On a six-dimensional Artificial Neural Network Model

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Abstract: This work introduces a new six-dimensional system with chaotic and periodic solutions. For special values of parameters, we calculate the Kaplan-Yorke dimension and we show the dynamics of Lyapunov exponents. Some definitions and propositions are given. Visualizations where possible, are provided.

Keywords: artificial neural network, Kaplan-Yorke dimension, Lyapunov exponents, chaotic solution, periodic solution

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1 Introduction

Mathematical modeling of nonlinear dynamic systems is an interdisciplinary tool for studying various processes in nature and society. Artificial neural network (ANN in short) models are a simplification of human neural systems. ANN consists of computing units. These blocks are called artificial neurons. Artificial neurons are similar to neurons in the biological nervous system, [1]. The ANNs model is the most common emerging tool for modeling environmental concerns, particularly, in water quality modeling, [2]. Also, the ANN model for the crude oil distillation column was constructed based on the results of simulations, [3]. ANN are used in agriculture, medicine, marketing and other industries. Their mathematical models can be formulated in terms of systems of differential equations of the form (1)

$$\begin{cases} x_1' = \tanh(w_{11}x_1 + \ldots + w_{1n}x_n) - b_1x_1, \\ x_2' = \tanh(w_{21}x_1 + \ldots + w_{2n}x_n) - b_2x_2, \\ \ldots \\ x_n' = \tanh(w_{n1}x_1 + \ldots + w_{nn}x_n) - b_nx_n. \end{cases} \tag{1}$$

Each dependent variable is associated with a neuron. It accepts signals from other neurons and elaborates its signal which is sent to a network, [4]. In such systems, periodic solutions, quasi-periodic solutions, and also chaotic solutions are possible. This article uses the Lyapunov exponents and the Kaplan-Yorke formula to determine the chaotic solutions of the system (1).

2 Lyapunov exponents and Kaplan-Yorke dimension

Lyapunov exponents are a useful tool to distinguish between regular and chaotic dynamics. The generally accepted convention is to write the Lyapunov exponents in descending order

$$\lambda_1 \geq \lambda_2 \geq \ldots \lambda_n.$$

Lyapunov exponent measures how quickly an infinitesimally small distance between two initially close states grows over time

$$F^t(x_0 + \epsilon) - F^t(x_0) \approx \epsilon e^{\lambda t}.$$

The left-hand side is the distance between two initially close states after t steps, and the right-hand side is the assumption that the distance grows exponentially over time. The exponent λ measured for a long period

of time is the Lyapunov exponent. Each indicator can be interpreted as indicator of the rate of stretching (if $\lambda > 0$), small distances grow indefinitely over time. If $\lambda < 0$, small distances don't grow indefinitely, i.e., the system settles down into a periodic trajectory, [5].
Some facts about Lyapunov exponents:

- The number of Lyapunov exponents is equal to the number of phase space dimensions, or the order of the system of differential equations, [6], [7].
- The largest Lyapunov exponent of a stable system does not exceed zero, [8].
- A hyperchaotic system is defined as a chaotic system with at least two positive Lyapunov exponents. Combined with one null exponent and one negative exponent, the minimal dimension for a hyperchaotic system is four, [9].
- A strictly positive maximal Lyapunov exponent is often considered as a definition of deterministic chaos.

Knowing the Lyapunov exponents allows us to conclude how the system develops over time. Often enough to know the sign the highest exponent, and the sum of the Lyapunov exponents.
In 1979, the Kaplan-Yorke formula was proposed to estimate the fractal size - in terms of Lyapunov exponents, [10], [11],[12].

$$D_{KY} = j + \frac{1}{|\lambda_{j+1}|} \sum_{j=1}^{j} \lambda_j \tag{2}$$

with j representing the index such that

$$\sum_{i=1}^{j} \lambda_j > 0, \sum_{i=1}^{j+1} \lambda_j < 0.$$

The result obtained by this formula is called the Kaplan-Yorke dimension or the Lyapunov dimension. A valuable advantage of this dimension is that in order to calculate it, one needs to know only the spectrum of Lyapunov exponents of the given attractor, [13].

Definition 2.1. *A chaotic system is a deterministic system that exhibits irregular and unpredictable behavior, [8].*

Definition 2.2. *An attractor is the limiting trajectory of the representing point in the phase space, to which all initial modes tend [14].*

Each attractor has a basin of attraction that contains all the initial conditions which will generate trajectories joining asymptotically this attractor [15].

