Robust Tracking and Disturbance Rejection for Decentralized Neutral Distributed-Time-Delay Systems

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Abstract: - An important problem in control engineering is to design a controller to achieve robust asymptotic tracking of certain reference signals despite certain disturbance inputs. In the present work, this problem, which is known as the robust servomechanism problem, is considered for decentralized linear time-invariant (LTI) neutral systems with distributed time-delay. However, the system is also allowed to have discrete time-delays besides distributed time-delays. The reference signals and the disturbance input are assumed to satisfy a LTI neutral delay-differential equation with distributed and/or discrete time-delays. The necessary and sufficient conditions for the existence of a controller which solves this problem are derived. The structure of this controller (when it exists) is also presented.

Key-Words: - Servomechanism problem, Decentralized control, Neutral time-delay systems, Distributed time-delay, Robust control, Tracking, Disturbance rejection

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1 Introduction

Controller design has, in general, two aims: stability and performance. One of the most common performance requirements is to track given reference signals in the presence of certain disturbance inputs. The conroller design problem to achieve robust tracking and disturbance rejection, besides stability, is known as the *robust servomechanism problem*. This problem has been studied extensively for linear time-invariant (LTI) finite-dimensional systems (e.g., see, [1], [2], [3], [4], [5]; a system is said to be *finite-dimensional* if its state can be represented by a finite-dimensional vector). This problem has also been studied for finite-dimensional nonlinear, [6], [7], [8], and discrete-event, [9], systems.

For many systems, generally known as large-scale systems, it is not feasible, if not impossible, to collect all the measured outputs in a centralized place, calculate the control inputs centrally and then let various actuators apply these inputs. Thus, one has to use a *decentralized control structure* in such a case, [10]. When such a structure is imposed, the robust servomechanism problem is known as the *decentralized robust servomechanism problem*, which has also been extensively studied for finite-dimensional LTI systems (e.g., [11], [12], [13], [14], [15]).

Many systems, especially large-scale systems, however, may involve time-delays which must be taken into account during controller design. Such systems are generally called as *time-delay systems*. The state of a time-delay system can not be represented by a finite-dimensional vector. Thus, these systems are in the class of *infinite-dimensional systems*, [16]. The dynamics of time-delay systems can, in general, be described by delay-differential equations, [17]. If these equations do not involve the delayed derivative of the state vector, then the corresponding time-delay system is said to be *retarded*. Otherwise, it is said to be *neutral*. It is known that a retarded LTI time-delay system can have only finitely many modes in any right-half complex-plane, [18]. However, a neutral time-delay system can have chains of infinitely many modes going to infinity along a vertical axis. Thus, in general, it is more difficult to control a neutral system, compared to a retarded system.

The time-delays in a system can be discrete or distributed, [19]. In fact, many systems may involve both types of time-delays at the same time, [20]. However, it is possible to represent both discrete and distributed time-delays together using Dirac-delta functions in a distributed-time-delay formulation (see Section 2 below). Thus, distributed-time-delay systems are more general than discrete-time-delay systems.

Robust servomechanism problem has recently been considered for time-delay systems in [21], [22], [23], among others. The necessary and sufficient conditions for the solvability of the (centralized) robust servomechanism problem for LTI time-delay systems, has first been presented in [24]. In [24], however, only retarded discrete-time-delay systems were considered. Retarded systems with distributed time-delay were later considered in [25]. Neutral systems with discrete and distributed time-delays were respectively considered in [26], [27]. Results of [26], were then extended to descriptor-type systems in [28] (a time-delay system described by delay-algebraic-differential equations, rather than delay-differential equations, is said to be of descriptor-type).

Robust servomechanism problem for decentralized time-delay systems was first considered in [29] for retarded and in [30], for neutral systems. In [29], [30], however, only systems with discrete time-delays were considered. Retarded distributed-time-delay systems were recently considered in [31]. In the present work, we extend the results of [31], to neutral systems with distributed time-delay. However, we use a formulation in which we can represent discrete and distributed time-delays together. We consider references and disturbances which satisfy a LTI delay-differential equation of neutral type with distributed and/or discrete time-delays. As special cases, this formulation also allows references and disturbances which satisfy a LTI delay-differential equation of retarded type and/or a simple LTI differential equation. We first present the necessary and sufficient conditions for the existence of a controller which solves this problem. We then present the structure of this controller when it exists.

Throughout the paper, $\mathbf{R}$ and $\mathbf{C}$ respectively denote the sets of real and complex numbers. For $s \in \mathbf{C}$, $\mathrm{Re}(s)$ denotes the real part of s. For positive integers k and l, $\mathbf{R}^k$ and $\mathbf{R}^{k\times l}$ respectively denote the spaces of k dimensional real vectors and $k \times l$ dimensional real matrices. I_k and $0_{k\times l}$ respectively denote the $k \times k$ dimensional identity matrix and the $k \times l$ dimensional zero matrix. When the dimensions are apparent, I and 0 are used to denote, respectively, the identity and the zero matrices of appropriate dimensions. For $\xi : \mathcal{I} \to \mathbf{R}^k$, where $\mathcal{I}$ is an interval of the real line, $\dot{\xi}$, $\ddot{\xi}$, and $\xi^{(l)}$ respectively denote the first, the second, and the l^{th} derivative of ξ. $\det(\cdot)$ and $\mathrm{rank}(\cdot)$ respectively denote the determinant and the rank of the indicated matrix. $\mathrm{bdiag}(\cdots)$ denotes a block diagonal matrix with indicated matrices on the main diagonal, $\otimes$ denotes the Kronecker product, and $\cdot^T$ denotes the transpose.

