# Adaptive Refinement of the Variational Grid Method

Yu. K. DEM'YANOVICH, N. A. LEBEDINSKAYA, A. N. TEREKHOV
Mathematics & Mechanics Faculty,
Saint Petersburg State University,
7/9 Universitetskaya Embankment,
Saint Petersburg,
RUSSIA

*Abstract:* - In this paper, we have developed a new approach to the local improvement of an approximate solution which has been obtained with the finite element method. The proposed improvement is adaptive, and it minimizes the energy functional. The discussed algorithm can be implemented analytically as well as numerically. The proposed approach can be applied to the adaptive refinement of the previously calculated variational grid approximation. The proposed method allows you to expand the approximate space by adding new coordinate functions. The implementation of a local approximation requires a small amount of arithmetic operations. The number of arithmetic operations does not depend on the number of nodes of the previously calculated approximation. Examples of the implementation of the method are considered for a problem with strong degeneracy and for a problem without degeneracy.

*Key-Words:* - adaptive methods, refinement calculations, variational grid methods

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## 1 Introduction

Research in the field of artificial intelligence covers many areas of the development of Science and practice. These studies give opportunities to optimize efforts aimed at the effective solution of difficult problems. Here we give examples of some studies in the mentioned area. Paper [1] predicts the future of excitation energy transfer with artificial intelligence-based Quantum dynamics. In paper [2] the authors propose an algorithm with a faster convergence rate for federated edge learning. In paper [3] the authors propose a new algorithm of machine learning techniques. Based on the recent advancements in music structure analysis, the authors of paper [4] automate the evaluation process by introducing a collection of metrics. The development of a disagreement-based online learning algorithm is given in [5]. In [6] the authors develop a deep learning-based approach to model. Let us mention some works in which it is possible to effectively use learning systems and **the** means of artificial intelligence. Paper [7] is devoted to the solution of the nonstationary, integro-differential equation with a degenerate elliptic differential operator. It seems that the adaptive methods developed with the help of artificial intelligence, greatly simplifies the solution of problems of mathematical physics. Article [8] investigates the approximate solution to a nonlinear Volterra integro-differential equation. In article [9], the authors have proposed a highly efficient and accurate collocation method. It can also be assumed that the mentioned means would be useful in solving the problems considered in works [10] and [11].This work is devoted to the variational grid method for one-dimensional boundary value problems for differential equations of the second order. Such problems are often encountered in the study and modeling of various phenomena in physics and technology. The numerical solution of such problems with appropriate accuracy can be very labor intensive. The numerical solution of these tasks may require significant computer resource systems (with respect to memory, time and computational accuracy). Special difficulties arise in the case **when** solving degenerate equations. S.G. Michlin conducted many studies of these problems (see [12] - [13]). Computational stability issues are investigated in [12]. A comprehensive study of the approximation in variational grid methods in the case of uniform grids is given in the monograph [13]. A generation to irregular grids is available (see [14]). In well-known works, only the global improvement of the approximation was considered. In this case, large computing system resources (memory and run time) are required. The proposed local improvement requires fewer resources. The local improvement can be done analytically or numerically.

The aim of this work is to develop adaptive refinement methods based on the previously calculated variation-grid approximation. At the same time, an adaptive expansion of the projection space is discussed.

Let us dwell on the advantages of the proposed approach. The refinement under consideration is obtained as a result of a small number of arithmetic operations. The last one does not depend on the total number of nodes, used in obtaining the previously calculated numerical solution. The proposed approach supports adaptability, i.e. it automatically chooses the best version of the algorithm. The solution is refined locally. The refinement process involves only a few nodes of the original grid and their corresponding values of the approximate solution. The proposed method does not require global recalculation of the previously obtained approximate solution.

## 2 Background

Consider the one-dimensional boundary value problem

$-(p(x)u')' = f(x), \quad x \in (0,1),$ (1)

$u(0) = u(1) = 0,$ (2)

where $p(x)$ is a measurable bounded function $p(x) > p_0 = const > 0,\ x \in\ (0,1), f \in L_2(0,1)$. The generalized solution of the problem (1) -- (2) is the solution of the problem to a minimum of functional

$F(u) = \int_0^1 [\,p(x)\big(u'^{(x))^2} - 2u\, f(x)\big]dx$ (3)

on the energy space of the operator (1) -- (2) (see [1]). For an approximate solution of problem (1) -- (2) use **the** minimization of the functional (3) on one or another subspace (the Ritz methods, finite elements, etc.). We take the space of piecewise linear functions using a finite set of nodes on the segment [0,1] and vanishing at the ends of this segment. So, let the grid be given on the interval [0,1]

$X\!:\ 0 = x_0 < x_1 < \cdots < x_{M-1} < x_M = 1.$

The variant of the variational grid method under consideration consists of functional minimization

$F(\tilde{u}) = \sum_{j=0}^{M-1} \int_{x_j}^{x_{j+1}} \left[p(x)|\frac{v_{j+1}-v_j}{x_{j+1}-x_j}|^2 - 2f(x)\ \times \left(v_j + \frac{v_{j+1}-v_j}{x_{j+1}-x_j}(x - x_j)\right)\right] dx;$ (4)

as functions of the variables $v_1, v_2, \dots, vR_{M-1.}$ Here $v_0 = v_M = 0,$

$\tilde{u} = \tilde{u}(x) = v_j + \frac{v_{j+1}-v_j}{x_{j+1}-x_j}(x - x_j)\ \forall x \in (x_j, x_{j+1}),$

$j \in \{0,1, \dots, M-1\}.$

To reduce the error of the approximate solution for a given number of grid nodes, it is important to find their successful location. In those parts of the domain where the solution changes slowly, the number of nodes may not be large. However, in those parts of the domain where the solution changes quickly, you should **use** significantly more nodes. The measure of quality for the choice of nodes is the value of the functional (4). In this way, functional (4) is desirable to minimize both variables $v_1, v_2, \dots, v_{M-1}$, and variables $x_1, x_2, \dots, x_{M-1}$, **while** observing the condition $x_1 < x_2 < \cdots < x_{M-1}$.