Definition 2.3. *A chaotic attractor is an attractor that exhibits sensitivity to initial conditions, [16].*

Proposition 2.1. *In dynamical systems that include three or more equations, there may be even more unusual attractors, which are commonly called strange or chaotic attractors.*

Floris Takens (1940-2010) a Dutch mathematician known for contributions to the theory of differential equations, the theory of dynamical systems, chaos theory and fluid mechanics. Introduced the concept of a "strange attractor". He was the first to show how chaotic attractors could be learned by neural networks [17].

Proposition 2.2. *It is possible to find a chaotic attractor in differential systems presenting chaotic behavior [18], [19].*

2.1 The six-dimensional systems

Consider the system

$$\begin{cases} x_1' = \tanh(w_{11}x_1 + w_{12}x_2 + w_{13}x_3 + w_{14}x_4 + w_{15}x_5 + w_{16}x_6) - b_1x_1, \\ x_2' = \tanh(w_{21}x_1 + w_{22}x_2 + w_{23}x_3 + w_{24}x_4 + w_{25}x_5 + w_{26}x_6) - b_2x_2, \\ x_3' = \tanh(w_{31}x_1 + w_{32}x_2 + w_{33}x_3 + w_{34}x_4 + w_{35}x_5 + w_{36}x_6) - b_3x_3, \\ x_4' = \tanh(w_{41}x_1 + w_{42}x_2 + w_{43}x_3 + w_{44}x_4 + w_{45}x_5 + w_{46}x_6) - b_4x_4, \\ x_5' = \tanh(w_{51}x_1 + w_{52}x_2 + w_{53}x_3 + w_{54}x_4 + w_{55}x_5 + w_{56}x_6) - b_5x_5, \\ x_6' = \tanh(w_{61}x_1 + w_{62}x_2 + w_{63}x_3 + w_{64}x_4 + w_{65}x_5 + w_{66}x_6) - b_6x_6 \end{cases} \quad (3)$$

and the regulatory matrix

$$W = \begin{pmatrix} 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 \end{pmatrix}. \quad (4)$$

The initial conditions are

$$x_1(0) = 1.1;\ x_2(0) = 0.5;\ x_3(0) = 1.1;$$

$$x_4(0) = -1;\ x_5(0) = -1;\ x_6(0) = -1. \quad (5)$$

Parameters are $b_1 = b_2 = b_3 = b_4 = b_5 = b_6 = 0.042$.

The graph of the system (3) with the regulatory matrix (4) is considered in Figure 1.

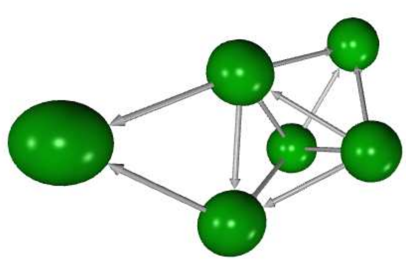

Figure 1: The graph of the system (3) with the regulatory matrix (4).

The projections of 6D trajectories on three-dimensional subspace (x_1, x_4, x_5) are in Figure 2. Solutions $(x_1(t), x_2(t), x_3(t), x_4(t), x_5(t), x_6(t))$ of the system (3) with the regulatory matrix (4) are shown in Figure 3.

The dynamics of Lyapunov exponents are shown in Figure 4.

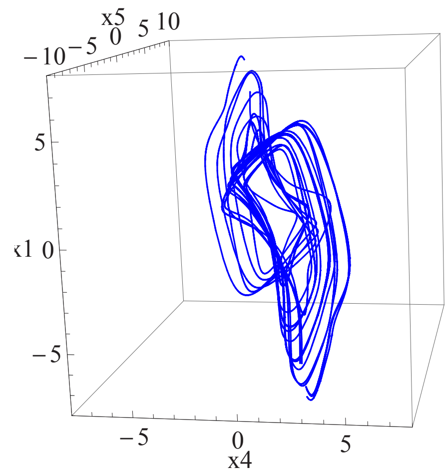

Figure 2: The projection of 6D trajectories to 3D subspace (x_1, x_4, x_5), $b = 0.042$.

