Bounding periodic orbits in second order systems

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Abstract: - This paper provides an upper bound for the number of periodic orbits in planar systems. The research results in, [7], and, [8], allows one to produce a bound on the number of periodic orbits/limit cycles.
Introducing the concept of Maximal Grade and Maximal Number of Periodic Orbits, a simple algebraic calculation leads to an upper bound on the number of periodic trajectories for general second order systems. In particular, it also applies to polynomial ODE's.
As far as the author is aware, such a powerful result is not available in the literature. Instead, the methods in this paper provide a tool to determine an upper bound on the periodic orbits/limit cycles for a wide range of dynamical systems.

Key-Words: - Periodic orbit, Limit cycles, Nonlinear ODE.

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1 Introduction

In recent papers, [7] and [8], a necessary and sufficient condition for the existence of periodic orbits in general second order oscillators (non-conservative) were presented. Moreover, the periodic orbit's phase portrait can be constructed (at least for a trajectory portion) by solving a first order singular nonlinear ODE.

However, sometimes the exact determination of a periodic orbit is not required and a bound on the number of them is enough, [1], [9], [10]. In this direction, even simple polynomial systems can oppose resistance to the possibility of finding bounds for their number of limit cycles, [6], [11].

This paper aims to provide an upper bound for the number of periodic orbits in a second order oscillator to the form:

$$\ddot{x}(t) = f(x(t), \dot{x}(t), a_1, a_2, \dots, a_m) \qquad (1)$$
$$x(t) \in \Re, f \in C^{n+1}(\Re)$$

Where $\{a_1, a_2, \dots, a_m\}$ is a real set of numeric parameters and n is the minimal derivative order such that:

$$\frac{d^n \left(\frac{f(x(t),\dot{x}(t),a_1,a_2,\dots,a_m)}{\dot{x}(t)} |_{\dot{x}=\phi(x)} \right)}{dx^n} = g\left(x(t)\right)$$

This means that, after n derivatives, a function $g(.)$ is obtained that does not depend on the parameters $\{a_1, a_2, \dots, a_m\}$.

Despite simplified versions of the nonlinear oscillators considered in this paper, up to the author's knowledge, no other available method is able to find an explicit upper bound for the number of periodic orbits.

The paper is organized as follows: Section 2 presents all the machinery necessary and the main results, Section 3 presents some examples of application and finally Section 4 some conclusions.

2 A bound for the number of periodic orbits

This section introduces some machinery needed to prove the main theorem:

Lemma 1 *Given a function $f \in C^{n+1}(\Re)$, and a function $\phi(x) \in C^0(\Re)$, that is a solution of:*

$$\begin{cases} \frac{d\phi(x)}{dx} = \frac{f(x,\phi(x),a_1,a_2,\dots,a_m)}{\phi(x)}, & \phi(A_i) = 0 \\ \frac{d\phi(x)}{dx}|_{x=A_i} \to \infty, & i = 1, \dots L \end{cases}$$

Then, $\phi(x) \in C^{n+1}(\Re \backslash A_i)$.

proof *By induction, taking a derivative:*

$$\frac{\frac{\partial f(x,\phi(x),a_1,a_2,\dots,a_m)}{\partial x}}{\phi(x)} + \\ -\frac{f(x,\phi(x),a_1,a_2,\dots,a_m)\cdot\frac{d\phi(x)}{dx}}{\phi(x)^2} \in C^0(\Re \backslash A_i)$$

In other words: $\frac{d^2\phi(x)}{dx^2} \in C^0(\Re \backslash A_i)$. Then, assuming that $\phi(x) \in C^n(\Re \backslash A_i)$ and taking an extra derivative:

$$\frac{d}{dx}\left[\underbrace{\frac{d^{n-1}}{dx^{n-1}}\left(\frac{f(x,\phi(x),a_1,a_2,\ldots,a_m)}{\phi(x)}\right)}_{C^{n-1}(\Re\backslash A_i)}\right]$$