The space of the piecewise linear continuous functions of the form $\tilde{u} = \tilde{u}(x) = v_j + \frac{v_{j+1}-v_j}{x_{j+1}-x_j}(x - x_j)\ \forall x \in (x_j, x_{j+1}), j \in \{0, \dots, M-1\}$ is denoted by $\tilde{S}(X)$. Here $v_i \in \mathcal{R}^1$ are arbitrary numbers with $i \in \{0, \dots, M-1\}$, and $v_0 = v_M = 0$. The space $\tilde{S}(X)$ is a subspace of the energy space $H_A$.

The basis of the space $\tilde{S}(X)$ consists of the functions

$\tilde{\omega}_j(x) = \frac{x-x_{j-1}}{x_j-x_{j-1}}\ for\ x \in (x_{j-1}, x_j),$

$\tilde{\omega}_j(x) = \frac{x_{j+1}-x}{x_{j+1}-x_j}\ for\ x \in (x_j, x_{j+1}),$

$\tilde{\omega}_j(x) = 0\ for\ x \in [a,b] \setminus [x_j, x_{j+1}].$

where $j = 1,2, \dots, M-1$. Minimizing the functional (3) on the space $\tilde{S}(X)$,

$F(\tilde{u}) ==> \min_{\tilde{u}\,\in\,\tilde{S}(X)} F(\tilde{u})$ (5)

we obtain a certain version of the variational grid method.

By $\tilde{u}_* denote$ the solution of problem (5).

Denote by $X_\xi$ $t$he grid, obtained from the grid X with adding the node $\xi\ \epsilon (x_k, x_{k+1})$. We have $X_\xi = X \cup \{\xi\}$. It is clear to see that

$\tilde{S}(X) \subset \tilde{S}(X_\xi), \tilde{S}(X_\xi) = L\{\tilde{S}(X), \omega_\xi\},$

where

$\omega_\xi(x) = \frac{x-x_k}{\xi-x_k}\ for\ x \in (x_k, \xi),$

$\omega_\xi(x) = \frac{x_{k+1}-x}{x_{k+1}-\xi}\ for\ x \in (\xi, x_{k+1}),$

$\omega_\xi(x) = 0\ for\ x \in [0,1] \setminus [x_k, x_{k+1}].$

Here $L\{\dots\}$ denotes the linear span of objects contained in curly braces. Let $\xi \in (x_k, x_{k+1})$, and the number $v_*(\xi)$ satisfies the condition

$\mathrm{I}u_* - (\tilde{u}_* + v_*(\xi)\omega_\xi\mathrm{I} = \min_{v \in R^1} \mathrm{I}u_* - (\tilde{u}_* + v\omega_\xi\mathrm{I}.$

Obviously the ratio $\min_{v \in R^1} \mathrm{I}u_* - (\tilde{u}_* + v\omega_\xi\mathrm{I} \le \mathrm{I}u_* - (\tilde{u}_* + w\omega_\xi\mathrm{I}\ \ \forall w \in R^1$. Hence

$\mathrm{I}u_* - (\tilde{u}_* + v_*(\xi)\omega_\xi\mathrm{I} \le \mathrm{I}u_* - \tilde{u}_*\mathrm{I}.$

Let $\xi_*$ be solution of the problem

$\min_{\xi \in\, [x_k, x_{k+1}]} \mathrm{I}u_* - (\tilde{u}_* + v_*(\xi)\omega_\xi\mathrm{I}.$

It is clear that finding the node $\xi_*$ in a certain sense optimizes the approximation of the solution $u_*$ of problem (1) -- (2).

The functional $F(\tilde{u})$ can be represented as

$$F(\tilde{u}) = \sum_{j=0}^{M-1} \int_{x_j}^{x_{j+1}} \left[ p(x) \left( \frac{v_{j+1}-v_j}{x_{j+1}-x_j} \right)^2 + q(x) \left( v_j + \frac{v_{j+1}-v_j}{x_{j+1}-x_j} (x - x_j) \right)^2 - 2f(x) \left( v_j + \frac{v_{j+1}-v_j}{x_{j+1}-x_j} (x - x_j) \right) \right] dx \qquad (6)$$

Let's raise the question of adaptively **by** adding a new node in the advanced fixed interval $(x_k, x_{k+1})$.

## 3 Case of One Node

At this point, we will assume $M = 2$. Let us introduce the notation $x_0 = 0$, $\xi = x_1$, $x_2 = 1$.