2 Problem Statement

Consider a decentralized LTI neutral distributed-time-delay system with μ control agents, which is described by:

$$\dot{x}(t) + \int_{-\bar{\tau}}^{0} F(\tau)\dot{x}(t+\tau)d\tau = \int_{-\bar{\tau}}^{0} \left[A(\tau)x(t+\tau) + \sum_{k=1}^{\mu} B_k(\tau)u_k(t+\tau) + E(\tau)w(t+\tau) \right] d\tau \tag{1}$$

$$z_k(t) = \int_{-\bar{\tau}}^{0} \Big[C_k(\tau)x(t+\tau) + \sum_{l=1}^{\mu} D_{k,l}(\tau)u_l(t+\tau) + G_k(\tau)w(t+\tau) \Big] d\tau \ , \quad k = 1,\ldots,\mu \tag{2}$$

$$y_k(t) = \int_{-\bar{\tau}}^{0} \Big[H_k(\tau)x(t+\tau) + \sum_{l=1}^{\mu} J_{k,l}(\tau)u_l(t+\tau) + L_k(\tau)w(t+\tau) \Big] d\tau \ , \quad k = 1,\ldots,\mu \tag{3}$$

Here $x(t) \in \mathbf{R}^n$ is the state vector and $w(t) \in \mathbf{R}^r$ is the *disturbance input* at time t. Furthermore, $u_k(t) \in \mathbf{R}^{p_k}$ is the *control input*, $z_k(t) \in \mathbf{R}^{q_k}$ is the *output*, and $y_k(t) \in \mathbf{R}^{m_k}$ is the *measurement* at time t, for the k^{th} control agent, $k = 1,\ldots,\mu$. Finally, $\bar{\tau} > 0$ is the *maximum time-delay* in the system. Moreover, $A : [-\bar{\tau}, 0] \to \mathbf{R}^{n\times n}$, $F : [-\bar{\tau}, 0] \to \mathbf{R}^{n\times n}$, $B_k : [-\bar{\tau}, 0] \to \mathbf{R}^{n\times p_k}$, $C_k : [-\bar{\tau}, 0] \to \mathbf{R}^{q_k\times n}$,

$D_{k,l} : [-\bar{\tau}, 0] \to \mathbf{R}^{q_k \times p_l}$, $H_k : [-\bar{\tau}, 0] \to \mathbf{R}^{m_k \times n}$, and $J_{k,l} : [-\bar{\tau}, 0] \to \mathbf{R}^{m_k \times p_l}$ ($k, l = 1, \ldots, \mu$) are matrix functions which are known nominally, and $E : [-\bar{\tau}, 0] \to \mathbf{R}^{n \times r}$, $G_k : [-\bar{\tau}, 0] \to \mathbf{R}^{q_k \times r}$, and $L_k : [-\bar{\tau}, 0] \to \mathbf{R}^{m_k \times r}$ ($k = 1, \ldots, \mu$) are arbitrary matrix functions, which are not necessarily known. It is assumed that all of these matrix functions are bounded, except that they may involve Dirac-delta functions, $\delta(\tau + h_i)$, in order to represent finitely many discrete time-delays $h_i \in [0, \bar{\tau}]$ ($i = 1, \ldots, l$, for some positive integer l; including zero delay when $h_i = 0$). However, it is assumed that $F(\tau)$ does not involve $\delta(\tau + h_i)$ for $h_i = 0$, but may involve $\delta(\tau + h_i)$ for $h_i > 0$. This assumption guarantees that the given system is not of descriptor-type. Otherwise, however, it does not impose any restrictions, since the system dynamics (1) already includes $\dot{x}(t)$.

Here, $\delta(\tau + h_i)$ is defined as

$$\int_{-\bar{\tau}}^{0} \delta(\tau + h_i)\chi(t + \tau)d\tau = \chi(t - h_i) \qquad (4)$$

for any $\chi : [-\bar{\tau}, \infty) \to \mathbf{R}$.