Then, according to the derivative process above, the $n-1$ derivatives will lead a quotient:

$$\frac{d^{n-1}}{dx^{n-1}}\left(\frac{f(x,\phi(x),a_1,a_2,\ldots,a_m)}{\phi(x)}\right)= =\frac{P(x)}{Q(x)},\quad \{P(x),Q(x)\}\in C^{n-1}(\Re\backslash A_i)$$

Finally, an extra derivative in this quotient will lead: $\frac{d^{n+1}\phi(x)}{dx^{n+1}}\in C^{n+1}(\Re\backslash A_i)$. *This completes the proof.*

Lemma 2 *If a function $\phi(x)\in C^{n+1}(\Re)$ is solution of:*

$$\begin{cases}\frac{d\phi(x)}{dx}=\frac{f(x,\phi(x),a_1,a_2,\ldots,a_m)}{\phi(x)}, & \phi(A_i)=0,\quad i=1,\ldots L\\ \frac{d^{(i)}\phi(x)}{dx^i}|_{x=A_i}\to\infty, & i=1,\ldots L\end{cases}$$

Allowing $L\to\infty$, then is solution of:

$$\begin{cases}\frac{d^{(n+1)}\phi(x)}{dx^{n+1}}=\frac{d^n\left(\frac{f(x(t),\phi(x),a_1,a_2,\ldots,a_m)}{\phi(x)}\right)}{dx^n}=g\left(x(t)\right)\\ \phi(A_i)=0,\quad \frac{d^{(i)}\phi(x)}{dx^i}|_{x=A_i}\to\infty,\quad i=1,\ldots L\end{cases}$$

proof *The proof is rather straightforward taking n derivatives from the given ODE, in the view of Lemma 1. This completes the proof.*

With this lemma, it is clear that any periodic orbit of (1), it is a solution contained in, [8]:

$$\begin{cases}\frac{d^{(n+1)}\phi(x)}{dx^{n+1}}=\frac{d^n\left(\frac{f(x(t),\phi(x),a_1,a_2,\ldots,a_m)}{\phi(x)}\right)}{dx^n}=g\left(x(t)\right)\\ \phi(A_i)=0,\quad \frac{d^{(i)}\phi(x)}{dx^i}|_{x=A_i}\to\infty,\quad i=1,\ldots L\end{cases}\tag{2}$$

However, this ODE does not depend on the parameters $\{a_1,a_2,\ldots,a_m\}$, or in other words, equation (2) serves as a universal ODE containing all the possible periodic orbits corresponding to (1).

The final preliminary result is about uniqueness of solutions:

Lemma 3 (Uniqueness of solutions) *Given an ODE:*

$$\frac{d\phi(x)}{dx}=\frac{f(x,\phi(x),a_1,a_2,\ldots,a_m)}{\phi(x)}$$
$$\phi(A_i)=0,\quad i=1,\ldots L$$

Every solution satisfying the initial condition is unique.

proof *Let's suppose that two different solutions $\{\phi_1(x),\phi_2(x)\}$, $\phi_1(x)\neq\phi_2(x)\forall x\neq A_i$ exists:*

$$\begin{cases}\frac{d\phi_1(x)}{dx}=\frac{f(x,\phi_1(x),a_1,a_2,\ldots,a_m)}{\phi_1(x)}\\ \frac{d\phi_2(x)}{dx}=\frac{f(x,\phi_2(x),a_1,a_2,\ldots,a_m)}{\phi_2(x)}\end{cases}$$

Integrating between x and A_i:

$$\begin{cases}-\frac{\phi_1(x)^2}{2}=\int_x^{A_i}f(\sigma,\phi_1(\sigma),a_1,a_2,\ldots,a_m)\cdot d\sigma\\ -\frac{\phi_2(x)^2}{2}=\int_x^{A_i}f(\sigma,\phi_2(\sigma),a_1,a_2,\ldots,a_m)\cdot d\sigma\end{cases}$$

Subtracting both equations and taking absolute value:

$$\frac{1}{2}\cdot\left|\phi_1(x)^2-\phi_2(x)^2\right|= =\left|\int_x^{A_i}(f(\sigma,\phi_1(\sigma),a_1,a_2,\ldots,a_m)+ -f(\sigma,\phi_2(\sigma),a_1,a_2,\ldots,a_m))\cdot d\sigma\right|$$