So, in the case under consideration $F(\tilde{u})$ represents a function of two variables. For this function, we introduce notation $\Phi(\xi, v)$, so that

$$\Phi(\xi, v) = F(\tilde{u}) = \int_0^\xi \left[ p(x) \left( \frac{v}{\xi} \right)^2 - 2f(x) \frac{v}{\xi} x \right] dx + \int_\xi^1 \left[ p(x) \left( \frac{v}{1-\xi} \right)^2 - 2f(x) \left( v - \frac{v}{1-\xi} (x - \xi) \right) \right] dx. \qquad (7)$$

From formula (7) we have

$$F(\tilde{u}) = v^2 [\xi^{-2} \int_0^\xi p(x)dx + (1-\xi)^{-2} \int_0^\xi p(x)dx] - 2v[\xi^{-1} \int_\xi^1 f(x)xdx + (1-\xi)^{-1} \int_\xi^1 f(x)(1-x)dx]. \qquad (8)$$

The minimum point of the quadratic form (8) for fixed $\xi$ is the point $v_* = v_*(\xi)$ defined by the formula

$$v_*(\xi) = [\xi^{-1} \int_\xi^1 f(x)xdx + (1-\xi)^{-1} \int_\xi^1 f(x)(1-x)dx] \times [\xi^{-2} \int_0^\xi p(x)dx + (1-\xi)^{-2} \int_\xi^1 p(x)dx]^{-1}.$$

The minimum of the mentioned quadratic form is equal to

$$\Phi(\xi, v_*(\xi)) = -[\xi^{-1} \int_\xi^1 f(x)xdx + (1-\xi)^{-1} \int_\xi^1 f(x)(1-x)dx]^2 \times [\xi^{-2} \int_0^\xi p(x)dx + (1-\xi)^{-2} \int_\xi^1 p(x)dx]^{-1} \qquad (9)$$

Relation (9) can be transformed into the form

$$\Phi(\xi, v_*(\xi)) = -[(1-\xi) \int_\xi^1 f(x)xdx + \xi \int_\xi^1 f(x)(1-x)dx]^2 \times [(1-\xi)^2 \int_0^\xi p(x)dx + \xi^2 \int_\xi^1 p(x)dx]^{-1}$$

In the case $p(x) = 1$, $f(x) = 2$, the solution to problem (1) -- (2) is the function $f(x) = x(1-x)$. Substituting $p(x) = 1$, $f(x) = 2$, into (9), we find $\Phi(\xi, v_*(\xi)) = -x(1-x)$. Calculating the minimum with respect to the variable $\xi$, we find the minimum point $\xi_* = 1/2$.

## 4 Adaptive Grid Refinement

Consider the question of the optimal location of the added node in the variational grid method for problem (1) - (2). Functional (6) should be minimized. Adding the node $\xi$ to the interval $(x_k, x_{k+1})$ and the value $v$ of the approximation at that node, we obtain the corresponding extension $\tilde{S}$ $(X_\xi)$. Our goal is to choose this node in the best way, i.e. so that the resulting approximation error of the desired solution is the smallest.

It is easy to see that it suffices to minimize the function

$$G_k(\xi, v) = \int_{x_k}^\xi [p(x) \left( \frac{v-v_k}{\xi-x_k} \right)^2 + q(x) \left( v_k + \frac{v-v_k}{\xi-x_k} (x - x_k) \right)^2 - 2f(x) \left( v_k + \frac{v-v_k}{\xi-x_k} (x - x_k) \right)] dx + \int_\xi^{x_{k+1}} [p(x) \left( \frac{v-v_{k+1}}{\xi-x_{k+1}} \right)^2 + q(x) \left( v_{k+1} + \frac{v-v_{k+1}}{\xi-x_{k+1}} (x - x_{k+1}) \right)^2 - 2f(x) \left( v_{k+1} + \frac{v-v_{k+1}}{\xi-x_{k+1}} (x - x_{k+1}) \right)] dx. \qquad (10)$$

Setting

$$P_k(\xi, v) = \left( \frac{v-v_k}{\xi-x_k} \right)^2 \int_{x_k}^\xi p(x)] dx + v^2 \int_{x_k}^\xi q(x)\, dx + 2v_k \frac{v-v_k}{\xi-x_k} \int_{x_k}^\xi q(x)(x-x_k)\, dx + \left( \frac{v-v_k}{\xi-x_k} \right)^2 \int_{x_k}^\xi q(x)\,(x-x_k)^2 dx - 2v_k \int_{x_k}^\xi f(x)\, d - 2\frac{v-v_k}{\xi-x_k} \int_{x_k}^\xi f(x)\,(x-x_k) dx, \qquad (11)$$

$$Q_k(\xi, v) = \left(\frac{v - v_{k+1}}{\xi - x_{k+1}}\right)^2 \int_{\xi}^{x_{k+1}} p(x)]dx + v^2 \int_{\xi}^{x_{k+1}} q(x)\,dx$$
$$+ 2v_{k+1}\frac{v - v_{k+1}}{\xi - x_{k+1}}\int_{\xi}^{x_{k+1}} q(x)(x - x_{k+1})\,dx$$
$$+ \left(\frac{v - v_{k+1}}{\xi - x_{k+1}}\right)^2 \int_{\xi}^{x_{k+1}} q(x)\,(x - x_{k+1})^2 dx$$
$$- 2v_{k+1}\int_{\xi}^{x_{k+1}} f(x)\,dx$$
$$- 2\frac{v - v_{k+1}}{\xi - x_k + 1}\int_{\xi}^{x_{k+1}} f(x)\,(x - x_{k+1})dx, \quad (12)$$

