|S_1-S_0\|$$
$$\le \beta^q[1+\beta+\cdots+\beta^{p-q-1}]\|S_1-S_0\|$$
$$\le \beta^q\left[\frac{1-\beta^{p-q}}{1-\beta}\right]\|y_1\|$$

Since, $0<\beta=L\sum_{i=0}^{m}\frac{1}{N^{n-\xi_i}}<1$, and $p>q$ *then*, $(1-\beta^{p-q})\le 1$. Consequently,

$$\|S_p-S_q\| \le \left[\frac{\beta^q}{1-\beta}\right]\|y_1\| \le \left[\frac{\beta^q}{1-\beta}\right]\max_{t\in J}|y_1(t)|$$

But, $|y_1(t)|<c$, and as $q\to\infty$ then, $\|S_p-S_q\|\to 0$, and hence, $\{S_p\}$ is a Cauchy sequence in Banach space E so, $\sum_{i=0}^{\infty}y_i(t)$ convergence.

Theorem 3.3. (Error analysis):
The max. absolute trunc. error of solution equation (6) to the problem (1) is estimated to be,

$$\left\|x-\sum_{i=0}^{q}x_i\right\| \le \left(\sum_{k=0}^{n-1}\frac{-T^k}{k!\,K^{n-k}}\right)\frac{\beta^q}{1-\beta}\|y_1\|.$$
if n is odd

And n is even

$$\left\|x-\sum_{i=0}^{q}x_i\right\| \le \left(\sum_{k=1}^{n-1}\frac{(-1)^{k+1}(T^k}{k!K^{n-k}}\right)\frac{\beta^q}{1-\beta}\|y_1\|$$

Proof:
By theorem (3.2), we have

$$\|S_p-S_q\| \le \frac{\beta^q}{1-\beta}\left(\max_{t\in J}\right)e^{-Nt}|y_1(t)|$$

but,

$$S_p=\sum_{i=0}^{p}y_i,$$

then

$$P\to\infty, \quad S_p\to y(t),$$

So,

$$\|y-S_q\| \le \frac{\beta^q}{1-\beta}\|x_1\|$$

So,

$$\left\|y-\sum_{i=0}^{q}y_i\right\| \le \frac{\beta^q}{1-\beta}\|y_1\| \tag{17}$$

From equation (1), we get

$$\sum_{i=0}^{\infty}x(t)=\sum_{j=0}^{n-1}c_j\frac{t^j}{j!}+I^n\sum_{j=0}^{n-1}y_i(t). \tag{18}$$

By using equation (1) and the above equation, we can get:

$$x(t)-\sum_{i=0}^{q}x_i(t)=I^n y(t)+I^n\sum_{j=0}^{q}y_i(t)$$

$$=\int_0^t \ldots n-fold \ldots \int_0^t \left(y(\tau)-\sum_{j=0}^{q}y_i(\tau)\right)d\tau\ldots d\tau$$

$$e^{-Nt}\left|x(t)-\sum_{i=0}^{q}x_i(t)=I^n\right| = \int_0^t \ldots n - fold \ldots \int_0^t e^{-N(t-\tau)}e^{-N\tau}\left(\left|y(\tau)-\sum_{j=0}^{q}y_i(\tau)\right|\right)d\tau\ldots d\tau$$

$$\left\|x(t)-\sum_{i=0}^{q}x_i(t)=I^n\right\| \le \left\|y(\tau)-\sum_{j=0}^{q}y_i(\tau)\right\|\int_0^t \ldots n - fold \ldots \int_0^t e^{-N(t-\tau)}d\tau\ldots d\tau$$

from the equation (17), we get:

$$\left\|x-\sum_{i=0}^{q} x_i\right\| \leq \left(\sum_{k=0}^{n-1} \frac{-T^k}{k!\,K^{n-k}}\right) \frac{\beta^q}{1-\beta} \|y_1\|.$$
If n is odd

(19)

and

$$\left\|x-\sum_{i=0}^{q} x_i\right\| \leq \left(\sum_{k=1}^{n-1} \frac{(-1)^{k+1}(T^k}{k!K^{n-k}}\right) \frac{\beta^q}{1-\beta} \|y_1\|$$
If n is even

(20)

This completes the proof.

