Stabilization of Linear Time-Invariant Systems by State-Derivative Feedback

KONSTADINOS H. KIRITSIS
Hellenic Air Force Academy,
Department of Aeronautical Sciences,
Division of Automatic Control,
Dekelia Air Base, PC 13671,
Acharnes, Attikis, Tatoi,
GREECE

Abstract: - In this paper is studied the stabilization problem by state-derivative feedback for linear time-invariant continuous-time systems. In particular, explicit necessary and sufficient conditions are established for the stability of a closed-loop system, obtained by state-derivative feedback from the given linear time-invariant continuous-time system. Furthermore a procedure is given for the computation of stabilizing state-derivative feedback. Our approach is based on properties of real and polynomial matrices.

Key-Words: - stabilization, state-derivative feedback, linear time-invariant continuous-time systems.

Received: October 15, 2022. Revised: January 17, 2023. Accepted: February 20, 2023. Published: March 24, 2023.

1 Introduction

What are the conditions under which the closed-loop system obtained by state-derivative feedback from a given linear time-invariant continuous-time system is stable? This simple question is known as stabilization of linear time-invariant continuous-time systems by state-derivative feedback. In [1] are established sufficient conditions for the solution of the stabilization problem by state-derivative feedback for linear time-invariant continuous-time systems. In particular is proven that if the given linear time-invariant continuous-time system is either controllable or uncontrollable with stable uncontrollable poles and all its controllable poles are nonzero then the closed-loop system obtained by state- derivative feedback from a given linear time-invariant continuous-time system is stable. In [2], see also [3], is proven that if the given linear time-invariant continuous-time system has at least one zero pole then the closed-loop system obtained by state- derivative feedback from a given linear time-invariant continuous-time system has also at least one zero pole; therefore the stabilization problem by state-derivative feedback has no solution. The state-derivative feedback design methods have been extensively studied over the last twenty years. The motivation for the study of these methods comes from some practical applications such as controlled vibration suppression of mechanical systems, for more complete references we refer the reader to [1-5] and references given therein.

To the best of our knowledge the stabilization problem by state-derivative state feedback for linear time-invariant continuous-time systems in its full generality, is still an open problem. This motivates the present study. In this paper, are established explicit necessary and sufficient conditions for the solution of the stabilization problem by state-derivative feedback for linear time-invariant continuous-time systems. In particular it is proved that the sufficient conditions of [1] for the solution of stabilization problem by state-derivative feedback for linear time-invariant continuous-time systems are also necessary. Furthermore a procedure is given for the computation of stabilizing state- derivative feedback.

2 Problem Formulation

Consider a linear time-invariant continuous-time system described by the following state-space equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \qquad (1)$$

where $\mathbf{A}$ and $\mathbf{B}$ are real matrices of size $(n \times n)$ and $(n \times m)$ respectively, $\mathbf{x}(t)$ is the state vector of dimensions $(n \times 1)$ and $\mathbf{u}(t)$ is the vector of inputs of dimensions $(m \times 1)$. Consider the control law

$$\mathbf{u}(t) = \mathbf{D}\dot{\mathbf{x}}(t) + \mathbf{v}(t) \qquad (2)$$

where $\mathbf{D}$ is a real matrix of size $(m \times n)$ and $\mathbf{v}(t)$ is the reference input vector of size $(m \times 1)$. By applying the state-derivative feedback (2) to the system (1), the state-space equations of closed–loop system are

$$[\mathbf{I} - \mathbf{BD}]\dot{\mathbf{x}}(t) = \mathbf{Ax}(t) + \mathbf{Bv}(t) \qquad (3)$$

Let $\mathcal{R}$ be the field of real numbers. Also let $\mathcal{R}[s]$ be the ring of polynomials with coefficients in. $\mathcal{R}$. The stabilization problem by state-derivative feedback considered in this paper can be stated as follows: Does there exists a state-derivative state feedback (2) such that

$$det[(\mathbf{I}s - (\mathbf{I} - \mathbf{BD})^{-1}\mathbf{A}] = c(s) \qquad (4)$$

where $c(s)$ is a monic, strictly Hurwitz polynomial over $\mathcal{R}[s]$ of degree n (i.e., all roots of $c(s)$ have negative real parts). If so, give conditions for existence and a procedure for the computation of matrix $\mathbf{D}$. It is pointed out that relationship (4) ensures that the closed-loop system (3) is a stable regular state-space system [6].

3 Basic Concepts and Preliminary Results

This section contains lemmas which are needed to prove the main results of this paper and some basic notions from linear control theory that are used throughout the paper. Let C be the field of complex numbers, also let C^+ be the set of complex numbers λ with $Re(\lambda) \geq 0$. A matrix whose elements are polynomials over $\mathcal{R}[s]$ termed polynomial matrix. A polynomial matrix $\mathbf{U}(s)$ over $\mathcal{R}[s]$ of dimensions $(k \times k)$ is said to be unimodular if and only if

$$\det[\mathbf{U}(s)] = \mu \qquad (5)$$

where μ is a finite nonzero real number; therefore every unimodular polynomial matrix has a polynomial inverse. Every polynomial matrix $\mathbf{W}(s)$ of size $(p \times m)$ with $rank[\mathbf{W}(s)]=r$, can be expressed as [7]

$$\mathbf{U}_1(s)\mathbf{W}(s)\mathbf{U}_2(s)=\mathbf{M}(s) \qquad (6)$$

The polynomial matrices $\mathbf{U}_1(s)$ and $\mathbf{U}_2(s)$ are unimodular and the matrix $\mathbf{M}(s)$ is given by

$$\mathbf{M}(s) = \begin{bmatrix} \mathbf{M}_r(s) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \qquad (7)$$