Moreover:

$$\frac{1}{2}\cdot\left|\phi_1(x)^2-\phi_2(x)^2\right|^2= =\frac{1}{2}\cdot\left|\phi_1(x)^2-\phi_2(x)^2\right|\cdot \left|\int_x^{A_i}(f(\sigma,\phi_1(\sigma),a_1,a_a,\ldots,a_m)+ -f(\sigma,\phi_2(\sigma),a_1,a_2,\ldots,a_m))\cdot d\sigma\right|$$

Then:

$$\frac{1}{2}\cdot\left|\phi_1(x)^2-\phi_2(x)^2\right|\leq\int_x^{A_i}\nu(\sigma)\cdot\left|\phi_1(x)^2-\phi_2(x)^2\right|\cdot d\sigma$$

where $\nu(x)=|f(\sigma,\phi_1(\sigma),a_1,a_a,\ldots,a_m)+ -f(\sigma,\phi_2(\sigma),a_1,a_a,\ldots,a_m)|$.

Considering the Gronwall-Bellman inequality, [2], pp. 45, with zero independent term, then:

$$\left|\phi_1(x)^2-\phi_2(x)^2\right|\leq 0\cdot e^{2\cdot\int_x^{A_i}\nu(\sigma)\cdot d\sigma}=0\to\phi_1(x)=\phi_2$$

This completes the proof.

Finally, noticing that successive derivatives in (1) up to the order $n+1$ will lead:

$$\begin{cases} \frac{d^{(n+1)}\phi(x)}{dx^{n+1}} = \frac{d^n\left(\frac{f(x(t),\dot{x}(t),a_1,a_2,\ldots,a_m)}{\dot{x}(t)}|_{\dot{x}=\phi(x)}\right)}{dx^n} = \\ = g\left(x(t),\dot{x}(t),\ddot{x}(t),\ldots,\frac{d^n x(t)}{dx^n}\right) \\ \phi(A_i)=0, \quad \frac{d^{(i)}\phi(x)}{dx^i}|_{x=A_i}\to\infty, \quad i=1,\ldots L \end{cases}$$

This motivates the following definition:

Definition 1 (Maximal Grade (MG)) *Given a second order ODE (1) with its complementary ODE:*

$$\begin{cases} \frac{d\phi(x)}{dx} = \frac{f(x,\phi(x),a_1,a_2,\ldots,a_m)}{\phi(x)} \\ \phi(A_i)=0, \quad i=1,\ldots L \\ \frac{d^{(i)}\phi(x)}{dx^i}|_{x=A_i}\to\infty, \quad i=1,\ldots L \end{cases}$$

The maximal grade (MG) is defined to be the minimal derivative order, such that:

$$\begin{cases} \frac{d^{(n+1)}\phi(x)}{dx^{n+1}} = \frac{d^n\left(\frac{f(x(t),\dot{x}(t),a_1,a_2,\ldots,a_m)}{\dot{x}(t)}\right)}{dx^n} = \\ = g\left(x(t),\dot{x}(t),\ddot{x}(t),\ldots,\frac{d^n x(t)}{dx^n}\right) \\ \phi(A_i)=0, \quad \frac{d^{(i)}\phi(x)}{dx^i}|_{x=A_i}\to\infty, \quad i=1,\ldots L \end{cases} \tag{3}$$

In the same manner, the maximal number of periodic orbits is defined to be:

Definition 2 (MNPO) *Given a second order ODE (1) with its maximal complementary ODE (3) and defining its solution by $\phi(x)=\zeta(x)\in\mathbb{C}^{n+1}(\Re)$, the maximal number of periodic orbits is defined as the number of real solutions of $\zeta(x)=0$.*

Then, the following theorem provides an upper bound for the number of periodic orbits:

Theorem 1 *A second order ODE : $\ddot{x}(t) = f(x(t),\dot{x}(t),a_1,a_2,\ldots,a_n)$, $x(t)\in\Re, f\in C^{n+1}(\Re)$ possessing L limit cycles: $\{x(0)=A_i\in\Re^+, x(T)=A_i, \dot{x}(0)=0, \quad i=1,\ldots,L\}$, with a $MG=M, \quad M\neq 0$ and $MNPO=R$ (possibly with $R\to\infty$) satisfies:*

$$L\leq min\{R,M\}$$

proof *Recalling that the solution to (3) is defined to be: $\phi(x)=\zeta(x)\in\mathbb{C}^{n+1}(\Re)$. It turns out that this solution does not depend on the set of parameters $\{a_1,a_2,\ldots,a_m\}$, moreover: $\zeta(A_i)=0, \quad \forall i=1,\ldots,L$.*

On the other hand, let's recall that according to, [8], given a second order oscillator, the periodic orbits can be computed by solving the following auxiliary ODE:

$$\begin{cases} \frac{d\phi(x)}{dx} = \frac{f(x,\phi(x),a_1,a_2,\ldots,a_m)}{\phi(x)} \\ \phi(A_i)=0, \quad i=1,\ldots L \end{cases}$$

Let's denote the solution to this ODE by $\phi(x,a_1,a_2,\ldots,a_m) = \zeta(x,a_1,a_2,\ldots,a_m) \in \mathbb{C}^{n+1}(\Re)$. Notice that in this case, the solution does depends on the set of parameters $\{a_1,a_2,\ldots,a_m\}$.

Performing an asymptotic approximation for the solution $\phi(x,a_1,a_2,\ldots,a_m)$ with $x\to A_i, \quad \forall i=1,\ldots,L$ to the form, [5]:

$$\zeta(x,a_1,a_2,\ldots,a_m)\sim\zeta(x)+\alpha_L(x,a_1,a_2,\ldots,a_m) \quad (x\to A_i) \tag{4}$$

where α_L is a polynomial of degree L such that: $\alpha_L(A_i,a_1,a_2,\ldots,a_m)=0$. Moreover, let's prove that this asymptotic approximation satisfies (solution) the ODE:

$$\begin{cases} \frac{d\phi(x)}{dx} = \frac{f(x,\phi(x),a_1,a_2,\ldots,a_m)}{\phi(x)} \\ \phi(A_i)=0, \quad \frac{\phi(x)}{dx}|_{x=A_i}\to\infty, \quad i=1,\ldots L \end{cases}$$

Asymptotically:

$$lim_{x\to A_i} \quad \frac{d\left(\zeta(x)+\alpha_L(x,a_1,a_2,\ldots,a_m)\right)}{dx} = \\ = lim_{x\to A_i} \quad \frac{f(x,\zeta(x)+\alpha_L(x,a_1,\ldots,a_m),a_1,\ldots,a_m)}{\zeta(x)+\alpha_L(x,a_1,a_2,\ldots,a_m)}$$

Rewriting this limit as:

$$lim_{x\to A_i} \quad \frac{d\left(\zeta(x)\cdot\left(1+\frac{\alpha_L(x,a_1,\ldots,a_m)}{\zeta(x)}\right)\right)}{dx} = \\ = lim_{x\to A_i} \quad \frac{f(x,\zeta(x)\cdot\left(1+\frac{\alpha_L(x,a_1,\ldots,a_m)}{\zeta(x)}\right),a_1,\ldots,a_m)}{\zeta(x)\cdot\left(1+\frac{\alpha_L(x,a_1,\ldots,a_m)}{\zeta(x)}\right)}$$

Since $\alpha_L(A_i,a_1,a_2,\ldots,a_m)=0$ and $\zeta(A_i)=0$, applying L'Hospital rule:

$$lim_{x\to A_i} \quad \frac{d\zeta(x)}{dx} = \\ lim_{x\to A_i} \quad \frac{f(x,\zeta(x))}{\zeta(x)}$$

Showing that $\zeta(x)$ is actually a solution to the given ODE confirming the universality of $\zeta(x)$.