we have

$$G_k(\xi, v) = P_k(\xi, v) + Q_k(\xi, v)\,. \quad (13)$$

Let us introduce the notation

$$p_{k,0}(\xi) = \int_{x_k}^{\xi} p(x)\,dx,$$
$$q_{k,0}(\xi) = \int_{x_k}^{\xi} q(x)\,dx, \quad (14)$$
$$q_{k,1}(\xi) = \int_{x_k}^{\xi} q(x)(x - x_k)\,dx,$$
$$q_{k,2}(\xi) = \int_{x_k}^{\xi} q(x)(x - x_k)^2\,dx, \quad (15)$$
$$f_{k,0}(\xi) = \int_{x_k}^{\xi} f(x)\,dx,$$
$$f_{k,1}(\xi) = \int_{x_k}^{\xi} f(x)(x - x_k)\,dx, \quad (16)$$
$$\bar{p}_{k+1,0}(\xi) = \int_{\xi}^{x_{k+1}} p(x)\,dx,$$
$$\bar{q}_{k+1,0}(\xi) = \int_{\xi}^{x_{k+1}} q(x)\,dx, \quad (17)$$
$$\bar{q}_{k+1,1}(\xi) = \int_{\xi}^{x_{k+1}} q(x)(x - x_{k+1})\,dx,$$
$$\bar{q}_{k+1,2}(\xi) = \int_{\xi}^{x_{k+1}} q(x)(x - x_{k+1})^2\,dx, \quad (18)$$
$$\bar{f}_{k+1,0}(\xi) = \int_{\xi}^{x_{k+1}} f(x)\,dx,$$
$$\bar{f}_{k+1,1}(\xi) = \int_{\xi}^{x_{k+1}} f(x)(x - x_{k+1})\,dx, \quad (19)$$

In addition, let's put

$$\alpha = (\xi - x_k)^{-2}\left[p_{k,0}(\xi) + q_{k,2}(\xi)\right], \quad (20)$$
$$\beta = 2(\xi - x_k)^{-1}\left[v_k q_{k,1}(\xi) + f_{k,1}(\xi)\right], \quad (21)$$
$$\gamma = v_k\left[v_k q_{k,0}(\xi) - 2f_{k,0}(\xi)\right], \quad (22)$$
$$\bar{\alpha} = (\xi - x_{k+1})^{-2}\left[\bar{p}_{k+1,0}(\xi) + \bar{q}_{k+1,2}(\xi)\right], \quad (23)$$
$$\bar{\beta} = 2(\xi - x_{k+1})^{-1} \times \left[v_{k+1}\bar{q}_{k+1,1}(\xi) + \bar{f}_{k+1,1}(\xi)\right], \quad (24)$$
$$\bar{\gamma} = v_{k+1}\left[v_{k+1}\bar{q}_{k+1,0}(\xi) - 2\bar{f}_{k+1,0}(\xi)\right], \quad (25)$$

**Theorem 1.** *The next formulas are right*

$$G_k(v, \xi) = A(\xi)v^2 + \mathrm{B}(\xi)\,v + \mathrm{C}(\xi), \quad (26)$$

*where*

$$A(\xi) = \alpha + \bar{\alpha}, \qquad \mathrm{B}(\xi) = \beta + \bar{\beta} - 2v_k\alpha - 2v_{k+1}\bar{\alpha}, \quad (27)$$
$$\mathrm{C}(\xi) = v_k^2\alpha + v_{k+1}^2\bar{\alpha} - v_k\beta - v_{k+1}\bar{\beta} + \gamma + \bar{\gamma}. \quad (28)$$

**Proof.** By relations (11) and (14) - (16) we have

$$P_k(\xi, v) = \left(\frac{v - v_k}{\xi - x_k}\right)^2 p_{k,0}(\xi) + v_k^2 q_{k,0}(\xi) + 2v_k\frac{v - v_k}{\xi - x_k}q_{k,1}(\xi) + \left(\frac{v - v_k}{\xi - x_k}\right)^2 q_{k,2}(\xi) - 2v_k f_{k,0}(\xi) - 2\frac{v - v_k}{\xi - x_k}f_{k,1}(\xi). \quad (12) \quad (29)$$

From relations (12) and (17) - (19) we find

$$Q_k(\xi, v) = \left(\frac{v - v_{k+1}}{\xi - x_{k+1}}\right)^2 \bar{p}_{k+1,0}(\xi) + v_{k+1}^2\bar{q}_{k+1,0}(\xi) + 2v_{k+1}\frac{v - v_{k+1}}{\xi - x_{k+1}}\bar{q}_{k+1,1}(\xi) + \left(\frac{v - v_{k+1}}{\xi - x_{k+1}}\right)^2 \bar{q}_{k+1,2}(\xi) - 2v_{k+1}\bar{f}_{k+1,0}(\xi) - \frac{v - v_{k+1}}{\xi - x_{k+1}}\bar{f}_{k+1,1}(\xi). \quad (30)$$

Formulas (29) -- (30) can be represented

$$P_k(\xi, v) = \left(\frac{v - v_k}{\xi - x_k}\right)^2 (p_{k,0}(\xi) + q_{k,2}(\xi)) + 2\frac{v - v_k}{\xi - x_k}(v_k q_{k,1}(\xi) - f_{k,1}(\xi)) + v_k(v_k q_{k,0}(\xi) - 2f_{k,0}(\xi)), \quad (31)$$

$$Q_k(\xi, v) = \left(\frac{v - v_{k+1}}{\xi - x_{k+1}}\right)^2 (\bar{p}_{k+1,0}(\xi) + \bar{q}_{k+1,2}(\xi)) + v_{k+1}\,(v_{k+1}\,\bar{q}_{k+1,0}(\xi) - 2\bar{f}_{k+1,0}(\xi)) + \frac{v - v_{k+1}}{\xi - x_{k+1}}(2v_{k+1}\bar{q}_{k+1,1}(\xi) - \bar{f}_{k+1,1}(\xi)). \quad (32)$$

Using formulas (13) and (31) -- (32), we obtain representation (26) -- (28). The theorem has been proven.

*Remark 2. The minimum point of the quadratic function (26) is the point*

$$v_*(\xi) = -\frac{\mathrm{B}(\xi)}{2\mathrm{A}(\xi)}. \quad (33)$$

*Thus*

$$\min_{v \in R^1} G_k(\xi, v) = G_k(\xi, v_*(\xi)) = -\frac{\mathrm{B}^2(\xi)}{4\mathrm{A}(\xi)} + \mathrm{C}(\xi). \quad (34)$$

Consider the function $G_k(\xi, v_*(\xi))$ depending on ξ, putting $R_k(\xi) = G_k(\xi, v_*(\xi))$.