Also consider a LTI delay-differential operator of neutral type with differential degree ν, defined as

$$\mathcal{D}\chi(t) := \frac{d^\nu}{dt^\nu}\chi(t) + \sum_{l=0}^{\nu} \frac{d^l}{dt^l} \int_{-\bar{\tau}}^{0} \alpha_l(\tau)\chi(t + \tau)d\tau \qquad (5)$$

where $\chi : [-\bar{\tau}, \infty) \to \mathbf{R}$ is any at least ν times differentiable function. Here, $\alpha_l : [-\bar{\tau}, 0] \to \mathbf{R}$, $l = 0, \ldots, \nu$, are bounded functions, except that they may involve Dirac-delta functions $\delta(\tau + \gamma_i)$ for finitely many $\gamma_i \in [0, \bar{\tau}]$, which represent discrete time-delays. It is assumed, however, that $\alpha_\nu(0)$ is bounded, i.e., $\alpha_\nu(\tau)$ does not involve $\delta(\tau)$, but may involve $\delta(\tau + \gamma_i)$, for $\gamma_i > 0$.

In our problem, the k^{th} control input, $u_k(t)$, is to be determined by the k^{th} control agent based on the k^{th} measurement, $y_k(t)$, and the k^{th} *reference*, $r_k(t)$ ($k = 1, \ldots, \mu$), so that the overall closed-loop system is asymptotically stable and the k^{th} *tracking error*

$$e_k(t) := z_k(t) - r_k(t) \qquad (6)$$

satisfies

$$\lim_{t \to \infty} e_k(t) = 0\ , \quad k = 1, \ldots, \mu \qquad (7)$$

None of the references is necessarily known in advance, however, it is assumed that they satisfy

$$\mathcal{D}r_k(t) = 0\ , \quad k = 1, \ldots, \mu \qquad (8)$$

The disturbance input, $w(t)$, which is not generally available, is also assumed to satisfy

$$\mathcal{D}w(t) = 0 \qquad (9)$$

Remark 1: It is of course possible that the disturbance input $w(t)$ and each reference $r_k(t)$ satisfy different delay-differential equations. That is, there may exist $\mu + 1$ different LTI delay-differential operators of neutral type, $\mathcal{D}^0$, $\mathcal{D}^1$, ..., $\mathcal{D}^\mu$, such that $\mathcal{D}^0 w(t) = 0$, $\mathcal{D}^1 r_1(t) = 0$, ..., and $\mathcal{D}^\mu r_\mu(t) = 0$. In such a case, however, it is possible to find LTI delay-differential operators, $\mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_\mu$, such that (8) and (9) are satisfied, where $\mathcal{D} := \mathcal{D}_0\mathcal{D}^0 = \mathcal{D}_1\mathcal{D}^1 = \ldots = \mathcal{D}_\mu\mathcal{D}^\mu$. It is also possible that the maximum time-delay in the system and in $\mathcal{D}$ are different. In such a case, however, maximum of the two delays can be taken as $\bar{\tau}$ and the functions in (5) can be extended by zero functions (in case the system has a larger maximum delay) or the matrix functions in (1)–(3) can be extended by zero matrix functions (in case the operator (5) has a larger maximum delay).

Remark 2: In (5) we assumed the most general form of LTI neutral delay-differential operators. However, as special cases, this operator also includes LTI retarded delay-differential operators (in which case $\alpha_\nu(\tau) = 0$, $\tau \in [-\bar{\tau}, 0]$), as well as simple LTI differential operators. In fact, simple LTI differential operators may be more common in practice. As an example, if the the references and the disturbance are constant signals, then we should have $\mathcal{D} = \frac{d}{dt}$. This, however, can be written as in (5) with $\nu = 1$ and $\alpha_0(\tau) = \alpha_1(\tau) = 0$, $\tau \in [-\bar{\tau}, 0]$. If any one of these signals also involve a sinusoidal signal of frequency f_o, then $\mathcal{D} = \frac{d^3}{dt^3} + \omega_0^2 \frac{d}{dt}$, which can be written as in (5) with $\nu = 3$, $\alpha_0(\tau) = \alpha_2(\tau) = \alpha_3(\tau) = 0$, and $\alpha_1(\tau) = \omega_0^2 \delta(\tau)$, $\tau \in [-\bar{\tau}, 0]$, where $\omega_0 = 2\pi f_o$.

Our problem can now be stated as follows:

Problem P: Find μ decentralized LTI feedback controllers (from y_k to u_k, $k = 1, \ldots, \mu$) for the system (1)–(3), such that the closed-loop system is asymtotically stable and, for all initial condtions of the system (1), for all disturbances $w(t)$ satisfying (9), for all references $r_k(t)$, satisfying (8), and for all non-destabilizing matrix functions appearing in (1)–(3), (7) is satisfied.

3 Preliminaries

Let $\bar{F}(s) := I + \int_{-\bar{\tau}}^{0} F(\tau)e^{s\tau}d\tau$ and $\bar{A}(s) := \int_{-\bar{\tau}}^{0} A(\tau)e^{s\tau}d\tau$. The function

$\phi(s) := \det\left(s\bar{F}(s) - \bar{A}(s)\right)$ is then known as the *characteristic function* of (1). A *mode* of (1) is any $\lambda \in \mathbf{C}$ which satisfies $\phi(\lambda) = 0$. If $\mathrm{Re}(\lambda) \geq 0$, then such a mode is known as an *unstable mode* of (1). Asymptotic stability of (1) is equivalent to the condition that (1) has no unstable modes, [32]. In general, there exist infinitely many modes of (1). However, there exist only finitely many modes (if any) λ of (1) with $\mathrm{Re}(\lambda) > \gamma_f$, for some finite $\gamma_f \in \mathbf{R}$, [32].