Note: *The polynomial derivative: $\alpha'_L = \frac{d\alpha_L(x,a_1,\ldots,a_m)}{dx}$ is another polynomial with one degree less, so this function is bounded. Moreover, the derivative: $\frac{d\zeta(x)}{dx} = \frac{f(x,\zeta)}{\zeta(x)}$, in the view* *of the solution's $\zeta(x)$ universality. In this case, $lim_{x\to A_i} \quad \frac{\alpha'_L\cdot\zeta(x)}{f(x,\zeta(x)} = 0$.*

On the other hand, according to Lemma 3 a solution $\zeta(x, a_1, a_2, \ldots, a_m)$ is unique, so let's prove that the polynomial $\alpha_L(x, a_1, a_2, \ldots, a_m))$ is, in fact, asymptotically unique.

Moreover, it is not difficult to conclude this assumption: two polynomials with identical roots ($A_i, \quad i = 1, \ldots, L$) are identical up to a constant.

Concluding that, asymptotically, the polynomial α_L is unique. Next, by virtue of Lemma 2, the solution $\zeta(x) + \alpha_L(x, a_1, a_2, \ldots, a_m)$ should provide an asymptotic solution to:

$$\frac{d^{(M+1)}\left(\zeta(x) + \alpha_L(x, a_1, \ldots, a_m)\right)}{dx^{M+1}} = \\ = \frac{d^M\left(\frac{f(x(t),\zeta(x)+\alpha_L(x,a_1,a_2,\ldots,a_m),a_1,\ldots,a_m)}{\zeta(x)+\alpha_L(x,a_1,a_2,\ldots,a_m)}\right)}{dx^M}, \quad (x \to A_i)$$

That is:

$$\frac{d^{(M+1)}\zeta(x)}{dx^{M+1}} + \frac{d^{M+1}\alpha_L(x, a_1, \ldots, a_m)}{dx^{M+1}} = \\ = \frac{d^M\left(\frac{f(x(t),\zeta(x)+\alpha_L(x,a_1,\ldots,a_m),a_1,\ldots,a_m)}{\zeta(x)+\alpha_L(x,a_1,a_2,\ldots,a_m)}\right)}{dx^M}$$

Moreover, by virtue of the universality of $\zeta(x)$:

$$\frac{d^{(M+1)}\zeta(x)}{dx^{M+1}} = \frac{d^M\left(\frac{f(x(t),\zeta(x),a_1,a_a,\ldots,a_m)}{\zeta(x)}\right)}{dx^M}$$

Then:

$$\frac{d^M\left(\frac{f(x(t),\zeta(x),a_1,\ldots,a_m)}{\zeta(x)}\right)}{dx^M} + \\ + \frac{d^{M+1}\alpha_L(x, a_1, \ldots, a_m)}{dx^{M+1}} = \\ = \frac{d^M\left(\frac{f(x(t),\zeta(x)+\alpha_L(x,a_1,a_2,\ldots,a_m),a_1,\ldots,a_m)}{\zeta(x)+\alpha_L(x,a_1,\ldots,a_m)}\right)}{dx^M}$$

Asymptotically:

$$\frac{d^M\left(\frac{f(x(t),\zeta(x),a_1,\ldots,a_m)}{\zeta(x)}\right)}{dx^M} + \frac{d^{M+1}\alpha_L(x, a_1, \ldots, a_m)}{dx^{M+1}} = \\ \frac{d^M\left(\frac{f(x(t),\zeta(x)+\alpha_L(x,a_1,\ldots,a_m),a_1,a_a,\ldots,a_m)}{\zeta(x)+\alpha_L(x,a_1,a_2,\ldots,a_m)}\right)}{dx^M} \sim \\ \frac{d^{M+1}\alpha_L(x, a_1, a_2, \ldots, a_m)}{dx^{M+1}} = 0 \quad (x \to A_i)$$

Having considered that: $\zeta(x) + \alpha_L(x, a_1, a_2, \ldots, a_m) \sim \zeta \quad (x \to A_i)$. This equivalence shows that all the derivatives above the order M will lead: $\frac{d^p\alpha_L(x,a_1,a_2,\ldots,a_m)}{dx^p} = 0, \quad p \geq M+1$, then at most M distinct roots are possible for the polynomial $\alpha_L(x, a_1, a_2, \ldots, a_m)$.