From formulas (33) -- (34) we obtain the relation

$$R_k(\xi) = \frac{1}{2}v_*(\xi)\mathrm{B}(\xi) + \mathrm{C}(\xi). \quad (35)$$

The problem of finding the optimal position of the point ξ goes to finding the minimum of the function $R_k(\xi)$ on the interval $[x_k, x_{k+1}]$.

Let's consider several special cases.

# 5 Non-degenerate Boundary Value Problem

Consider the boundary value problem

$$-u'' = 2, \quad u(0) = u(1) = 0.$$

The solution to this problem is the function $u(x) = x(1 - x)$.

In the case under consideration we have

$$p(x) = 1, \ q(x) = 0, \ f(x) = 2.$$

According to formulas (14) - (19) we have

$$p_{k,0}(\xi) = \xi - x_k,$$
$$q_{k,0}(\xi) = q_{k,1}(\xi) = q_{k,2}(\xi) = 0, \qquad (36)$$
$$f_{k,0}(\xi) = 2(\xi - x_k), \quad f_{k,1}(\xi) = (\xi - x_k)^2, \qquad (37)$$
$$\bar{p}_{k+1,0}(\xi) = x_{k+1} - \xi,$$
$$\bar{q}_{k+1,0}(\xi) = \bar{q}_{k+1,1}(\xi) = \bar{q}_{k+1,2}(\xi) = 0, \qquad (38)$$
$$\bar{f}_{k+1,0}(\xi) = x_{k+1} - \xi,$$
$$\bar{f}_{k+1,1}(\xi) = -(x_{k+1} - \xi)^2. \qquad (39)$$

Using formulas (36) - (39) in relations (31) - (32), we find

$$P_k(\xi, v) = \frac{(v - v_k)^2}{\xi - x_k} - 2(v - v_k)(\xi - x_k) - 4v_k(\xi - x_k),$$

$$Q_k(\xi, v) = \frac{(v - v_{k+1})^2}{x_{k+1} - \xi} + 2(v - v_{k+1})(\xi - x_{k+1}) + 4v_{k+1}(\xi - x_{k+1}).$$

In accordance with formulas (20) - (25) we have

$$\alpha = (\xi - x_k)^{-1}, \ \beta = -2(\xi - x_k), \ \gamma = -4v_k(\xi - x_k),$$
$$\bar{\alpha} = (x_{k+1} - \xi)^{-1}, \ \bar{\beta} = -2(x_{k+1} - \xi), \ \bar{\gamma} = -4v_{k+1}(x_{k+1} - \xi).$$

**Theorem 2.** *Relation*

$$v_*(\xi) = (\xi - x_k) \quad (x_{k+1} - \xi) + [v_k(x_{k+1} - \xi) + v_{k+1}(\xi - x_k)](x_{k+1} - x_k)^{-1}. \qquad (40)$$

*is correct.*

**Proof**. In view of relations (27) - (28) we obtain

$$\mathrm{A}(\xi) = (\xi - x_k)^{-1} + (x_{k+1} - \xi)^{-1} = (\xi - x_k)^{-1}(x_{k+1} - \xi)^{-1}(x_{k+1} - x_k),$$

$$\mathrm{B}(\xi) = -2(x_{k+1} - x_k) - 2v_k(\xi - x_k)^{-1} - 2v_{k+1}(x_{k+1} - \xi)^{-1}.$$

Note that for $v_k > 0$, $v_{k+1} > 0$ and $\xi \epsilon (x_k, x_{k+1})$ we have $\mathrm{B}(\xi) < 0$. Expression $\mathrm{B}(\xi)$ can be written as

$$\mathrm{B}(\xi) = -2(\xi - x_k)^{-1} \quad (x_{k+1} - \xi)^{-1}[(\xi - x_k)(x_{k+1} - \xi)(x_{k+1} - x_k) + v_k(x_{k+1} - \xi) + v_{k+1}(\xi - x_k)].$$

Similarly, we find

$$\mathrm{C}(\xi) = v_k^2(\xi - x_k)^{-1} + v_{k+1}^2 - 2v_k(\xi - x_k) - 2v_{k+1}(x_{k+1} - \xi). \qquad (41)$$

From formula (33) we obtain relation (40). This concludes the proof.

Corollary 1. *For $\xi \epsilon (x_k, x_{k+1})$, $v_k > 0, v_{k+1} > 0$ the inequalities $v_*(\xi) > 0$, $B(\xi) < 0$ hold. In addition, the formulas*

$$v_*(x_k) = v_k, \quad v_*(x_{k+1}) = v_{k+1}$$

*are right.*

The proof obviously follows from formula (40).

**Theorem 3.** *The formula*

$$\min_{\xi \epsilon [x_k, x_{k+1}]} R_k(\xi) = R_k\left(\frac{x_k + x_{k+1}}{2}\right)$$

*holds.*

**Proof.** By relation (35) we have

$$R_k(\xi) = \frac{1}{2} v_*(\xi)\mathrm{B}(\xi) + \mathrm{C}(\xi) \qquad (42)$$

The second term in (42) is defined by relation (41), while the first term can be represented as

$$v_*(\xi)\mathrm{B}(\xi)/2 = -(\xi - x_k)^{-1}(x_{k+1} - \xi)^{-1} (x_{k+1} - x_k)^{-1} \times [(\xi - x_k)(x_{k+1} - \xi)(x_{k+1} - x_k) + v_k(x_{k+1} - \xi) + v_{k+1}(\xi - x_k)]^2.$$

To find the critical point of function $R_k(\xi)$ from (42) we have

$$v_*'(\xi)\mathrm{B}(\xi) + v_*(\xi)\mathrm{B}'(\xi) + 2C'(\xi) = 0.$$

It is easy to check that this equation is satisfied by the value $\xi = \xi_* = \frac{x_k + x_{k+1}}{2}$.