The non-singular polynomial matrix $\mathbf{M}_r(s)$ of size $(r \times r)$ in (7) is given by

$$\mathbf{M}_r(s) = \text{diag}[a_1(s), a_2(s), \ldots, a_r(s)] \qquad (8)$$

The nonzero polynomials $a_i(s)$ for $i=1,2,\ldots,r$ are termed invariant polynomials of $\mathbf{W}(s)$ and have the following property

$$a_i(s) \text{ divides } a_{i+1}(s), \text{ for } i = 1,2,\ldots, r-1 \qquad (9)$$

The relationship (6) is called Smith-McMillan form of $\mathbf{W}(s)$ over $\mathcal{R}[s]$. Since the matrices $\mathbf{U}_1(s)$ and $\mathbf{U}_2(s)$ are unimodular and the polynomial matrix $\mathbf{M}_r(s)$ given by (8) is non-singular, from (6) and (7) it follows that

$$rank[\mathbf{W}(s)] = rank[\mathbf{M}_r(s)] = r \qquad (10)$$

Let $\mathbf{A}(s)$, be a polynomial matrix over $\mathcal{R}[s]$ if there are polynomial matrices $\mathbf{P}(s)$ and $\mathbf{Q}(s)$ of appropriate dimensions such that

$$\mathbf{A}(s) = \mathbf{P}(s)\,\mathbf{Q}(s) \qquad (11)$$

Then the polynomial matrix $\mathbf{P}(s)$ over $\mathcal{R}[s]$ termed the left divisor of $\mathbf{A}(s)$ [7]. Let $\mathbf{A}(s)$ and $\mathbf{B}(s)$, be polynomial matrices over $\mathcal{R}[s]$ if

$$\mathbf{A}(s) = \mathbf{D}(s)\,\mathbf{M}(s) \qquad (12)$$

$$\mathbf{B}(s) = \mathbf{D}(s)\,\mathbf{N}(s) \qquad (13)$$

for polynomial matrices $\mathbf{M}(s)$, $\mathbf{N}(s)$ and $\mathbf{D}(s)$ over $\mathcal{R}[s]$, then $\mathbf{D}(s)$ termed the common left divisor of polynomial matrices $\mathbf{A}(s)$ and $\mathbf{B}(s)$ [7]. A greatest common left divisor of two polynomial matrices $\mathbf{A}(s)$ and $\mathbf{B}(s)$ is a common left divisor which is a right multiple of every common left divisor [8]. Let $\mathbf{A}$ and $\mathbf{B}$ be real matrices of size $(n \times n)$ and $(n \times m)$ respectively. Then there always exists a unimodular matrix $\mathbf{U}(s)$ over $\mathcal{R}[s]$ such that

$$[\mathbf{I}s - \mathbf{A}, \mathbf{B}] = [\mathbf{V}(s), \mathbf{0}]\,\mathbf{U}(s) \qquad (14)$$

The non-singular polynomial matrix $\mathbf{V}(s)$ of size $(n \times n)$ is a greatest common right divisor of the polynomial matrices $[\mathbf{I}s - \mathbf{A}]$ and $\mathbf{B}$ [8]. Since the polynomial matrix $\mathbf{U}(s)$ is unimodular from (14) it follows that

$$rank[\mathbf{I}s - \mathbf{A}, \mathbf{B}] = rank[\mathbf{V}(s), \mathbf{0}] = rank[\mathbf{V}(s)] = n \qquad (15)$$

Definition 1: The nonzero polynomial $c(s)$ over $\mathcal{R}[s]$ is said to be strictly Hurwitz if and only if $c(s) \neq 0, \forall s \in C^+$.

Definition 2: Let $\mathbf{V}(s)$ be a non-singular matrix over $\mathcal{R}[s]$ of size $(n \times n)$. Also let $c_i(s)$ for $i=1,2,\ldots,n$ be the invariant polynomials of polynomial matrix $\mathbf{V}(s)$. The polynomial matrix $\mathbf{V}(s)$ is said to be strictly Hurwitz if and only if the polynomials $c_i(s)$ are strictly Hurwitz for every $i=1,2,\ldots,n$, or alternatively if and only if $det[\mathbf{V}(s)]$ is a strictly Hurwitz polynomial.

Definition 3: The matrix $\mathbf{A}$ over $\mathcal{R}$ matrices of size $(n \times n)$, is said to be Hurwitz stable if and only if all eigenvalues of the matrix $\mathbf{A}$ have negative real parts or alternatively if and only if the characteristic polynomial of matrix **A** is a strictly Hurwitz polynomial.

Definition 4: Let **A** and **B** be matrices over $\mathcal{R}$ matrices of size *(n x n)* and (*n x m*), respectively. Then the pair (**A**, **B**) is said to be stabilizable if and only if there exists a real matrix **K** of size (*m x n*), such that the matrix [**A**+**BK**] is Hurwitz stable [9]. The following Lemma is taken from [10].