Next, suppose that the following decentralized LTI static feedbacks:

$$u_k(t) = K_k y_k(t)\ , \quad k = 1, \ldots, \mu \qquad (10)$$

are applied to the system (1,3). Here $K_k \in \mathbf{R}^{p_k \times m_k}$ are arbitrary, except that they satisfy $\det(I - KJ_0) \neq 0$, where $K := \mathrm{bdiag}(K_1, \ldots, K_\mu)$ and $J_0 := \lim_{s\to\infty} \bar{J}(s)$, where the limit is taken along the positive real axis and

$$\bar{J}(s) := \begin{bmatrix} \bar{J}_{1,1}(s) & \cdots & \bar{J}_{1,\mu}(s) \\ \vdots & & \vdots \\ \bar{J}_{\mu,1}(s) & \cdots & \bar{J}_{\mu,\mu}(s) \end{bmatrix}$$

where $\bar{J}_{k,l}(s) := \int_{-\bar{\tau}}^{0} J_{k,l}(\tau)e^{s\tau}d\tau$. This condition guarantees the well-posedness of the closed-loop system, [33]. A mode of (1) which remains a mode of the closed-loop system under all controls of the form (10) is known as a *decentralized fixed mode (DFM)* of (1,3), [34]. A necessary and sufficient condition for $\lambda \in \mathbf{C}$ to be a DFM of (1,3) is that for some $\kappa \in \{0, \ldots, \mu\}$ and $\{k_1, \ldots, k_\kappa\} \subset \{1, \ldots, \mu\}$, where $k_1, \ldots, k_\kappa$ are distinct ($\{k_1, \ldots, k_\kappa\} = \emptyset$ if $\kappa = 0$),

$$\mathrm{rank} \begin{bmatrix} \bar{A}(\lambda) - s\bar{F}(\lambda) & \bar{B}_{k_1}(\lambda) & \cdots & \bar{B}_{k_\kappa}(\lambda) \\ \bar{H}_{k_{\kappa+1}}(\lambda) & \bar{J}_{k_{\kappa+1},k_1}(\lambda) & \cdots & \bar{J}_{k_{\kappa+1},k_\kappa}(\lambda) \\ \vdots & \vdots & & \vdots \\ \bar{H}_{k_\mu}(\lambda) & \bar{J}_{k_\mu,k_1}(\lambda) & \cdots & \bar{J}_{k_\mu,k_\kappa}(\lambda) \end{bmatrix} < n \qquad (11)$$

where $\{k_{\kappa+1}, \ldots, k_\mu\} := \{1, \ldots, \mu\} \setminus \{k_1, \ldots, k_\kappa\}$, $\bar{B}_k(s) := \int_{-\bar{\tau}}^{0} B_k(\tau)e^{s\tau}d\tau$, and $\bar{H}_k(s) := \int_{-\bar{\tau}}^{0} H_k(\tau)e^{s\tau}d\tau$, [33]. A necessary and sufficient condition for the existence of a (possibly dynamic) decentralized LTI feedback controller which asymptotically stabilizes (1,3) is that (1,3) must not have any unstable DFMs, [33].

4 Main Results

In this section we will present the necessary and sufficient conditions for the existence of a controller which solves Problem P. We will then present the structure of this controller when these conditions are satisfied. For this purpose, we first define

$$\mathcal{A}(\tau) := \begin{bmatrix} 0 & 0 & \cdots & 0 & -\alpha_0(\tau) \\ \delta(\tau) & 0 & & & -\alpha_1(\tau) \\ 0 & \delta(\tau) & & & -\alpha_2(\tau) \\ \vdots & & \ddots & & \vdots \\ 0 & & & \delta(\tau) & -\alpha_{\nu-1}(\tau) \end{bmatrix}$$

and

$$\mathcal{F}(\tau) := \begin{bmatrix} 0_{\nu-1\times\nu-1} & 0_{\nu-1\times1} \\ 0_{1\times\nu-1} & \alpha_\nu(\tau) \end{bmatrix}$$

where $\alpha_l(\cdot)$, $l = 0, \ldots, \nu$, are the functions appearing in (5).

Next, let us define a *fictitious system*:

$$\dot{p}(t) + \int_{-\bar{\tau}}^{0} \mathcal{F}(\tau)\dot{p}(t+\tau)d\tau = \int_{-\bar{\tau}}^{0} \mathcal{A}(\tau)p(t+\tau)d\tau \qquad (12)$$

Then, we can write any $r_k(t)$ and any $w(t)$, respectively satisfying (8) and (9), as

$$r_k(t) = C_k^r p(t)\ , \quad k = 1, \ldots, \mu, \qquad w(t) = C^w p(t) \qquad (13)$$

for some arbitrary constant matrices C_k^r, $k = 1, \ldots, \mu$, and C^w. These matrices are arbitrary, since, apart from the fact that they respectively satisfy (8) and (9), $r_k(t)$ and $w(t)$ are assumed to be unknown. Because of the same reason, the initial condition of the system (12) is also arbitrary.