The study of the second derivative shows that point $\xi_*$ is the minimum point of the function $R_k(\xi)$ on the segment $[x_k, x_{k+1}]$.. This completes the proof.

Corollary 2. *In the case under consideration, for the optimal refinement of the original grid, the grid intervals should be divided in half.*

**Theorem 4**. *Ratio*

$$R_k(x_k + 0) = -2(v_k + v_{k+1})(x_{k+1} - x_k) + \frac{v_{k+1}^2 + v_k^2 - v_k v_{k+1}}{x_{k+1} - x_k} \qquad (43)$$

*is true.*

**Proof**. Assuming $\eta = \xi - x_k$ we have

$$v_*(x_k + \eta) = \eta(x_{k+1} - x_k - \eta) + [v_k(x_{k+1} - x_k - \eta) + v_{k+1}\,\eta(x_{k+1} - x_k)^{-1}].$$

Hence $v_*(x_k + \eta) = [v_k + \eta\left(x_{k+1} - x_k + \frac{v_{k+1} - v_k}{x_{k+1} - x_k}\right) - \eta^2$.

In a similar way, we receive the presentation

$$B(x_k + \eta) = -2\eta^{-1}(x_{k+1} - x_k - \eta)^{-1} [\eta\,(x_{k+1} - x_k - \eta)(x_{k+1} - x_k) + v_k(x_{k+1} - x_k - \eta) + \eta v_{k+1}].$$

After elementary transformations, we find $\mathrm{B}(x_k + \eta) = -2v_k\,\eta^{-1} - 2[x_{k+1} - x_k + v_{k+1}(x_{k+1} - x_k - \eta)^{-1}]$.

For function $\mathrm{C}(\xi)$, we similarly obtain

$$\mathrm{C}(x_k + \eta) = v_k^2\eta^{-1} + v_{k+1}^2(x_{k+1} - x_k - \eta)^{-1} - 2v_k\,\eta - 2v_{k+1}(x_{k+1} - x_k - \eta).$$

Using representation (42), we have

$$R(x_k + \eta) = \frac{1}{2}\left\{v_k + \eta\left(x_{k+1} - x_k + \frac{v_{k+1} - v_k}{x_{k+1} - x_k}\right) - \eta^2\right\}\left\{-\frac{2v_k}{\eta} - 2[x_{k+1} - x_k + v_{k+1}(x_{k+1} - x_k - \eta)^{-1}]\right\} + \left\{\frac{v_k^2}{\eta} + v_{k+1}^2(x_{k+1} - x_k - \eta)^{-1} - 2v_k\eta - 2v_{k+1}(x_{k+1} - x_k - \eta)\right\}.$$

This implies the limit relation

$\lim_{\eta\to+0} R_k(x_k+\eta) = -v_k\left(x_{k+1}-x_k+\frac{v_{k+1}-v_k}{x_{k+1}-x_k}\right) - v_k\left(x_{k+1}-x_k+\frac{2v_{k+1}}{x_{k+1}-x_k}\right)+\frac{v_{k+1}^2}{x_{k+1}-x_k} - 2v_{k+1}(x_{k+1}-x_k)$,

The limit relation is equivalent to formula (43).

This concludes the proof.

For $\eta = \xi - x_k$, we have

$v_*(x_k+\eta) = \eta(x_{k+1}-x_k-\eta) + [v_k(x_{k+1}-x_k-\eta) + v_{k+1}\eta]\,(x_{k+1}-x_k)^{-1}$.

Let us determine the function $R_k(\xi)$ at the point $\xi = x_k$. Extending the function $R_k(\xi)$ by continuity to the point $x_k$, we have

$R_k(x_k) = -2(v_k+v_{k+1})\,(x_{k+1}-x_k) +$ =- $+\frac{v_{k+1}^2+v_k^2-v_kv_{k+1}}{x_{k+1}-x_k}$.

*Remark 2. Setting* $k=0, x_0=0, x_1=1, v_0=v_1=0$, *we get*

$P_0(\xi,v) = \frac{v^2}{\xi} - 2v\xi$,

$Q_0(\xi,v) = \frac{v^2}{1-\xi} - 2v(\xi-1)$, so that according to (10) *and (19) we have*

$\Phi(\xi,v) = G_0(\xi,v) = P_0(\xi,v) + Q_0(\xi,v) = \frac{v^2}{\xi(1-\xi)}$.

## 6 Degenerate Boundary Value Problem

Consider a model problem with strong degeneracy

$$-(xu')' + qu = \frac{1}{2}(qx^2-4x-q), u(1)=0, (44)$$

where $q = const > 0$.

The generalized solution of the problem is obtained by minimizing the functional (3) on the energy space of the operator (44).

The solution $u_*(x)$ of problem (44) is

$u_*(x) = \frac{1}{2}(x^2-1)$.

For the construction of the variational grid method, we use projection space $S^d(X)$ which was proposed in [13]. This space differs from space $S(X)$ by changing the coordinate function $\omega_1(x)$ to **a** function $\omega_1^d(x)$,

$\omega_1^d(x) = 1$ for $x\epsilon[x_0,x_1)$,

$\omega_1^d(x) = \frac{x_2-x}{x_2-x_1}$ for $x\epsilon[x_1,x_2)$

$\omega_1^d(x) = 0$ for $x\epsilon R^1\backslash[x_0,x_2)$.

