Calculation of Determinant of a two-variable Polynomial Matrix in Complex Basis

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Abstract: - In this paper, an innovation in evaluation part of Evaluation Interpolation technique (E-I) is presented. The variables of this technique have been defined in Complex domain. The use of complex numbers instead of real ones in evaluation part, provides us with the opportunity of using conjugate properties. With these properties, the number of evaluations in evaluation part is being decreased. The interpolation part is common for both methods.

Key-Words: - Complex Basis, Polynomial Matrix, Evaluation - Interpolation, Two-variable polynomials

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1 Introduction

A lot of solutions among Control Theory problems consists of polynomials. Some of them are calculation of the determinant of a polynomial matrix [1], [2], calculation of the greatest common divisor among polynomials [3], computation of the generalised inverse of a polynomial matrix [4], [5], [6], polynomial matrices as solutions of diophantine equations [7], transfer function computation for multidimensional systems [8].The need for solving Control Theory problems has led to the development of several numerical computational methods. Evaluation - Interpolation technique is one of these methods, and is used for solving problems whose solution is a polynomial. It consists of two parts. In the first part (evaluation part), a set of fixed required points is defined and evaluated, and the second part (interpolation part) is for finding the unique polynomial that passes through all these points. The main advantage of this technique is the use of numerical analysis instead of analytical solutions, which makes it easier for algorithms to handle.

The word 'interpolation' has been introduced by J. Wallis in early 1655. Nevertheless polynomial interpolation in several variables is a relatively new topic that has been raised in the second half of the last century. Multivariate polynomial interpolation is a basic subject both in Approximation Theory and in Numerical Analysis, as it has many applications in mathematical problems [9], [10]. That's why it has been the concern of many scientists. Some of the most known multivariate polynomial interpolation methods are: the use of Vandermonde matrix in addition with LU factorization, Lagrange interpolation, Hermite - Birkhoff interpolation, Newton interpolation, Discrete Fourier Transform (DFT) and Fast Fourier Transform algorithms (FFT) [2] [11] [12].

This work is based on Newton two-variable polynomial interpolation method. First of all a set of fixed points is defined (evaluation part), and then with the interpolation method that has been just referred, the exact polynomial that approximates these points is being constructed. We study the special case, where the upper bounds of the degree of each variable is known, so we use a rectangular basis. In addition we define the required set of points in Complex domain. Taking into account conjugate properties, we prove that the number of required points can be reduced even up to the half. We perform this consideration through examples such as the calculation of the determinant of a two variable polynomial matrix both in Real and in Complex domain. As a conclusion, the number of required operations for this process is reduced.

2 Calculation of Determinant in Real basis

The following algorithm is based on Determinant Calculation with Evaluation-Interpolation technique [2].

2.1 A general algorithm

Let the two-variable polynomial matrix $A \in \mathbb{R}[x, y]^{n\times n}$

$$A(x, y) = \begin{bmatrix} a_{1,1}(x, y) & \cdots & a_{1,n}(x, y) \\ \vdots & \ddots & \vdots \\ a_{n,1}(x, y) & \cdots & a_{n,n}(x, y) \end{bmatrix}$$

Step 1:(Calculation of matrice's degree in each variable)

Degree matrices of x and y variable respectively, are

$$D^x = \begin{bmatrix} deg_x a_{1,1}(x,y) & \cdots & deg_x a_{1,n}(x,y) \\ \vdots & \ddots & \vdots \\ deg_x a_{n,1}(x,y) & \cdots & deg_x a_{n,n}(x,y) \end{bmatrix}$$

and

$$D^y = \begin{bmatrix} deg_y a_{1,1}(x,y) & \cdots & deg_y a_{1,n}(x,y) \\ \vdots & \ddots & \vdots \\ deg_y a_{n,1}(x,y) & \cdots & deg_y a_{n,n}(x,y) \end{bmatrix}$$

where $deg_x a_{i,j}(x,y),\ i = 1,\cdots,n,\ j = 1,\cdots,n$ is the degree of $a_{i,j}(x,y)$ element of A matrix with respect to x variable and $deg_y a_{i,j}(x,y),\ i = 1,\cdots,n,\ j = 1,\cdots,n$ is the degree of $a_{i,j}(x,y)$ element of A matrix with respect to y variable.
Let dc_s^x, dr_s^x, dc_s^y and dr_s^y the sum of the maximum element of each column and row of the matrices above.