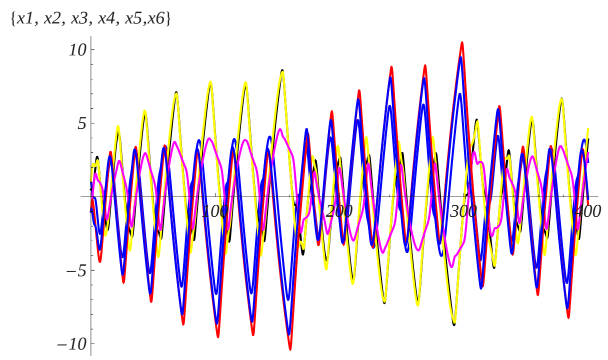

Figure 3: Solutions $(x_1(t), x_2(t), x_3(t), x_4(t), x_5(t), x_6(t))$ of the system (3) with the regulatory matrix (4), $b = 0.042$.

Lyapunov exponents are $\lambda_1 = 0.03; \lambda_2 = 0.00; \lambda_3 = -0.07; \lambda_4 = -0.13; \lambda_5 = -0.15$ and $\lambda_6 = -0.24$.

$$\lambda_1 + \lambda_2 > 0;$$

$$\lambda_1 + \lambda_2 + \lambda_3 < 0.$$

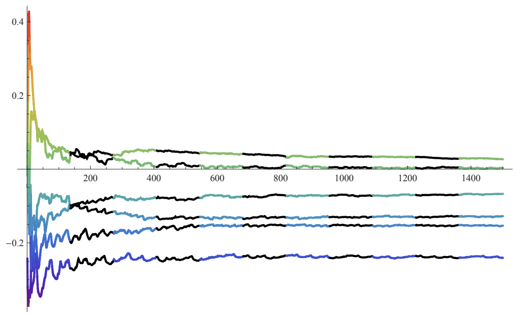

Figure 4: The dynamics of Lyapunov exponents.

The Kaplan-Yorke dimension is

$$D_{KY} = 2 + \frac{\lambda_1 + \lambda_2}{|\lambda_3|} = 2.29.$$

The presence of a positive λ_1 indicates the chaotic nature of the dynamics. Lyapunov exponents $\lambda_3, \lambda_4, \lambda_5$ and λ_6 are negative and are responsible for compression phase volume and the approximation of phase trajectories to the attractor. An estimate of the Kaplan-Yorke dimension from spectrum of Lyapunov exponents gives for a given attractor $D_{KY} = 2.29$.
Let change the value of the element w_{11} from 0 to 1 of the regulatory matrix (4).
The graph of the system (3) with the regulatory matrix (4), $w_{11} = 1$, is considered in Figure 5.

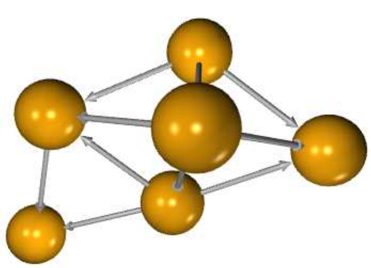

Figure 5: The graph of the system (3) with the regulatory matrix (4), $w_{11} = 1$.

The projections of 6D trajectories on two-dimensional subspace (x_5, x_6) are in Figure 6. Solutions $(x_1(t), x_2(t), x_3(t), x_4(t), x_5(t), x_6(t))$ of the system (3) with the regulatory matrix (4), $w_{11} = 1$, are shown in Figure 7.

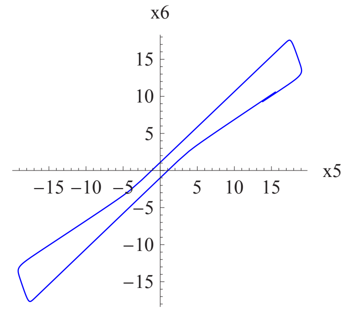

Figure 6: The projection of 6D trajectories to 2D subspace (x_5, x_6), $b = 0.042$.