As before, let's add the node $\xi$ to one of the intervals $(x_k,x_{k+1})$, where $k\epsilon\{1,2,3,\dots,M-1\}$.

In what follows, for convenience, we supply the superscript $d$ for objects related to the strong degeneration problem.

Let us proceed to the calculation of the functions (14) -- (19) under the conditions

$p(x) = x,\ q(x) = q = const,$

$$f^d(x) = \frac{1}{2}(qx\hat{}2 - 4x - q).$$

For the functions (14) -- (16) we successively obtain

$$p_{k,0}(\xi) = \int_{x_k}^{\xi} p(x)\,dx = \frac{1}{2}\left(\xi^2 - x_k^2\right),$$

$q_{k,0}(\xi) = \int_{x_k}^{\xi} q(x)\,dx = q(\xi - x_k)$,

$$q_{k,1}(\xi) = \int_{x_k}^{\xi} q(x)(x-x_k)\,dx = \frac{q}{2}(\xi-x_k)^2,$$

$$q_{k,2}(\xi) = \int_{x_k}^{\xi} q(x)(x-x_k)^2\,dx = \frac{q}{3}(\xi-x_k)^3,$$

$f_{k,0}^d(\xi) = \int_{x_k}^{\xi} f^d(x)\,dx] = \frac{1}{6}(\xi-x_k)\times[q\xi^2 + \xi(qx_k-6) + qx_k^2 - 6x_k - 3q]$,

,

$f_{k,1}^d(\xi) = \int_{x_k}^{\xi} f^d(x)(x-x_k)\,dx = \frac{1}{2}(\xi-x_k)\times \left[\frac{q}{4}\left(\xi^3+\xi^2x_k+\xi x_k^2+x_k^3\right) - \frac{qx_k+4}{3}\left(\xi^2+\xi x_k + x_k^2\right) + \frac{4x_k-q}{2}(\xi+x_k) - qx_k\right]$.

Similarly, we find the functions (17) -- (19).

$$\bar{p}_{k+1,0}(\xi) = \int_{\xi}^{x_{k+1}} p(x)\,dx = \frac{1}{2}(x_{k+1}^2 - \xi^2),$$

$$\bar{q}_{k+1,0}(\xi) = \int_{\xi}^{x_{k+1}} q(x)\,dx = q(x_{k+1}-\xi),$$

$\bar{q}_{k+1,1}(\xi) = \int_{\xi}^{x_{k+1}} q(x)(x-x_{k+1})\,dx = -\frac{q}{2}(x_{k+1}-\xi)^2$,

$$\bar{q}_{k+1,2}(\xi) = \int_{\xi}^{x_{k+1}} q(x)(x-x_{k+1})^2\,dx = \frac{q}{3}(x_{k+1}-\xi)^3,$$

$$\bar{f}_{k+1,0}^d(\xi) = \int_{\xi}^{x_{k+1}} f^d(x)\,dx = \frac{1}{6}(x_{k+1}-\xi)$$

$\times[q\xi^2 + \xi(qx_{k+1}-6) + qx_{k+1} - 6x_{k+1} - 3q]$,

$\bar{f}_{k+1,1}^d(\xi) = \int_{\xi}^{x_{k+1}} f^d(x)(x-x_{k+1})\,dx = \frac{1}{2}(x_{k+1}-\xi)[\frac{q}{4}(\xi^3+\xi^2x_{k+1}+\xi x_{k+1}^2+x_{k+1}^3) - \frac{4+qx_{k+1}}{3}\left(\xi^2+\xi x_{k+1}+x_{k+1}^2\right) + \frac{4x_{k+1}-q}{2}(x_{k+1}+\xi) + qx_{k+1}]$.

Now let's find expressions for $\alpha, \beta$, and $\gamma$, using formulas (20) - (22).. We have

$\alpha^d = (\xi-x_k)^{-2}[p_{k,0}(\xi) + q_{k,2}(\xi)] = \left[\frac{1}{2}(\xi+x_k) + \frac{q}{3}(\xi-x_k)^2\right](\xi-x_k)^{-1}$, $\beta^d = 2(\xi-x_k)^{-1}\left[v_kq_{k,1}(\xi) - f_{k,1}^d(\xi)\right] = v_kq(\xi-x_k) - [\frac{q}{4}(\xi^3+\xi^2x_k+\xi x_k^2+x_k^3) - \frac{qx_k+4}{3}(\xi^2+\xi x_k+x_k^2) + \frac{4x_k-q}{2}(\xi+x_k) - qx_k]$, $\gamma^d = v_k\left[v_kq_{k,0}(\xi) - 2f_{k,0}^d(\xi)q_{k,0}(\xi)\right] = -\frac{1}{3}v_k(\xi -$

$x_k)[-3v_k q + q\xi^2 + \xi(qx_k - 6) + qx_k^2 - 6x_k - 3q]$.

To find expressions for $\bar{\alpha}^d, \bar{\beta}^d, \bar{\gamma}^d$, we use formulas (23) -- (25),.