Next, let us state the following assumptions:

Assumption 1: System (12) together with the output

$$\begin{bmatrix} r(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} C^r \\ C^w \end{bmatrix} p(t) \qquad (14)$$

where $C^r := \begin{bmatrix} C_1^r \\ \vdots \\ C_\mu^r \end{bmatrix}$, is observable.

Assumption 2: $\begin{bmatrix} E(\tau) \\ G(\tau) \\ L(\tau) \end{bmatrix} \chi = 0$, $\forall \tau \in [-\bar{\tau}, 0]$, where χ is a constant vector, implies that $\chi = 0$, where $G(\cdot) := \begin{bmatrix} G_1(\cdot) \\ \vdots \\ G_\mu(\cdot) \end{bmatrix}$ and $L(\cdot) := \begin{bmatrix} L_1(\cdot) \\ \vdots \\ L_\mu(\cdot) \end{bmatrix}$.

Assumption 3: For any solution $p(t)$ of (12), $\lim_{t\to\infty} p(t) = 0$, only if $p(\tau) = 0$, $\forall \tau \in [-\hat{\tau}, 0]$, where $\hat{\tau} \geq 0$ is the maximum time-delay in $\mathcal{D}$ (which may be smaller than $\bar{\tau}$ - see Remark 1).

Assumption 4: $B(\tau)\chi = 0$, $\forall \tau \in [-\bar{\tau}, 0]$, where χ is a constant vector, implies that $\chi = 0$, where $B(\cdot) := \begin{bmatrix} B_1(\cdot) & \cdots & B_\mu(\cdot) \end{bmatrix}$.

Assumption 5: $\chi^T C(\tau) = 0$, $\forall \tau \in [-\bar{\tau}, 0]$, where χ is a constant vector, implies that $\chi = 0$, where $C(\cdot) := \begin{bmatrix} C_1(\cdot) \\ \vdots \\ C_\mu(\cdot) \end{bmatrix}$.

Remark 3: Assumptions 1–5 can be made without loss of generality. They are made only to avoid triviality. Assumption 1 implies that the fictitious system does not include any dynamics whose effect do not appear in both the reference and the disturbance. If so, these dynamics can be deleted. Assumption 2 implies that the shown matrix function does not have any linearly dependent columns. If it does, these columns, as well as the corresponding elements of w, can be deleted. Assumption 3 implies that the fictitious system does not include any stable dynamics. Since our problem is asymptotic tracking, stable dynamics are irrelevant and can be deleted. Assumption 4 implies that $B(\cdot)$ does not have any linearly dependent columns. If it does, these columns, together with the corresponding elements of $u := \begin{bmatrix} u_1 \\ \vdots \\ u_\mu \end{bmatrix}$ can be deleted. Finally, Assumption 5 implies that $C(\cdot)$ does not have any linearly dependent rows. If it does, these rows, together with the corresponding elements of $z := \begin{bmatrix} z_1 \\ \vdots \\ z_\mu \end{bmatrix}$ can be deleted.

Next, let $\hat{\mathcal{A}}(\cdot) := \mathcal{A}(\cdot) \otimes I_q$, $\hat{\mathcal{F}}(\cdot) := \mathcal{F}(\cdot) \otimes I_q$, and $\mathcal{B} := \begin{bmatrix} I_q \\ 0_{\hat{q}\times q} \end{bmatrix}$, where $q := \sum_{k=1}^{\mu} q_k$ and $\hat{q} := (\nu - 1)q$. Also, for $k = 1, \ldots, \mu$, let $D_k(\cdot) := \begin{bmatrix} D_{1,k}(\cdot) \\ \vdots \\ D_{\mu,k}(\cdot) \end{bmatrix}$ and $\mathcal{C}_k := \begin{bmatrix} 0_{q_k\times q_k^-} & I_{q_k} & 0_{q_k\times q_k^+} \end{bmatrix}$, where $q_k^- := \sum_{l=1}^{k-1} q_k$ ($q_k^- = 0$ if $k = 1$) and $q_k^+ := \sum_{l=k+1}^{\mu} q_k$ ($q_k^+ = 0$ if $k = \mu$). Finally, let $\hat{\mathcal{C}}_k := I_\nu \otimes \mathcal{C}_k$.