Lemma 1: Let **A** and **B** be matrices over $\mathcal{R}$ of size (*n x n*) and (*n xm*), respectively. The pair (**A**, **B**) is stabilizable if and only if the following condition holds:

(a) $rank[\mathbf{I}s - \mathbf{A},\ \mathbf{B}] = n,\ \forall s \in C^+$

Lemma 2: Let $\mathbf{V}(s)$ be a non-singular polynomial matrix over $\mathcal{R}[s]$, of size *(n x n)*. Also let $c_i(s)$ for $i=1,2,\ldots,n$ be the invariant polynomials of the polynomial matrix $\mathbf{V}(s)$. The polynomial matrix $\mathbf{V}(s)$ is strictly Hurwitz if and only if the following condition holds

(a) $rank[\mathbf{V}(s)] = n,\ \forall s \in C^+$

Proof: Let $\mathbf{V}(s)$ be a non-singular and strictly Hurwitz polynomial matrix of size *(n x n)* with invariant polynomials $c_i(s)$ for $i=1,2,\ldots,n$ From Definition 2 it follows that the polynomials $c_i(s)$ are strictly Hurwitz for every $i=1,2,\ldots,n$ and therefore from Definition 1 it follows that

$$c_i(s) \neq 0,\ \forall s \in C^+,\ \forall\ i=1,2,\ldots,n \quad (16)$$

we define the polynomial matrix

$$\mathbf{V}_n(s) = \text{diag}\,[c_1(s), c_2(s), \ldots, c_n(s)] \quad (17)$$

From (16) and (17) it follows that

$$rank[\mathbf{V}_n(s)] = rank[diag[c_1(s), c_2(s), \ldots, c_n(s)]] = n,\ \forall s \in C^+ \quad (18)$$

The Smith-McMillan form of polynomial matrix $\mathbf{V}(s)$ over $\mathcal{R}[s]$ is given by

$$\mathbf{K}(s)\,\mathbf{V}(s)\,\mathbf{L}(s) = \mathbf{V}_n(s) \quad (19)$$

where $\mathbf{K}(s)$ and $\mathbf{L}(s)$ are unimodular matrices. Since the matrices $\mathbf{K}(s)$, $\mathbf{L}(s)$ are unimodular, from (10), (17), (18) and (19) it follows that

$$rank[\mathbf{V}(s)] = n,\ \forall s \in C^+ \quad (20)$$

This is condition (a) of the Lemma. To prove sufficiency, we assume that condition (a) holds. Using (10) from (17) and (19) we have that

$$rank[\mathbf{V}(s)] = rank[\mathbf{V}_n(s)] = rank[diag[c_1(s), c_2(s), \ldots, c_n(s)]] = n \quad (21)$$

Since by assumption condition (a) holds we have that

$$rank[\mathbf{V}(s)] = n,\quad \forall s \in C^+ \quad (22)$$

Relationships (21) and (22) imply

$$rank[\mathbf{V}_n(s)] = rank[diag[c_1(s), c_2(s), \ldots, c_n(s)]] = n,\ \forall s \in C^+ \quad (23)$$

From (23) it follows that

$$c_i(s) \neq 0,\ \forall s \in C^+,\ \forall\ i=1,2,\ldots,n \quad (24)$$

Relationship (24) and Definition 1 imply that polynomials $c_i(s)$ are strictly Hurwitz for every $i=1,2,\ldots,n$, and therefore according to Definition 2 the non-singular polynomial matrix $\mathbf{V}(s)$ over $\mathcal{R}[s]$, is strictly Hurwitz. This completes the proof.

Lemma 3: Let **A** and **B** be matrices over $\mathcal{R}$ matrices of size *(n x n)* and (*n x m*), respectively and **B** not zero. Further let $\mathbf{V}(s)$ be a greatest common left divisor of polynomial matrices [**I**s-**A**] and **B** of size (*n x n*). The pair (**A**, **B**) is stabilizable if and only if the following condition holds:

(a) The polynomial matrix $\mathbf{V}(s)$ is strictly Hurwitz.

Proof: Let the pair (**A**, **B**) is stabilizable. Then from Lemma 1 it follows that

$$rank[\mathbf{I}s - \mathbf{A}, \mathbf{B}] = n,\ \forall s \in C^+ \quad (25)$$

Since by assumption the polynomial matrix $\mathbf{V}(s)$ is the greatest common left divisor of polynomial matrices [**I**s-**A**] and **B**, from (14) it follows that there exists a unimodular matrix $\mathbf{U}(s)$ such that

$$[\mathbf{I}s - \mathbf{A}, \mathbf{B}] = [\mathbf{V}(s, \mathbf{0}]\ \mathbf{U}(s) \quad (26)$$

Since the polynomial matrix $\mathbf{U}(s)$ is unimodular from (15) and (26) it follows that

$$rank[\mathbf{I}s - \mathbf{A}, \mathbf{B}] = rank[\mathbf{V}(s, \mathbf{0}] = rank[\mathbf{V}(s)] = n \quad (27)$$