$$dc_j^x = \max\{deg_x a_{i,j}(x,y) \mid i = 1,\cdots,n\}, \quad for\ j = 1,\cdots,n$$

$$dc_s^x = \sum_{j=1}^{n} dc_j^x$$

$$dr_i^x = \max\{deg_x a_{i,j}(x,y) \mid j = 1,\cdots,n\}, \quad for\ i = 1,\cdots,n$$

$$dr_s^x = \sum_{i=1}^{n} dr_i^x$$

$$dc_j^y = \max\{deg_y a_{i,j}(x,y) \mid i = 1,\cdots,n\}, \quad for\ j = 1,\cdots,n$$

$$dc_s^y = \sum_{j=1}^{n} dc_j^y$$

$$dr_i^y = \max\{deg_y a_{i,j}(x,y) \mid j = 1,\cdots,n\}, \quad for\ i = 1,\cdots,n$$

$$dr_s^y = \sum_{i=1}^{n} dr_i^y$$

Thus, according to [2] we have

$$k_1 = \min\{dc_s^x, dr_s^x\}$$
$$k_2 = \min\{dc_s^y, dr_s^y\}$$

Therefore, the number of required points is $N = (k_1+1)(k_2+1)$.
Step 2: (Evaluation Part)
Step 2a: Determination of the set of N random fixed points

$$S^{(k_1,k_2)} = \{(x_i, y_j) \mid i = 0,\ldots,k_1,\ j = 0,\ldots,k_2\}$$

Step 2b: Evaluation of all points of $S^{(k_1,k_2)}$ set in the numerical matrix A and computation of their determinant $\det(A(x_i,y_j))$ where $i = 0,\ldots,k_1,\ j = 0,\ldots,k_2$. The interpolation set is determined as follows:
$\{(A(x_i,y_j), \det(A(x_i,y_j))) \mid i = 0,\cdots,k_1\ j = 0,\cdots,k_2\}$
Step 3: (Interpolation Part)
Step 3a: Construction of the zero order table ($k = 0$) of required points (interpolation set).
Step 3b: Computation of the k-th order table of divided differences, where $k = 1,\cdots,n$ and $n = \max\{k_1,k_2\}$ according to [12]
Step 3c: Construction of the Newton polynomial

$$p(x,y) = X^T \cdot P \cdot Y$$

where

$$X = \begin{bmatrix} 1 \\ (x-x_0) \\ (x-x_0)\cdot(x-x_1) \\ \vdots \\ (x-x_0)\cdot(x-x_1)\cdot\cdots\cdot(x-x_{k_1-1}) \end{bmatrix}$$

P is the n-th order table of divided differences and

$$Y = \begin{bmatrix} 1 \\ (y-y_0) \\ (y-y_0)\cdot(y-y_1) \\ \vdots \\ (y-y_0)\cdot(y-y_1)\cdot\cdots\cdot(y-y_{k_2-1}) \end{bmatrix}$$

The calculated Newton polynomial is the determinant of the two-variable polynomial matrix, namely, $p(x,y) = \det(A(x,y))$.

3 Calculation of Determinant in Complex basis

In this case we are following the same procedure, just the way we did it above. The only difference is that we use interpolation points of the complex domain, which allow us to reduce their number due to the conjugate properties of the complex domain.

Proposition 1. *Let the two-variable polynomial*

$$p(x,y) = \sum_{n=0}^{k_1}\sum_{m=0}^{k_2} c_{n,m} x^n y^m$$

where k_1 is the degree of x variable and k_2 is the degree of y variable.
Let the point (ai, b), where i is the imaginary unit, and $a, b \in \mathbb{Z}$, then holds

$$p(\overline{ai, b}) = \overline{p(ai, b)} \tag{1}$$

Proof. We compute the value of $p(ai, b)$

$$\begin{aligned} p(ai, b) &= \sum_{n=0}^{k_1} \sum_{m=0}^{k_2} c_{n,m} (ai)^n b^m \\ &= \sum_{n=0}^{k_1} \sum_{m=0}^{k_2} c_{n,m} a^n b^m i^n \end{aligned}$$

$$= \begin{Bmatrix} \sum\limits_{n=0}^{k_1} \sum\limits_{m=0}^{k_2} c_{n,m} a^n b^m & \text{for} \quad n = 4k \\ \sum\limits_{n=0}^{k_1} \sum\limits_{m=0}^{k_2} c_{n,m} a^n b^m i & \text{for} \quad n = 4k+1 \\ \sum\limits_{n=0}^{k_1} \sum\limits_{m=0}^{k_2} c_{n,m} a^n b^m (-1) & \text{for} \quad n = 4k+2 \\ \sum\limits_{n=0}^{k_1} \sum\limits_{m=0}^{k_2} c_{n,m} a^n b^m (-i) & \text{for} \quad n = 4k+3 \end{Bmatrix}$$