Now, we can present the necessary and sufficient conditions for the existence of a controller which solves Problem P:

Theorem 1: Under Assumptions 1–5, there exists a controller which solves Problem P if and only if the following hold:

1) For $k = 1, \ldots, \mu$, the output z_k, given in (2), is contained in the measurement y_k, given in (3). That is $z_k(t) = T_k y_k(t)$, $\forall t \geq 0$, for some $T_k \in \mathbf{R}^{q_k\times m_k}$.
2) The system

$$\dot{\xi}(t) + \int_{-\bar{\tau}}^{0} \begin{bmatrix} F(\tau) & 0 \\ 0 & \hat{\mathcal{F}}(\tau) \end{bmatrix} \dot{\xi}(t+\tau)d\tau = \int_{-\bar{\tau}}^{0} \left(\begin{bmatrix} A(\tau) & 0 \\ \mathcal{B}C(\tau) & \hat{\mathcal{A}}(\tau) \end{bmatrix} \xi(t+\tau) + \sum_{k=1}^{\mu} \begin{bmatrix} B_k(\tau) \\ \mathcal{B}D_k(\tau) \end{bmatrix} \tilde{u}_k(t+\tau) \right) d\tau \quad (15)$$

$$\tilde{v}_k(t) = \int_{-\bar{\tau}}^{0} \left(\begin{bmatrix} H_k(\tau) & 0 \\ 0 & \delta(\tau)\hat{\mathcal{C}}_k \end{bmatrix} \xi(t+\tau) + \sum_{l=1}^{\mu} \begin{bmatrix} J_{k,l}(\tau) \\ 0 \end{bmatrix} \tilde{u}_l(t+\tau) \right), \quad k = 1, \ldots, \mu \quad (16)$$

has no unstable DFMs.

Proof: Let us first prove the *only if* part. Since the problem requires robust tracking (i.e., (7) must be satisfied robustly), $e_k(t)$ must be available to the k^{th} control agent for each $k = 1, \ldots, \mu$. However, since the k^{th} control agent can only access $r_k(t)$ and $y_k(t)$, in order to calculate $e_k(t)$ from (6), $z_k(t)$ must be contained in $y_k(t)$. This implies the necessity of Condition 1.

To prove the necessity of Condition 2, let

$$\tilde{e}(t) := \begin{bmatrix} e^{\nu-1}(t) \\ \vdots \\ e^1(t) \\ e(t) \end{bmatrix} \quad (17)$$

where $e(t) := \begin{bmatrix} e_1(t) \\ \vdots \\ e_\mu(t) \end{bmatrix}$,

$$e^1(t) := \dot{e}(t) + \int_{-\bar{\tau}}^{0} \alpha_\nu(\tau)\dot{e}(t+\tau)d\tau + \int_{-\bar{\tau}}^{0} \alpha_{\nu-1}(\tau)e(t+\tau)d\tau$$

$$e^2(t) := \ddot{e}(t) + \frac{d}{dt}\int_{-\bar{\tau}}^{0} \alpha_\nu(\tau)\dot{e}(t+\tau)d\tau + \frac{d}{dt}\int_{-\bar{\tau}}^{0} \alpha_{\nu-1}(\tau)e(t+\tau)d\tau + \int_{-\bar{\tau}}^{0} \alpha_{\nu-2}(\tau)e(t+\tau)d\tau$$

$$\vdots$$

$$e^{\nu-1}(t) := e^{(\nu-1)}(t) + \frac{d^{\nu-2}}{dt^{\nu-2}}\int_{-\bar{\tau}}^{0} \alpha_\nu(\tau)\dot{e}(t+\tau)d\tau + \frac{d^{\nu-2}}{dt^{\nu-2}}\int_{-\bar{\tau}}^{0} \alpha_{\nu-1}(\tau)e(t+\tau)d\tau + \frac{d^{\nu-3}}{dt^{\nu-3}}\int_{-\bar{\tau}}^{0} \alpha_{\nu-2}(\tau)e(t+\tau)d\tau + \ldots + \int_{-\bar{\tau}}^{0} \alpha_1(\tau)e(t+\tau)d\tau$$

Then, we obtain

$$\dot{e}(t) + \int_{-\bar{\tau}}^{0} \alpha_\nu(\tau)\dot{e}(t+\tau)d\tau = e^1(t) - \int_{-\bar{\tau}}^{0} \alpha_{\nu-1}(\tau)e(t+\tau)d\tau \ , \quad (18)$$

$$\dot{e}^k(t) = e^{k+1}(t) - \int_{-\bar{\tau}}^{0} \alpha_{\nu-k-1}(\tau)e(t+\tau)d\tau \ , \quad k = 1,\ldots,\nu-2 \ , \quad (19)$$

and

$$\dot{e}^{\nu-1}(t) = \mathcal{D}e(t) - \int_{-\bar{\tau}}^{0} \alpha_0(\tau)e(t+\tau)d\tau \quad (20)$$

Now, let $\tilde{x}(t) := \mathcal{D}x(t)$, $\tilde{u}_k(t) := \mathcal{D}u_k(t)$, $k = 1,\ldots,\mu$, and $\xi(t) := \begin{bmatrix} \tilde{x}(t) \\ \tilde{e}(t) \end{bmatrix}$. Then, use (1), (2), (6), (8), and (9) in (18)–(20) to obtain (15) and

$$e_k(t) = \begin{bmatrix} 0_{q_k \times (n+\hat{q})} & \mathcal{C}_k \end{bmatrix} \xi(t) \ , \quad k = 1,\ldots,\mu \quad (21)$$