From relationships (25) and (27) it follows that

$$rank[\mathbf{V}(s)] = n,\ \forall s \in C^+ \quad (28)$$

Relationship (28) and Lemma 2 imply that the polynomial matrix $\mathbf{V}(s)$ is strictly Hurwitz. This is condition (a) of the Lemma. To prove sufficiency, we assume that the polynomial matrix $\mathbf{V}(s)$ is strictly Hurwitz. Then from Lemma 2 it follows that

$$rank[\mathbf{V}(s)] = n,\ \forall s \in C^+ \quad (29)$$

Since the polynomial matrix $\mathbf{V}(s)$ is the greatest common left divisor of polynomial matrices [**I**s-**A**] and **B,** from (29) and (27) it follows that

$$rank[\mathbf{I}s - \mathbf{A}, \mathbf{B}] = n\ ,\ \forall s \in C^+ \qquad (30)$$

Lemma 1 and (30) imply that the pair (**A**, **B**) is stabilizable. This completes the proof.

Lemma 4: Let **A** and **B** be matrices over $\mathcal{R}$ matrices of size *(n x n)* and (*n x m*), respectively and **B** not zero. Further let **V**(*s*) be a greatest common left divisor of polynomial matrices [**I**s-**A**] and **B** of size (*n x n*). Further let **D** be a matrix over $\mathcal{R}$ of size (*m x n*) such that $det[\mathbf{I} - \mathbf{BD}] \neq 0$. Then the following condition holds:

(a) The polynomial matrix $[(\mathbf{I} - \mathbf{BD})^{-1}\ \mathbf{V}(s)$ is a left divisor of the matrix $[\mathbf{I}s - (\mathbf{I} - \mathbf{BD})^{-1}\mathbf{A}]$.

Proof: Since by assumption $det[\mathbf{I} - \mathbf{BD}] \neq 0$ the matrix $[\mathbf{I} - \mathbf{BD}]$ is non-singular and therefore the matrix $[\mathbf{I}s - (\mathbf{I} - \mathbf{BD})^{-1}\mathbf{A}]$ can be rewritten as follows

$$[\mathbf{I}s - (\mathbf{I} - \mathbf{BD})^{-1}\mathbf{A}] =$$

$$=[\mathbf{I} - \mathbf{BD}]^{-1}[(\mathbf{I} - \mathbf{BD})s - \mathbf{A}] =$$

$$= [\mathbf{I} - \mathbf{BD}]^{-1}[\mathbf{I}s - \mathbf{A}, \mathbf{B}]\begin{bmatrix} \mathbf{I} \\ -\mathbf{D}s \end{bmatrix} \qquad (31)$$

Since by assumption the polynomial matrix **V**(s) is the greatest common left divisor of polynomial matrices [**I**s-**A**] and **B**, from (12) and (13) it follows that

$$[\mathbf{I}s - \mathbf{A}] = \mathbf{V}(s)\ \mathbf{X}(s) \qquad (32)$$

$$\mathbf{B} = \mathbf{V}(s)\ \mathbf{Y}(s) \qquad (33)$$

for polynomial matrices **X**(s) and **Y**(s) over $\mathcal{R}[s]$ of appropriate dimensions. Using (32) and (33) and after simple algebraic manipulations, the relationship (31) can be rewritten as

$$[\mathbf{I}s - (\mathbf{I} - \mathbf{BD})^{-1}\mathbf{A}] =$$

$$= [\mathbf{I} - \mathbf{BD}]^{-1}\mathbf{V}(s)[\mathbf{X}(s) - \mathbf{Y}(s)(\mathbf{D}s] \qquad (34)$$

Using (11) from (34) it follows that the matrix $[\mathbf{I} - \mathbf{BD}]^{-1}\mathbf{V}(s)$ is a left divisor of the polynomial matrix $[\mathbf{I}s - (\mathbf{I} - \mathbf{BD})^{-1}\mathbf{A}]$. This is condition (a) of the Lemma and the proof is complete.

Lemma 5: Let **A** be a Hurwitz stable matrix over $\mathcal{R}$ of size *(n x n)*. Then the following condition holds:

(a) The matrix **A** is non-singular.

Proof: Let **A** be a Hurwitz stable matrix over $\mathcal{R}$ of size *(n x n)*. The characteristic polynomial $c(s)$ of matrix **A** is given by [11]

$$det[(\mathbf{I}s - \mathbf{A}] = c(s) \qquad (35)$$

From Definition 3 it follows that $c(s)$ is a strictly Hurwitz polynomial over $\mathcal{R}[s]$ of degree *n*. Let ξ_i for *i=1,2,…,n*, be the roots of $c(s)$. Then

$$c(\xi_i) = 0\ ,\forall\ i=1,2,\ldots,n \qquad (36)$$

Since $c(s)$ is a strictly Hurwitz polynomial of degree *n* from Definition 1 and (36) it follows that

$$Re(\xi_i) \leq 0\ ,\forall\ i=1,2,\ldots,n \qquad (37)$$

Since the polynomial $c(s)$ in (35) is the characteristic polynomial of the matrix **A**, the complex numbers ξ_i for *i=1,2,…,n*, are the eigenvalues of the matrix **A.** From (37) it follows that

$$\xi_i \neq 0\ ,\forall\ i=1,2,\ldots,n \qquad (38)$$

From (38) it follows that all eigenvalues of the matrix **A** are nonzero and therefore the matrix **A** is non-singular [11]. This is condition (a) of the Lemma and the proof is complete.