4 Numerical Examples

Example (4.1), [16]:

The Bagley-Torvik equation

$$a\frac{d^2x(t)}{dt} + b\frac{d^{\frac{3}{2}}x(t)}{dt} + cx(t) = c(t+1), \qquad (21)$$

$$0 \leq t \leq 5,$$
$$x(0) = x' = 1,$$

x(t)=(t+1),

is the valid solution

By using Adomian decomposition method, we will solve it:

X(t)=x(t)-t-1,

equation (21):

$$a\frac{d^2X(t)}{dt} + b\frac{d^{\frac{3}{2}}X(t)}{dt} + cX(t) = 0, \qquad (22)$$

$$X(0) = X' = 1,$$

Applying the Adomian decomposition method gets:

$$y_0(t) = 0, \qquad (23)$$

$$y_i(t) = \frac{-1}{a}\left(bI^{\frac{1}{2}}y_i + cI^2y_i\right), i \geq 1 \qquad (24)$$

From the equation (23) and (24), the solution:

$$\mathrm{y(t)} = \sum_{\mathrm{i=0}}^{\mathrm{m}} \mathrm{y_i(t)} \qquad (25)$$

The solution of equation (22) is:

$$X(t) = I^2y(t) = 0.$$

Finally, the above equation (21) is:

$$x(t) = X(t) + (1+t) = (1+t).$$

is the valid solution.

Example (4.2):

Let non-linear FDE,

$$\frac{d^3x(t)}{dt} = 6 - \frac{72}{5\pi}t + \frac{1}{10}\left(\frac{d^{5/2}x(t)}{dt}\right)^2 \qquad (26)$$

$$x = x' = x'' = 0,$$

which has x(t)=t^3.

By using the equations (23) and (24), the solution is:

$$y = 6 - \frac{72}{5\pi}t + \frac{1}{10}\left(I^{1/2}y(t)\right)^2,$$

Applying the Adomian decomposition method gets:

$$y_0(t) = 6 - \frac{72}{5\pi}t$$

$$y_i(t) = \frac{1}{10}(A_{i-1}), \quad i \geq 1.$$

Where A_i are Adomain polynomials of non-linear $\left(I^{1/2}y(t)\right)^2$. Finally, the solution (26):

$$x(t) = \left(I^3y(t)\right) = I^3\sum_{\mathrm{i=0}}^{\mathrm{m}} \mathrm{y_i(t)} \qquad (27)$$

Table 1. illustrates the absolute error of the ADM solution, while Table 2. illustrates the max. absolute sectioned. The figure shows ADM and exact solutions (when m=15).

Table 1. Absolute Error

m	max. error (N=5)
5	0.00418546
10	0.000325204
15	0.0000366795

Table 2. Maximum absolute error

m	**max. error (N=5)**
5	**0.0093013**
10	**0.00050822**
15	**0.0000277689**

4.1 Second Method: Proposed Numerical Method

This method is a disadvantage to numerail method. It solve only fractional differential equations with initial condition, we get past this shortcoming by using PNM.

The solution of steps:
Step 1: Use the transform the initial conditions to homogenous.

By the substitution for eqs. (1) and (2),

$$x=\sum_{j=0}^{n-1} c_j \frac{t^j}{j!}+X, \tag{28}$$

Step 2: Acquire solution algorithm.
By using the next equations:

$$D^{\xi_i}X=h^{-\xi_i}\sum_{j=0}^{m} w_j^{\xi_i} X_{m-j}$$

$$w_j^{\xi_i}=-1^j \frac{\Gamma(\xi_i+1)}{\Gamma(\xi_i+1-j)},$$

$$t_m=mh(m=0,1,2,\dots), \qquad X(t)=X〛\_m(t_m)$$

obtain values of X_m.

Step 3: coming back to the main new conditions. By relevance (28), we receive:

$$x_m=\sum_{j=0}^{n-1} c_j \frac{(t_m)^j}{j!}+X_m.$$

5 Numerical Examples

Example (5.1):
Let non-linear FDE:

$$\frac{dx(t)}{dt}+\frac{d^{\frac{1}{2}}x(t)}{dt}-2x^2(t)=0, \quad x(0)=c. \tag{29}$$

Set

X(t)=x(t)-c,

The equation (27) will be:

$$\frac{dX(t)}{dt}+\frac{d^{\frac{1}{2}}X(t)}{dt}-2X^2(t)-4cX(t)-2x^2=0,$$
$$X(0)=0. \tag{30}$$

The solution algorithm of equation (30) is:

$$X_m=\frac{(h^{-1}+4c)X_{m-1}-h^{\frac{-1}{2}}\sum_{j=1}^{m} w_j^{\frac{-1}{2}} X_{m-j}+2(X_{m-1})^2+2c^2}{h^{-1}-h^{\frac{-1}{2}}}$$
$$X_0=0, \quad m=1,2,\dots.$$

The solution of equation (29) is
$$x_m=X_m+c.$$

Figure 1, shows PNM and ADM solutions (when m=5, h=0.01).