Therefore, in order to achieve (7), the part of the system (15) which is observable through (21) must be stabilized using inputs $\tilde{u}_k(t)$. Since the part of (15) which corresponds to $\tilde{e}(t)$, however, is observable through (21) and the remaining part, which corresponds to $\tilde{x}(t)$ (which is simply the given system (1)) must also be stabilized as a problem requirement, the whole system (15) must be stabilized. Furthermore, this stabilization must be achieved by decentralized LTI feedback, where the k^{th} control agent can access $y_k(t)$ and $r_k(t)$. By Condition 1, however, using $y_k(t)$, the k^{th} control agent can obtain $z_k(t)$ and, using $z_k(t)$ and $r_k(t)$, can also obtain $e_k(t)$. Thus, both $\tilde{y}_k(t) := \mathcal{D}y_k(t)$ and $\tilde{e}_k(t) := \hat{\mathcal{C}}_k\tilde{e}(t)$ can be obtained by the k^{th} control agent. However, since $\begin{bmatrix} \tilde{y}_k(t) \\ \tilde{e}_k(t) \end{bmatrix} = \tilde{v}_k(t)$, given in (16), this means that the k^{th} control agent, which is to determine $\tilde{u}_k(t)$, can access $\tilde{v}_k(t)$. Therefore, to solve Problem P under Assumptions 1–5, a decentralized LTI controller which asymptotically stabilizes the system (15,16) must exist. This, however, is equivalent to Condition 2. This proves the *only if* part.

We will prove the *if* part constructively. As remarked above, due to Condition 1, the k^{th} controller can use

$$e_k(t) = T_k y_k(t) - r_k(t) \quad (22)$$

Then, the k^{th} control agent can build:

$$\dot{s}_k(t) + \int_{-\bar{\tau}}^{0} \hat{\mathcal{F}}_k(\tau)\dot{s}_k(t+\tau)d\tau = \int_{-\bar{\tau}}^{0} \hat{\mathcal{A}}_k(\tau)s_k(t+\tau)d\tau + \mathcal{B}_k e_k(t) \quad (23)$$

which is to be called as the k^{th} *servocompensator*. Here, $\hat{\mathcal{A}}_k(\cdot) := \mathcal{A}(\cdot) \otimes I_{q_k}$, $\hat{\mathcal{F}}_k(\cdot) := \mathcal{F}(\cdot) \otimes I_{q_k}$, and $\mathcal{B}_k := \begin{bmatrix} I_{q_k} \\ 0_{\hat{q}_k \times q_k} \end{bmatrix}$, where $\hat{q}_k := (\nu-1)q_k$, and $s_k(t) \in \mathbf{R}^{\nu q_k}$ is the state vector at time t.

Let $s(t) := \begin{bmatrix} s_1(t) \\ \vdots \\ s_\mu(t) \end{bmatrix}$ and $\hat{s}(t) := Ms(t)$, where

$$M := \begin{bmatrix} \text{bdiag}(M_{1,1},\ldots,M_{1,\mu}) \\ \vdots \\ \text{bdiag}(M_{\nu,1},\ldots,M_{\nu,\mu}) \end{bmatrix}$$

where $M_{i,k} := \begin{bmatrix} 0_{q_k \times q_k^{i-}} & I_{q_k} & 0_{q_k \times q_k^{i+}} \end{bmatrix}$, where $q_k^{i-} := (i-1)q_k$ and $q_k^{i+} := (\nu-i)q_k$, $k = 1,\ldots,\mu$, $i = 1,\ldots,\nu$. Note that $M \in \mathbf{R}^{\nu q \times \nu q}$ is invertible. Let $\eta(t) := \begin{bmatrix} x(t) \\ \hat{s}(t) \end{bmatrix}$. Then, we can describe the augmented dynamics of the plant (1) and the μ decentralized servocompensators (23) as:

$$\dot{\eta}(t) + \int_{-\bar{\tau}}^{0} \begin{bmatrix} F(\tau) & 0 \\ 0 & \hat{\mathcal{F}}(\tau) \end{bmatrix} \dot{\eta}(t+\tau) = \int_{-\bar{\tau}}^{0} \left(\begin{bmatrix} A(\tau) & 0 \\ \mathcal{B}C(\tau) & \hat{\mathcal{A}}(\tau) \end{bmatrix} \eta(t+\tau) + \sum_{k=1}^{\mu} \begin{bmatrix} B_k(\tau) \\ \mathcal{B}D_k(\tau) \end{bmatrix} u_k(t+\tau) \right.$$

$$+ \left[\begin{array}{c} E(\tau) \\ \mathcal{B}G(\tau) \end{array} \right] w(t+\tau) \Big) d\tau - \left[\begin{array}{c} 0 \\ \mathcal{B} \end{array} \right] r(t) \quad (24)$$

where $r(t) := \left[\begin{array}{c} r_1(t) \\ \vdots \\ r_\mu(t) \end{array} \right]$.