The following Lemma is partially based on the main results of [1] and [12]

Lemma 6: Let **A** and **B** be matrices over $\mathcal{R}$ matrices of size *(n x n)* and (*n x m*), respectively with **A** being non-singular and **B** not zero. Further let **D** be a matrix over $\mathcal{R}$ of size (*m x n*) such that

$$\det[\mathbf{I} - \mathbf{BD}] \neq 0.$$

Then there exists real matrices **F** of appropriate size and **D** given by

$$\mathbf{D} = \mathbf{F}[\mathbf{A} - \mathbf{BF}]^{-1}$$

such that the matrix $(\mathbf{I} - \mathbf{BD})^{-1}\mathbf{A}$ is Hurwitz stable if and only if the following condition holds:

(a) The pair (**A**, **B**) is stabilizable.

Proof: Let there exists real matrices **F** and **D** of appropriate dimensions with **D** given by

$$\mathbf{D} = \mathbf{F}[\mathbf{A} + \mathbf{BF}]^{-1} \qquad (39)$$

such that the matrix $(\mathbf{I} - \mathbf{BD})^{-1}\mathbf{A}$ is Hurwitz stable. Since by assumption the matrix **A** is non-singular, from (39) we have that

$$(\mathbf{I} - \mathbf{BD})^{-1}\mathbf{A} = [\mathbf{I} - \mathbf{BF}[\mathbf{A} + \mathbf{BF}]^{-1}]^{-1}\ \mathbf{A}=$$

$$= [\mathbf{A} + \mathbf{BF}]\ [\mathbf{A} - \mathbf{BF} + \mathbf{BF}]^{-1}\mathbf{A}=$$

$$= [\mathbf{A} + \mathbf{BF}]\mathbf{A}^{-1}\mathbf{A} = [\mathbf{A} + \mathbf{BF}\ ] \qquad (40)$$

Since by assumption the matrix $(\mathbf{I} - \mathbf{BD})^{-1}\mathbf{A}$ is Hurwitz stable, from (40) it follows that the matrix $[\mathbf{A} + \mathbf{BF}]$ is Hurwitz stable**.** Hurwitz stability of the matrix $[\mathbf{A} + \mathbf{BF}]$ and Definition 4 imply that the pair (**A**, **B**) is stabilizable**.** This is condition (a) of the Lemma. To prove sufficiency we assume that the pair (**A**, **B**) is stabilizable. Stabilizability of the pair (**A**, **B**) and Definition 4 imply the existence of matrix **F** of appropriate size such that the matrix $[\mathbf{A}+\mathbf{BF}]$ is Hurwitz stable. Hurwitz stability of the matrix $[\mathbf{A}+\mathbf{BF}]$ and Lemma 5 imply that the matrix $[\mathbf{A}+\mathbf{BF}]$ is non-singular and therefore invertible. Invertibility of the real matrix $[\mathbf{A}+\mathbf{BF}]$ implies the possibility of calculation of the matrix **D** given by (39).Taking in the mind the above, from relationship (40) we conclude that the real matrix

$$(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{A} \tag{41}$$

with **D** given by (39) is Hurwitz stable. This completes the proof.
The following Lemma was first published by Wonham in [9] and can be also found in any standard text of linear control theory.

Lemma 7. Let **A** and **B** be matrices over $\mathcal{R}$ matrices of size *(n x n)* and (*n x m*), respectively. Then the pair (**A**, **B**) is controllable if and only if for every monic polynomial $c(s)$ over $\mathcal{R}$[s] of degree n there exists a matrix **F** over $\mathcal{R}$ of size *m x n*, such that the matrix [**A**+**BF**] has characteristic polynomial $c(s)$.

The standard decomposition of uncontrollable systems given in the following Lemma was first published by Kalman in [13] and can be also found in any standard text of linear control theory.
The following Lemma is taken from [7].

Lemma 8: Let **A** and **B** be matrices over $\mathcal{R}$ matrices of size *(n x n)* and (*n x m*), respectively Further let the pair (**A**, **B**) is uncontrollable and **B** not zero. Then there exists a non-singular matrix **T** such that

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T}=\begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{0} & \mathbf{A}_{22}\end{bmatrix},$$
$$\mathbf{T}^{-1}\mathbf{B}=\begin{bmatrix}\mathbf{B}_1\\ \mathbf{0}\end{bmatrix}$$

The pair $(\mathbf{A}_{11},\ \mathbf{B}_{11})$ is controllable and the eigenvalues of the matrix $\mathbf{A}_{22}$ are the uncontrollable eigenvalues of the pair (**A**, **B**).

Lemma 9: Let **A** and **B** be matrices over $\mathcal{R}$ matrices of size *(n x n)*, (*n x m*), respectively and **B** not zero. Further let

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T}=\begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{0} & \mathbf{A}_{22}\end{bmatrix},\ \mathbf{T}^{-1}\mathbf{B}=\begin{bmatrix}\mathbf{B}_1\\ \mathbf{0}\end{bmatrix}$$

with $(\mathbf{A}_{11},\ \mathbf{B}_1)$ controllable. The pair (**A**, **B**) is stabilizable if and only if the following condition holds:

(a) The matrix $\mathbf{A}_{22}$ is Hurwitz stable or alternatively all uncontrollable eigenvalues of the pair (**A**, **B**) are stable (i.e., the eigenvalues of the matrix $\mathbf{A}_{22}$ have negative real parts).