$\bar{\alpha}^d = (x_{k+1} - \xi)^{-1}\left[\frac{1}{2}(\xi + x_{k+1}) + \frac{q}{3}(\xi - x_{k+1})^2\right]$, $\bar{\beta}^d = v_{k+1}q(x_{k+1} - \xi) + [\frac{q}{4}(\xi^3 + \xi^2 x_{k+1} + \xi x_{k+1}^2 + x_{k+1}^3) - \frac{qx_{k+1}+4}{3}(\xi^2 + \xi x_{k+1} + x_{k+1}^2) + \frac{4x_{k+1}-q}{2}(\xi + x_{k+1}) + qx_{k+1}]$,

$\bar{\gamma}^d = -\frac{1}{3}v_{k+1}(x_{k+1} - \xi)[-3v_{k+1}q + q\xi^2 + \xi(qx_{k+1} - 6) + qx_{k+1}^2 - 6x_{k+1} - 3q]$.

We introduce the notation

$$G_k^d(\xi, v) = \int_{x_k}^{\xi}\left[x\left(\frac{v-v_k}{\xi-x_k}\right)^2 + q\left(v_k + \frac{v-v_k}{\xi-x_k}(x-x_k)\right)^2 - 2f^d(x)\left(v_k + \frac{v-v_k}{\xi-x_k}(x - x_k)\right)\right]dx + \int_{\xi}^{x_{k+1}}\left[x\left(\frac{v-v_{k+1}}{\xi-x_{k+1}}\right)^2 + q\left(v_{k+1} + \frac{v-v_{k+1}}{\xi-x_{k+1}}(x - x_{k+1})\right)^2 - 2f^d(x)\left(v_{k+1} + \frac{v-v_{k+1}}{\xi-x_{k+1}}(x - x_{k+1})\right)\right]dx.$$

**Theorem 5.** *In degenerate boundary value problem (44) formula*

$$G_k^d(v, \xi) = A^d(\xi)v^2 + B^d(\xi)v + C^d(\xi) \quad (45)$$

*is right. Here*

$A^d(\xi) = \alpha^d + \bar{\alpha}^d$,

$B^d(\xi) = -2v_k\alpha^d - 2v_{k+1}\ \bar{\alpha}^d + \beta^d + \bar{\beta}^d$,

$C^d(\xi) = v_k^2\alpha^d + v_{k+1}^2\bar{\alpha}^d - v_k\beta^d - v_{k+1}\bar{\beta}^d + \gamma^d + \bar{\gamma}^d$,

**Proof** of the stated statement is similar to Theorem 1's proof. The only difference is the above calculation of the functions $A^d(\xi)$, $B^d(\xi)$, $C^d(\xi)$.

The critical point $v_*^d(\xi)$ of the quadratic form (45) is

$$v_*^d(\xi) = -\frac{B^d(\xi)}{2A^d(\xi)}.$$

Thus

$$\min_v G_k^d(\xi, v) = G_k^d\left(\xi, v_*^d(\xi)\right) = -\frac{B^d(\xi)}{2A^d(\xi)} + C^d(\xi)\,. \quad (46)$$

Consider the function $G_k^d\left(\xi, v_*^d(\xi)\right)$, depending on $\xi$. We denote

$$R_k^d(\xi) = G_k^d\left(\xi, v_*^d(\xi)\right). \quad (47)$$

From formulas (46) - (47) we obtain the relation

$$R_k^d(\xi) = \frac{1}{2}v_*^d(\xi)B^d(\xi) + C^d(\xi).$$

It is easy to see that in order to find the minimum point of the function, one should find the roots of the equation $\frac{dR_k^d}{d\xi}(\xi) = 0$. In this case, the last one is a quadratic equation. We do not present the corresponding calculations.

# 7 Discussion

In this paper, we have developed a new approach to the local improvement of approximate solution which has been obtained with the variational grid method. The proposed method is adaptive, and it minimizes the energy functional. The discussed algorithm can be implemented analytically as well as numerically. The algorithm contains a small number of arithmetic operations. The number of the mentioned operations does not depend on the number of grid nodes of the original variational grid method. The proposed approach can be applied to the adaptive refinement of previously calculated variational grid approximations.The proposed method allows you to expand the approximate space by adding new coordinate functions. The implementation of a local approximation requires a small amount of arithmetic operations. The number of arithmetic operations does not depend on the number of nodes of the previously calculated approximation. Examples of the implementation of the method are considered for problems with strong degeneracy and for problems without degeneracy. In this paper, we consider a new approach to the refinement approximate solution obtained with the variational grid method. This approach is useful when you want to obtain a local refinement of the solution. In other words, the proposed approach is useful in cases where local refinement is required for the previously obtained approximate solution. The benefits of the new approach are as follows: 1) **t**he solution is refined locally**;** 2) **t**he refinement process involves only a few nodes of the original grid and their corresponding values of the approximate solution**;** 3) **t**he proposed method does not require global recalculation of the previously obtained approximate solution**;** 4) **t**he refinement under consideration is obtained as a result of a small number of arithmetic operations**;** 5) **t**he last one does not depend on the total number of nodes used in obtaining the previously calculated numerical solution**;** 6) **t**he proposed approach supports adaptability, i.e. it automatically chooses the best version of the algorithm.

# 8 Conclusion

One-dimensional boundary value problems are often found in the study and in the modeling of various phenomena in physics and technology. The numerical solution of such problems with appropriate accuracy can be very labor intensive. Known numerical methods for solving such problems are reduced to solving algebraic systems of a higher order. This may require significant computer system resources (with respect to memory, time and computational accuracy). Special difficulties arise when solving degenerate equations. They can be overcome by the modification of the projection space. To implement a local refinement requires a small amount of arithmetic, which does not depend on the dimension of the projection space. The illustrative examples given in the sixth and seventh sections of this paper show the effectiveness of the proposed approach. In these examples, the integrals were calculated explicitly, because they were tabular integrals. In the case of a complex structure of the coefficients of equation (1), it is proposed to use the approximate methods for calculated integrals. Similar studies are expected to be carried out on other boundary value problems in the future.

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