$$= \begin{Bmatrix} \sum\limits_{n=0}^{k_1} \sum\limits_{m=0}^{k_2} c_{n,m} a^n b^m & \text{for} \quad n = 4k \\ \left(\sum\limits_{n=0}^{k_1} \sum\limits_{m=0}^{k_2} c_{n,m} a^n b^m \right) i & \text{for} \quad n = 4k+1 \\ -\left(\sum\limits_{n=0}^{k_1} \sum\limits_{m=0}^{k_2} c_{n,m} a^n b^m \right) & \text{for} \quad n = 4k+2 \\ -\left(\sum\limits_{n=0}^{k_1} \sum\limits_{m=0}^{k_2} c_{n,m} a^n b^m \right) i & \text{for} \quad n = 4k+3 \end{Bmatrix}$$

where $k \in \mathbb{N}$.
Similarly, we compute the value of $p(\overline{ai, b})$

$$\begin{aligned} p(\overline{ai, b}) &= p(\overline{ai}, b) = p(-ai, b) \\ &= \sum_{n=0}^{k_1} \sum_{m=0}^{k_2} c_{n,m} (-ai)^n b^m \\ &= \sum_{n=0}^{k_1} \sum_{m=0}^{k_2} (-1)^n c_{n,m} a^n b^m i^n \end{aligned}$$

$$= \begin{Bmatrix} \sum\limits_{n=0}^{k_1} \sum\limits_{m=0}^{k_2} (+1) c_{n,m} a^n b^m & \text{for} \quad n = 4k \\ \sum\limits_{n=0}^{k_1} \sum\limits_{m=0}^{k_2} (-1) c_{n,m} a^n b^m i & \text{for} \quad n = 4k+1 \\ \sum\limits_{n=0}^{k_1} \sum\limits_{m=0}^{k_2} (+1) c_{n,m} a^n b^m (-1) & \text{for} \quad n = 4k+2 \\ \sum\limits_{n=0}^{k_1} \sum\limits_{m=0}^{k_2} (-1) c_{n,m} a^n b^m (-i) & \text{for} \quad n = 4k+3 \end{Bmatrix}$$

$$= \begin{Bmatrix} \sum\limits_{n=0}^{k_1} \sum\limits_{m=0}^{k_2} c_{n,m} a^n b^m & \text{for} \quad n = 4k \\ -\left(\sum\limits_{n=0}^{k_1} \sum\limits_{m=0}^{k_2} c_{n,m} a^n b^m \right) i & \text{for} \quad n = 4k+1 \\ -\left(\sum\limits_{n=0}^{k_1} \sum\limits_{m=0}^{k_2} c_{n,m} a^n b^m \right) & \text{for} \quad n = 4k+2 \\ \left(\sum\limits_{n=0}^{k_1} \sum\limits_{m=0}^{k_2} c_{n,m} a^n b^m \right) i & \text{for} \quad n = 4k+3 \end{Bmatrix}$$

where $k \in \mathbb{N}$.
As we can see, it holds $p(\overline{ai, b}) = \overline{p(ai, b)}$ □

Thus, because of the fact that the determinant of a polynomial matrix is also a polynomial, it comes out that

$$\det(A(-ai, b)) = \det(A(\overline{ai, b})) = \overline{\det(A(ai, b))} \tag{2}$$

where a,b $\in \mathbb{Z}$

3.1 A general algorithm

Let the two-variable polynomial matrix $A \in \mathbb{R}[x, y]^{n \times n}$

$$A(x, y) = \begin{bmatrix} a_{1,1}(x, y) & \cdots & a_{1,n}(x, y) \\ \vdots & \ddots & \vdots \\ a_{n,1}(x, y) & \cdots & a_{n,n}(x, y) \end{bmatrix}$$

Step 1:(Calculation of matrice's degree in each variable)
The **Step 1** is the same with the algorithm in subsection 2.1. Then

$$\begin{aligned} k_1 &= \min\{dc_s^x, dr_s^x\} \\ k_2 &= \min\{dc_s^y, dr_s^y\} \end{aligned}$$