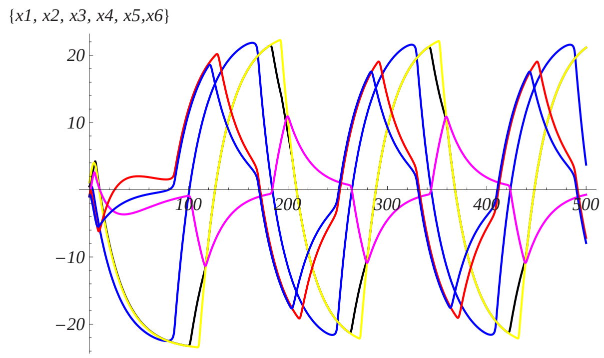

Figure 7: Solutions $(x_1(t), x_2(t), x_3(t), x_4(t), x_5(t), x_6(t))$ of the system (3) with the regulatory matrix (4),$w_{11} = 1$, $b = 0.042$.

The dynamics of Lyapunov exponents are shown in Figure 8.
Lyapunov exponents are $\lambda_1 = 0.00; \lambda_2 = -0.03; \lambda_3 = -0.04; \lambda_4 = -0.05; \lambda_5 = -0.54$ and $\lambda_6 = -0.64$.
$\lambda_1 = 0$; $\lambda_2 < 0$ the system (3) with the regulatory matrix (4), $w_{11} = 1$, has periodic solutions.
In dissipative dynamical system, the values of all Lyapunov exponents should sum to a negative number, [6],[7] . The system (3) with the regulatory matrix (4), $w_{11} = 1$, is a dissipative dynamical system $\lambda_1 + \ldots + \lambda_6 < 0$.
Let change the value of the element w_{11} from 0 to -1 of the regulatory matrix (4).
The graph of the system (3) with the regulatory matrix (4), $w_{11} = -1$, is considered in Figure 9.

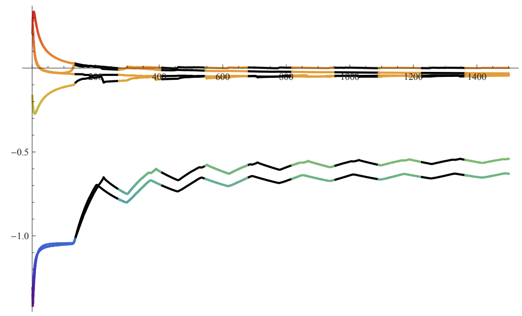

Figure 8: The dynamics of Lyapunov exponents.

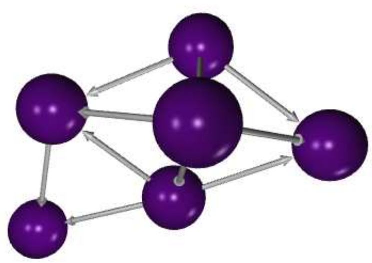

Figure 9: The graph of the system (3) with the regulatory matrix (4), $w_{11} = -1$.

The dynamics of Lyapunov exponents are shown in Figure 10.

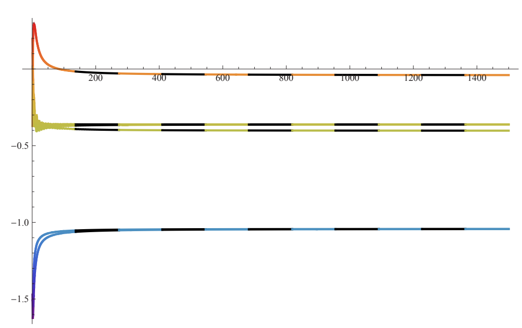

Figure 10: The dynamics of Lyapunov exponents.

Lyapunov exponents are $\lambda_1 = -0.04; \lambda_2 = -0.36; \lambda_3 = -0.36; \lambda_4 = -0.40; \lambda_5 = -1.04$ and $\lambda_6 = -1.04$.
$\lambda_1 < 0$ the system (3) with the regulatory matrix (4), $w_{11} = -1$ has stable fixed point. The system (3) with the regulatory matrix (4), $w_{11} = -1$, is a dissipative dynamical system $\lambda_1 + \ldots + \lambda_6 < 0$.

3 Conclusions

The six-dimensional system with different regulatory matrices is considered. The three examples are provided. Changing the value of an element w_{11} changes the system (3) solutions. In this paper, the periodic and the chaotic attractor is considered. Some definitions and propositions about Lyapunov exponents and Kaplan-Yorke dimension is given. The dynamics of Lyapunov exponents are shown. Visualizations of periodic, chaotic solutions of the system (3) are shown. Projections of $6D$ trajectories to $2D$ or $3D$ subspaces are shown.

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