Proof: From the statement of the Lemma we have that

$$\mathbf{A}=\mathbf{T}\begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{0} & \mathbf{A}_{22}\end{bmatrix}\mathbf{T}^{-1},\ \mathbf{B}=\mathbf{T}\begin{bmatrix}\mathbf{B}_1\\ \mathbf{0}\end{bmatrix} \tag{42}$$

with $(\mathbf{A}_{11},\mathbf{B}_1)$ controllable. If the pair (**A**, **B**) is stabilizable, then from Definition 4 it follows that there exists a matrix **F** such that the matrix [**A** +**BF**] is Hurwitz stable. Using (42) we have that

$$\mathbf{A}\ +\mathbf{BF}=\mathbf{T}\begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{0} & \mathbf{A}_{22}\end{bmatrix}\mathbf{T}^{-1}+\mathbf{T}\begin{bmatrix}\mathbf{B}_1\\ \mathbf{0}\end{bmatrix}\mathbf{F}=$$

$$=\mathbf{T}\{\begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{0} & \mathbf{A}_{22}\end{bmatrix}+\begin{bmatrix}\mathbf{B}_1\\ \mathbf{0}\end{bmatrix}\mathbf{FT}\}\mathbf{T}^{-1} \tag{43}$$

Let

$$\mathbf{FT}=[\mathbf{F}_1,\mathbf{F}_2] \tag{44}$$

Substituting (44) to (43) and after simple algebraic manipulations we have that

$$\mathbf{A}+\mathbf{BF}=\mathbf{T}\begin{bmatrix}\mathbf{A}_{11}+\mathbf{B}_1\mathbf{F}_1 & \mathbf{A}_{12}+\mathbf{B}_1\mathbf{F}_2\\ \mathbf{0} & \mathbf{A}_{22}\end{bmatrix}\mathbf{T}^{-1} \tag{45}$$

From (45) it follows that the matrices

$$[\mathbf{A}+\mathbf{BF}],\ \begin{bmatrix}\mathbf{A}_{11}+\mathbf{B}_1\mathbf{F}_1 & \mathbf{A}_{12}+\mathbf{B}_1\mathbf{F}_2\\ \mathbf{0} & \mathbf{A}_{22}\end{bmatrix} \tag{46}$$

are similar; therefore Hurwitz stability of [**A**+**BF**] implies Hurwitz stability of $\mathbf{A}_{22}$**.** Since the matrix $\mathbf{A}_{22}$ is Hurwitz stable, from Lemma 8 and Definition 3 it follows that all uncontrollable eigenvalues of the pair (**A**, **B**) are stable. This is condition (a) of the Lemma. To prove sufficiency we assume that condition (a) holds. Controllability of the pair $(\mathbf{A}_{11},\ \mathbf{B}_1)$ and Lemma 7 imply the existence of matrix $\mathbf{F}_1$ of appropriate size such that the matrix

$$\det[\mathbf{I}s-\mathbf{A}_{11}-\mathbf{B}_1\mathbf{F}_1]\ ]=\varphi(s) \tag{47}$$

where $\varphi(s)$ is an arbitrary monic, strictly Hurwitz polynomial over $\mathcal{R}$[s] of appropriate degree. The matrix $\mathbf{F}_1$ can be calculated using known methods for the solution of pole assignment problem by state feedback [7]. Let

$$\mathbf{F}=[\mathbf{F}_1,\mathbf{0}]\mathbf{T}^{-1} \tag{48}$$

Substituting (48) to (45) we have that

$$\mathbf{A}+\mathbf{BF}=\mathbf{T}\begin{bmatrix}\mathbf{A}_{11}+\mathbf{B}_1\mathbf{F}_1 & \mathbf{A}_{12}\\ \mathbf{0} & \mathbf{A}_{22}\end{bmatrix}\mathbf{T}^{-1} \quad (49)$$

Using (47), from (49) it follows that Hurwitz stability of $\mathbf{A}_{22}$ implies Hurwitz stability of [**A**+**BF**]. Hurwitz stability of [**A**+**BF**] and Definition 4 imply stabilizability of the pair (**A**, **B**). This completes the proof.

4 Problem Solution

The following theorem is the main result of this paper and gives explicit necessary and sufficient conditions for the solution over $\mathscr{R}$ of the stabilization problem by state-derivative feedback for linear time-invariant continuous-time systems with state-space equations given by (1).

Theorem 1. The stabilization problem by state-derivative feedback for linear time-invariant continuous-time systems with state-space equations given by (1) has solution over $\mathscr{R}$ if and only if the following condition holds:

(a) The matrix **A** is non-singular.

(b) The pair (**A**, **B**) is stabilizable.