The key conclusion is about L and M: $L \leq M$. So the expression for the asymptotic equivalence can be written as follows:

$$\begin{cases} \zeta(x, a_1, a_2, \ldots, a_m) \sim \zeta(x) + \alpha_M(x, a_1, a_2, \ldots, a_m) \\ (x \to A_i) \end{cases}$$

On the other hand, by virtue of (4), it is clear that, asymptotically:

$$\begin{cases} \zeta(x, a_1, a_2, \ldots, a_m) = 0 \sim \alpha_M(x, a_1, a_2, \ldots, a_m) = 0 \\ (x \to A_i) \end{cases}$$

Recalling that, by definition, $\zeta(A_i) = 0, \quad \forall i = 1, \ldots, L$. In other words, A_i is the amplitude of a periodic orbit if and only if: $\zeta(A_i, a_1, a_2, \ldots, a_m) = 0$ or if and only if: $\alpha_M = 0$, that is, at most M periodic orbits for a second order oscillator with $MG = M, \quad M \neq 0$.

The number of actual periodic orbits is, in fact, $L \leq M$. Therefore, not all the zeroes of α_M indicate the actual orbits, but this serves as an upper bound as requested. Furthermore, if $M = 0$, then $\alpha_M = 0$ provides no insight into this bound on the periodic orbits' number.

Finally, it should be noticed that the general solution to (3), that does not depend on the parameters $\{a_1, a_2, \ldots, a_m\}$, it contains all the possible periodic orbits for any possible combination of these parameters. Similarly, for a given oscillator, certain parameter selections will raise a center ($R \to \infty$) or, with another choice, limit cycles ($R < \infty$).

Looking for an upper bound on the limit cycles, if $R \to \infty$ the conclusion is clear: $L \leq M$, however, if $R < \infty$, two possibilities arise by virtue of the asymptotic equivalence (4):

- *$R > M$, then only the M zeroes of α_M could satisfy the asymptotic equivalence, so: $L \leq M$*
- *$R < M$, then only the zeroes of $\zeta(x)$ could satisfy the asymptotic equivalence, so: $L \leq R$*

In summary: $L \leq min\{R, M\}$. This completes the proof.

Finally, specializing this result to oscillators: $\ddot{x}(t) = -x(t) + \frac{dF(x(t))}{dx(t)} \cdot \dot{x}(t)$, with $F(x)$ a polynomial of degree N:

Corollary 1 *A second order ODE :* $\ddot{x}(t) = \ddot{x}(t) = -x(t) + \frac{dF(x(t))}{dx(t)} \cdot \dot{x}(t)$*, with* $F(x)$ *a polynomial of degree* N *and the number of isolated periodic orbits L, a bounded from above is given by:* $L \leq N + 1$*.*

proof *Applying Theorem 1:* $M = N + 1$*, in addition to setting all the coefficients to zero in* $F(x)$*, a center is obtained, then* $R = \infty$*. This completes the proof.*

3 Examples

3.1 Classical Van der Pol's Oscillator

The classical Van der Pol's equation can be written to be, [3], pp. 6:

$$\ddot{x}(t) = -x(t) + \epsilon \cdot \left(1 - x(t)^2\right) \cdot \dot{x}(t)$$

Then, it is not difficult to obtain:

$$\frac{d\phi(x)}{dx} = -\frac{x}{\phi(x)} + \epsilon \cdot \left(1 - x^2\right)$$

Setting $\epsilon = 0$ clearly results in a center, so: $R = \infty$. On the other hand, taking derivatives:

$$\frac{d^4\phi(x)}{dx^4} = \frac{d^3}{dx^3}\left(-\frac{x}{\phi(x)}\right)$$

In this case: $MG = 3$, then: $L \leq 3$. It is known that for a Van der Pol system $L = 1$.