From this augmented system, the k^{th} control agent, $k = 1, \ldots, \mu$, can measure

$$\begin{aligned} v_k(t) &:= \left[\begin{array}{c} y_k(t) \\ s_k(t) \end{array} \right] \\ &= \int_{-\bar{\tau}}^{0} \Big(\left[\begin{array}{cc} H_k(\tau) & 0 \\ 0 & \delta(\tau)\hat{\mathcal{C}}_k \end{array} \right] \eta(t+\tau) \\ &\quad + \sum_{l=1}^{\mu} \left[\begin{array}{c} J_{k,l}(\tau) \\ 0 \end{array} \right] u_l(t+\tau) \\ &\quad + \left[\begin{array}{c} L_k(\tau) \\ 0 \end{array} \right] w(t+\tau) \Big) d\tau \quad (25) \end{aligned}$$

Note that the systems (24,25) and (15,16) are equivalent except for the existence of external inputs w and r in (24,25) (which stay outside the loop when the control loop is closed). Therefore, by Condition 2, there exist μ decentralized controllers (to be called as the *stabilizing compensators*) that asymptotically stabilize the system (24,25). Furthermore, the same controllers also asymptotically stabilize the system (15,16). Thus, when these controllers are applied, we obtain $\lim_{t\to\infty} \xi(t) = 0$ (this is because, (15)–(16) has no external inputs). By (21), however, this means (7). Thus, these controllers achieve both asymptotic stability and tracking. This completes the proof. □

From the *if* part of the above proof, it is evident that each of the μ the decentralized controllers (when they exist) which solve Problem P are composed of two parts:

i) A *servocompensator* which is descried by (23) with its input (22).
ii) A *stabilizing compensator* with inputs $y_k(t)$ and $s_k(t)$ and output $u_k(t)$.

The servocompensators are responsible for achieving the tracking and rejecting the disturbance. The dynamics of these compensators are determined by the delay-differential equation (5) (equivalently by the fictitious system (12)) by which the references and the disturbance is satisfied.

The stabilizing compensators, on the other hand, are designed to asymptotically stabilize the augmented system (24,25). These compensators can be designed using any decentralized stabilizing controller design method developed for time-delay systems (e.g., see, [35], [36], [37], [38], and references therein). A recently developed software package, [39], may in particular be useful for this purpose.

Remark 4: It was shown in [40], that a LTI time-delay (centralized) system can be stabilized by LTI time-delay controllers if and only if it can be stabilized by LTI finite-dimensional controllers. This result was extended to the decentralized case in [33]. More specifically, it was shown in [33], that, a decentralized LTI time-delay system can be stabilized by LTI time-delay controllers if and only if it can be stabilized by LTI finite-dimensional controllers (however, time-delay controllers may be advantageous in some cases - see, [37], [41], [42]). Therefore, when a solution to Problem P exists, the stabilizing compensators can be designed as finite-dimensional systems. However, since the servocompensators are determined by the delay-differential operator (5), these compensators must be time-delay systems if (5) in fact involves time-delays. As stated in Remark 2, however, references and disturbance usually satisfy simple LTI differential equations, rather than delay-differential equations. In this case, then, when conditions of Theorem 1 hold, Problem P can be solved using LTI finite-dimensional controllers.

5 Conclusions

Robust tracking and disturbance rejection problem has been considered for LTI neutral systems with distributed-time-delay. The formulation, however, allows the representation of discrete time-delays together with distributed time-delay. Thus, the present results can also be used for systems with discrete time-delays, and/or for systems which involve both kinds of delays. The necessary and sufficient conditions for the existence of a controller which solves this problem have been derived. Although certain assumptions (1–5) have been made, as discussed in Remark 3, these assumptions were made only to avoid triviality and can be made without loss of generality. Furthermore, these assumptions are needed only for the necessity of the conditions given in Theorem 1. Thus, even if any one of these assumptions do not hold, there still exist decentralized LTI controllers which solve Problem P, as long as the conditions given in Theorem 1 hold.

It has also been shown that the controller which solves the problem (when exists) is composed of μ decentralized LTI controllers each of which consists of a *servocompensator* and a *stabilizing compensator*. This structure, as well as the conditions given in Theorem 1, are in fact, generalizations of the structure and the conditions given in [11], for LTI finite-dimensional systems with references and the disturbance satisfying a LTI differential equation. This structure and the conditions are also generalizations of the structure and the conditions given in [30], for neutral discrete-time-delay systems with references and the disturbance satisfying a LTI delay-differential equation of neutral type with discrete time-delays and of the structure and the conditions given in [31], for retarded distributed-time-delay systems with references and the disturbance satisfying a LTI delay-differential equation of retarded type with distributed and/or discrete time-delays. Furthermore, when $\mu = 1$, both the structure of the controller and the conditions of Theorem 1 reduce to the structure and the conditions given in [27], for the centralized case.

Finally. as explained in Remark 4, when the conditions of Theorem 1 hold and the references and the disturbance satisfy simple LTI differential equations, rather than delay-differential equations (which is the more common case as discussed in Remark 2), the problem can be solved using LTI finite-dimensional controllers.

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Conflicts of Interest

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