Proof: Let system (1) is stabilizable by state-derivative feedback. Then from (4) we have that

$$det[(\mathbf{I}s-(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{A}]=c(s) \quad (50)$$

where $c(s)$ is a monic, strictly Hurwitz polynomial over $\mathscr{R}[s]$ of degree *n*. Relationship (50) and Definition 3 imply that the matrix $[(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{A}]$ is Hurwitz stable. Hurwitz stability of the matrix $(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{A}$ and Lemma 5 imply non-singularity of $(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{A}$ and therefore non-singularity of matrix **A**. This is condition (a) of the Theorem. Let $\mathbf{V}(s)$ be a greatest common left divisor of polynomial matrices [**I**s-**A**] and **B** of size (*n x n)*. Then from Lemma 4 it follows that the polynomial matrix $[(\mathbf{I}-\mathbf{BD})^{-1}\ \mathbf{V}(s)$ is a left divisor of the polynomial matrix $[\mathbf{I}s-(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{A}]$ that is

$$[\mathbf{I}s-(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{A}]=[(\mathbf{I}-\mathbf{BD})^{-1}\ \mathbf{V}(s)\ \mathbf{X}(s) \quad (51)$$

where $\mathbf{X}(s)$ is a matrix over $\mathscr{R}$[s] of appropriate size. From (51) we have that

$$det[\mathbf{I}s-(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{A}]= \\ = det[(\mathbf{I}-\mathbf{BD})^{-1}]\ det[\mathbf{V}(s)]\ det[\mathbf{X}(s)] \quad (52)$$

From relationship (50) and (52) it follows that

$$det[\mathbf{V}(s)] \text{ divides } (c(s)) \quad (53)$$

Since by assumption $c(s)$ is a monic, strictly Hurwitz polynomial over $\mathscr{R}[s]$ of degree *n,* from (53) it follows that det[$\mathbf{V}(s)$] is a strictly Hurwitz polynomial over $\mathscr{R}$[s]; therefore by Definition 2 the polynomial matrix **V**(s) is strictly Hurwitz. Since **V**(s) is strictly Hurwitz, from Lemma 3 it follows that the pair (**A**, **B**) is stabilizable. This is condition (b) of the Theorem.

To prove sufficiency, we assume that conditions (a) and (b) hold. Stabilizability of the pair (**A**, **B**) imply that the pair (**A**, **B**) is either controllable or uncontrollable with stable uncontrollable eigenvalues (i.e. all uncontrollable eigenvalues have negative real parts).

If the pair (**A**, **B**) is controllable, then from Lemma 7 it follows that there exists a matrix **F** of appropriate size over $\mathscr{R}$ such that

$$\det[\mathbf{I}s-\mathbf{A}\ -\mathbf{BF}]=\det[\mathbf{I}s-\mathbf{A}-\mathbf{BF}]=\chi(s) \quad (54)$$

where $\chi(s)$ be an arbitrary monic, strictly Hurwitz polynomial over $\mathscr{R}[s]$ of degree *n.* The matrix **F** can be calculated using known methods for the solution of pole assignment problem by state feedback [7].

If the pair (**A**, **B**) is uncontrollable with stable uncontrollable eigenvalues, then from Lemma 8 and Lemma 9 it follows that there exists a matrix **T** such that

$$\mathbf{T}^{-1}\mathbf{AT}=\begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{0} & \mathbf{A}_{22}\end{bmatrix},\ \mathbf{T}^{-1}\mathbf{B}=\begin{bmatrix}\mathbf{B}_1\\ \mathbf{0}\end{bmatrix} \quad (55)$$

The pair $(\mathbf{A}_{11},\ \mathbf{B}_1)$ is controllable and the matrix $\mathbf{A}_{22}$ is Hurwitz stable. Controllability of the pair $(\mathbf{A}_{11},\ \mathbf{B}_1)$ and Lemma 7 imply the existence of a matrix $\mathbf{F}_1$ over $\mathscr{R}$ of appropriate dimensions such that

$$\det[\mathbf{I}s-\mathbf{A}_{11}-\mathbf{B}_1\mathbf{F}_1]\ ]=\varphi(s) \quad (56)$$

where $\varphi(s)$ is an arbitrary monic, strictly Hurwitz polynomial over $\mathscr{R}$[s] of appropriate degree. The matrix $\mathbf{F}_1$ can be calculated using known methods for the solution of pole assignment by state feedback [7]. According to (49) the matrix [**A**+**BF**] with **F** given by

$$\mathbf{F}=[\mathbf{F}_1,\mathbf{0}]\mathbf{T}^{-1} \quad (57)$$

is Hurwitz stable. Conditions (a) and (b) and Lemma 6 imply the existence of the matrix **D** given by

$$\mathbf{D}=\mathbf{F}[\mathbf{A}+\mathbf{BF}]^{-1} \quad (58)$$

such that the matrix $(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{A}$ is Hurwitz stable that is

$$det[(\mathbf{I}s-(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{A}]=c(s) \quad (59)$$

where $c(s)$ is a monic, strictly Hurwitz polynomial over $\mathscr{R}[s]$ of degree *n.* From (59) and (4) it follows that the closed-loop system (3) is a stable regular state-space system. This completes the proof.

The sufficiency part of the proof of Theorem 1 provides a construction of the matrix **D** of state-derivative feedback which stabilizes the system (1). The major steps of this construction are given below.