3.2 Polynomial systems

Considering the important case analysed by Dumortier, et. al, [4]:

$$\ddot{x}(t) = -\epsilon \cdot (x - \lambda) - \frac{\partial H(c, e, a, x)}{\partial x} \cdot \dot{x}$$

Where $H(c, e, a, x) = \left(x^2 - 1\right)^2 \cdot (c \cdot e \cdot x + 1) \cdot \left(x^2 + e \cdot x + \frac{1}{8}\right) - a \cdot x$. Considering polynomial's degree of $H(c, e, a, x)$ equal to 7 and according to Corollary 1: $M = 8$.

On the other hand, let's consider a generalization to the given polynomial $H(c, e, a, x)$ by:

$$H^*(c, e, a, x, d) = \left(x^2 - 1\right)^2 \cdot (c \cdot e \cdot x + d) \cdot \\ \cdot \left(x^2 + e \cdot x + \frac{1}{8}\right) - a \cdot x$$

It is clear that the conclusion about the number of limit cycles includes the case $d = 0$, or, in other words, the given system.

In this way, setting all the coefficients to zero: $\{a = 0, c = 0, d = 0, e = 0\}$:

$$\ddot{x}(t) = -\epsilon \cdot (x - \lambda)$$

Which is not more than a center: $R = \infty$. Then, the conclusion, according to Theorem 1 is: $L \leq 8$. This is in complete agreement with, [4], where the bound was: $L \geq 4$.

4 Conclusions

In this paper, in light of the recent necessary and sufficient conditions for periodic orbits in planar systems, [7], and, [8], a bound from above is presented for the number of periodic orbits, based on the concept of maximal grade (MG) and maximum number of periodic orbits (MNPO).

This result allowed, for instance, to establish an upper bound for polynomial systems based on a polynomial's degree, in particular, the influential case studied by Dumortier, et. al, was analysed and an upper bound obtained,[4].

In a future paper, oscillators without parameters or even bifurcation will be studied utilizing the method in this paper.

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References

[1] A. Lins Neto, W. de Melo, C.C. Pugh, On Liénard Equations, Proc. Symp. Geom. and Topol., Springer Lectures Notes in Math. Number 597, pp. 335–357, 1977.

[2] Linear System Theory. Wilson Rugh, Pearson, 2nd Edition. 1995.

[3] Oscillations in Planar Dynamic Systems (SERIES ON ADVANCES IN MATHEMATICS FOR APPLIED SCIENCES). R. E. Mickens. World Scientific. 1996.

[4] F. Dumortier, D. Panazzolo and R. Roussarie. More Limit Cycles than Expected i n Liénard Equations. Proceedings of the American Mathematical Society, Vol. 135, 6, 1895 1904, 2007.

[5] Asymptotic Methods in Analysis. N. G. De Brujin. Dover Publications.2010.

[6] Jaume Llibre and Xiang Zhang. Limit cycles of the classical Liènard differential systems: A survey on the Lins Neto, de Melo and Pugh's conjecture, Exposition. Math., vol.35, 386–299, 2017. DOI: [10.1016/j.exmath.2016.12.001]

[7] García, Andrés Gabriel (2019). First Integrals vs Limit Cycles. https://arxiv.org/abs/1909.07845, doi = 10.48550/ARXIV.1909.07845.

[8] He , J.-H. and García , A. (2021). The simplest amplitude-period formula for non-conservative oscillators. Reports in Mechanical Engineering, 2(1), 143-148. https://doi.org/10.31181/rme200102143h

[9] Rodrigo D Euzébio and Jaume Llibre and Durval J Tonon. (2022). Lower bounds for the number of limit cycles in a generalised Rayleigh–Liénard oscillator. IOP Publishing: Nonlinearity, 35(8), 3883. https://dx.doi.org/10.1088/1361-6544/ac7691

[10] Xianbo Suna, Wentao Huang (2017). Bounding the number of limit cycles for a polynomial Liénard system by using regular chains. Journal of Symbolic Computation, 79, 197-210.

[11] Yifan Hu, Wei Niu, Bo Huang (2021). Bounding the number of limit cycles for parametric Liénard systems using symbolic computation methods. Communications in Nonlinear Science and Numerical Simulation, 96, 105716. https://doi.org/10.1016/j.cnsns.2021.105716.