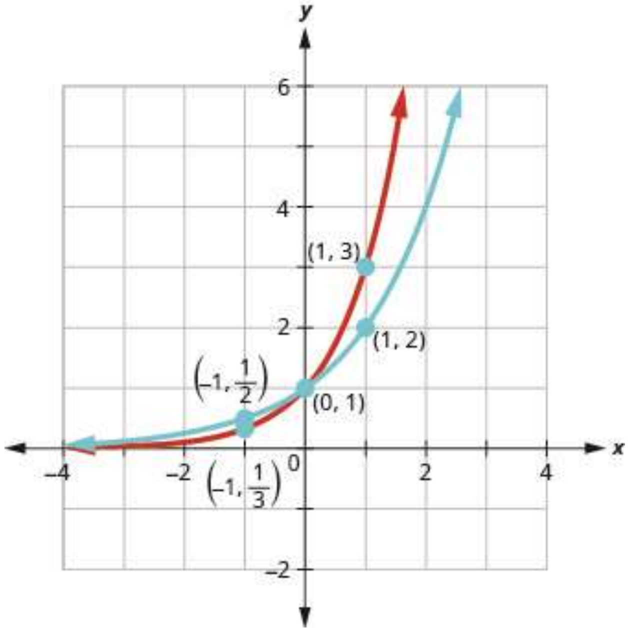

Fig. 1: PNM and ADM solutions.

Example (5.2):
Let non-linear FDE,

$$a\frac{d^3x(t)}{dt}+b\frac{d^{\alpha_2}X(t)}{dt}+c\frac{d^{\alpha_1}X(t)}{dt}+ex^2=f(t) \tag{31}$$

$$0<\alpha_1\le 1, 1<\alpha_2<2,$$

$$f(t)=\frac{ct^{1-\alpha_1}}{\Gamma(1-\alpha_1)}+et^2$$

$$x(0)=0, x'(0)=1, x''(0)=0$$

The exact solution x(t)=t^3.

X(t)=x(t)-t

Equation (31):

$$a\frac{d^3X(t)}{dt}+b\frac{d^{\alpha_2}X(t)}{dt}+c\frac{d^{\alpha_1}X(t)}{dt}+eX^2+2etX=0$$

$$X=X'=x''=0 \tag{32}$$

By algorithm of equation (32) is:

$$X_m = \frac{1}{ah^{-3} + bh^{-\alpha_2} + ch^{-\alpha_1}} \left(ah^{-3}(3X_{m-1} - 3X_{m-2} + X_{m-3}) - bh^{-\alpha_2} \sum_{j=1}^{m} w_j^{\alpha_2} X_{m-j} - c\, h^{-\alpha_1} \sum_{j=1}^{m} w_j^{\alpha_2} X_{m-j} - c - e(X_{m-1})^2 - 2et_m X_{m-1} \right),$$

$$X_0 = X_1 = X_2 = 0, m = 3,4, \dots.$$

Finally, the solution of equation (31) is

$$x_m = X_m + t_m.$$

Table 3. illustrates the results passed from PNM and ADM solutions.

Table 3. Absolute Error at (t=1)

PNM	ADM
h	N
0.1	2
0.01	4
0.001	6

We used two methods to solve FDEs, each method has an advantage over the other. If the solution is needed in a narrow interval, ADM is preferred to be used, as it gives a more accurate solution but if the solution is needed in a wide interval, PNM is preferred to be used (see the result in Table 3). We see that after we overcome the disadvantage of the numerical method it gives a more accurate solution than the numerical methods.

6 Conclusion

We used the ADM for solving the non-linear fractional differential equations, we introduced some new theorems are give the existence, uniqueness, convergence, and maximum absolute truncation error to the Adomian decomposition method series solution when applied to these equations. Some numerical examples are discussed and solved by using the Adomian decomposition method.

We see from the results that the exact error coincides with the approximate error obtained from using the theorems, see for example.

We use a numerical method for comparison, we see that after we overcome the disadvantage of this method. In the two methods that we used to solve fractional differential equations (ADM with numerical method), each method has an advantage over the other.

The method is still open for investigation, especially in fractional differential equations with higher orders.

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