Construction

Given: **A**, **B**
Find: **D**

Step 1: Check conditions (a) and (b) of Theorem 1. If these conditions are satisfied go to *Step 2.* If conditions (a) and (b) are not satisfied go to *Step 4.*

Step 2: Stabilizability of the pair (**A**, **B**) implies that the pair (**A**, **B**) is either controllable or uncontrollable with stable uncontrollable eigenvalues. If the pair (**A**, **B**) is controllable, then from Lemma 7 it follows that there exists a matrix **F** over $\mathcal{R}$ such that

$$\det[\mathbf{I}s - \mathbf{A} \; - \mathbf{BF}] = \det[\mathbf{I}s - \mathbf{A} - \mathbf{BF}] = \chi(s)$$

where $\chi(s)$ be an arbitrary monic and strictly Hurwitz polynomial over $\mathcal{R}$[s] of degree *n.* The matrix **F** can be calculated using known methods for the solution of pole assignment problem by state feedback [7].

If the pair (**A**,**B**) is uncontrollable with stable uncontrollable eigenvalues then from Lemma 8 and Lemma 9 it follows that there exists a matrix **T** such that

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix}$$

$$\mathbf{T}^{-1}\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix}$$

The pair $(\mathbf{A}_{11},\ \mathbf{B}_1)$ is controllable and the matrix $\mathbf{A}_{22}$ is Hurwitz stable. Controllability of the pair $(\mathbf{A}_{11},\ \mathbf{B}_1)$ and Lemma 7 imply the existence of a matrix $\mathbf{F}_1$ over $\mathcal{R}$ of appropriate dimensions such that

$$\det[\mathbf{I}s - \mathbf{A}_{11} - \mathbf{B}_1\mathbf{F}_1]\ ] = \varphi(s)$$

where $\varphi(s)$ is an arbitrary monic, strictly Hurwitz polynomial over $\mathcal{R}$[s] of appropriate degree. The matrix $\mathbf{F}_1$ can be calculated using known methods for the solution of pole assignment by state feedback [7]. According to (49) the matrix [**A**+**BF**] with **F** given by

$$\mathbf{F} = [\mathbf{F}_1,\ \mathbf{0}]\mathbf{T}^{-1}$$

is Hurwitz stable.

Step 3: Put

$$\mathbf{D} = \mathbf{F}[\mathbf{A} + \mathbf{BF}]^{-1}$$

Step 4: The stabilization problem by state-derivative state feedback has no solution.

5 Conclusions

In this paper the stabilization problem by state-derivative feedback for linear time-invariant continuous-time systems is studied and completely solved. The proof of the main results of this paper is constructive and furnishes a procedure for the computation of stabilizing state-derivative feedback. As far as we know the stabilization problem by state-derivative feedback for linear time-invariant continuous-time systems in its full generality, is still an open problem. This clearly demonstrates the originality of the contribution of the main results of this paper with respect to existing results.

References:

[1] T.H.S. Abdelaziz and M. Valasek, Direct algorithm for pole placement by state derivative feedback, for multi-input linear systems-nonsingular case, *Kybernetika,* Vol. 41, No. 5, 2004, pp. 637-660.

[2] T.H.S. Abdelaziz and M. Valasek, Pole assignment for SISO linear system by state derivative feedback, *IEE Proceedings Control Theory and Applications,* Vol. 151, No. 4, 2004, pp. 377-385.

[3] W. Michels, T.Vyhlidal, H. Huijberts and H. Nijmeijer, Stabilizability and stability robustness of state derivative controllers, *SIAM Journal of Control and Optimization,* Vol. 47, No. 6, 2009, pp. 3100-3117.

[4] T.H.S. Abdelaziz and M. Valasek, A direct algorithm for pole assignment for single input linear system by state derivative feedback, *Acta Polytechnika,* Vol. 43, No. 6, 2003, pp. 52-60.

[5] M.A. Beteto, M. E. Assuncao M.C.M. Teixeira, E.R.P, da Silva, L.F.S. Buzachero and R.P. Caun, Less conservative conditions for robust LTR-state-derivative controller: an LMI approach, *International Journal of Systems and Science,* Vol. 52, No.12, 2021, pp. 2518-2537.

[6] G.C. Verghese, R.C. Levy and T. Kailath, A generalized state-space system for singular systems, *IEEE Transactions Automatic Control,* Vol. 26, No. 4, 1981, pp. 811-831.

[7] V.Kucera, *Analysis and Design of Linear Control Systems*, Prentice Hall, London, 1991.

[8] W.A. Wolowich, *Linear Multivarible systems*, Springer Verlag, Berlin, New York, 1974.
[9] W. M. Wonham, On Pole assignment in multi-Input controllable linear systems. *IEEE Trans. Automat. Control,* Vol. 12, No.6, 1967, pp. 660-665.
[10] B.N. Datta, *Numerical Methods for Linear Control Systems Design and Analysis*, Elsevier, Academic Press, 2004.
[11] C. D. Meyer, *Matrix Analysis and Applied Linear Algebra,* SIAM Philadelphia, 2000
[12] R. Cardim, M.C.M. Teixeira, E. Assuncao and M.R. Covacic, Design of state derivative feedback controllers using a state feedback controller design, *Proceedings of 3rd IFAC Symposium on Systems Structure and Control,* Vol. 40, 2007, pp. 22-27.
[13] R.E. Kalman, Mathematical description of linear dynamical systems, *SIAM Journal of Control and Optimization,* Vol. 1, No.2, 1963, pp. 152-192.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The research of this manuscript was conducted by the author who contributed to the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The author has no conflict of interest to declare that is relevant to the content of this article.