Therefore, the number of required points is $N = (k_1 + 1)(k_2 + 1)$.
Step 2: (Evaluation Part)
Step 2a: Determination of the set of N random fixed points

$$S^{(k_1,k_2)} = \{(x_{l_1} \cdot i, y_{l_2}) \mid l_1 = 0, \ldots, k_1, \\ l_2 = 0, \ldots, k_2\}$$

where i is the imaginary unit.
Step 2b: From 2 we have that

$$\det(A(-ai, b)) = \det(A(\overline{ai, b})) = \overline{\det(A(ai, b))}$$

Thus, evaluation is executed only for $\lceil \frac{N}{2} \rceil$ complex points of $S^{(k_1,k_2)}$ set in the numerical matrix A and the computation of their determinant $\det(A(x_{l_1} \cdot i, y_{l_2}))$ follows. The determnants of the next $N - \lceil \frac{N}{2} \rceil$ points of $S^{(k_1,k_2)}$ set occur as conjugate numbers of the previous one. The interpolation set is determined as follows:

$$\{(A(x_{l_1}\cdot i, y_{l_2}), det(A(x_{l_1}\cdot i, y_{l_2}))) \mid l_1 = 0,\ldots,k_1, \quad l_2 = 0,\ldots,k_2\}$$

and i is the imaginary unit
Step 3: (Interpolation Part)
Step 3a: Construction of the zero order table ($k = 0$) of required points (interpolation set).
Step 3b: Computation of the k-th order table of divided differences, where $k = 1,\cdots,n$ and $n = \max\{k_1, k_2\}$ according to [12]
Step 3c: Construction of the Newton polynomial

$$p(x,y) = X^T \cdot P \cdot Y$$

where

$$X = \begin{bmatrix} 1 \\ (x-x_0) \\ (x-x_0)\cdot(x-x_1) \\ \vdots \\ (x-x_0)\cdot(x-x_1)\cdot\cdots\cdot(x-x_{k_1-1}) \end{bmatrix}$$

P is the n-th order table of divided differences and

$$Y = \begin{bmatrix} 1 \\ (y-y_0) \\ (y-y_0)\cdot(y-y_1) \\ \vdots \\ (y-y_0)\cdot(y-y_1)\cdot\cdots\cdot(y-y_{k_2-1}) \end{bmatrix}$$

The calculated Newton polynomial is the determinant of the two-variable polynomial matrix, namely, $p(x,y) = \det(A(x,y))$.

4 Examples

4.1 Example in Real basis

Let the two-variable polynomial matrix

$$A(x,y) = \begin{bmatrix} xy^2 & 1 & 2xy \\ -xy & -1 & 3x \\ 2y & -2x & xy \end{bmatrix}$$

Step 1:(Calculation of matrice's degree in each variable)
Degree matrices with respect to x and y variable, are

$$D^x = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D^y = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

We define dc_s^x, dr_s^x, dc_s^y and dr_s^y as the sum of the maximum element of each column and row for x and y variable respectively.

$$\left.\begin{aligned} dc_1^x &= max\{1,1,0\} = 1 \\ dc_2^x &= max\{0,0,1\} = 1 \\ dc_3^x &= max\{1,1,1\} = 1 \end{aligned}\right\} \Rightarrow$$

$$dc_s^x = dc_1^x + dc_2^x + dc_3^x = 3$$

and

$$\left.\begin{aligned} dr_1^x &= max\{1,0,1\} = 1 \\ dr_2^x &= max\{1,0,1\} = 1 \\ dr_3^x &= max\{0,1,1\} = 1 \end{aligned}\right\} \Rightarrow$$

$$dr_s^x = dr_1^x + dr_2^x + dr_3^x = 3$$

and

$$\left.\begin{aligned} dc_1^y &= max\{2,1,1\} = 2 \\ dc_2^y &= max\{0,0,0\} = 0 \\ dc_3^y &= max\{1,0,1\} = 1 \end{aligned}\right\} \Rightarrow$$

$$dc_s^y = dc_1^y + dc_2^y + dc_3^y = 3$$

and

$$\left.\begin{aligned} dr_1^y &= max\{2,0,1\} = 2 \\ dr_2^y &= max\{1,0,0\} = 1 \\ dr_3^y &= max\{1,0,1\} = 1 \end{aligned}\right\} \Rightarrow$$

$$dr_s^y = dr_1^y + dr_2^y + dr_3^y = 4$$

Thus, according [2] we have

$$\begin{aligned} k_1 &= min\{dc_s^x, dr_s^x\} = min\{3,3\} = 3 \\ k_2 &= min\{dc_s^y, dr_s^y\} = min\{3,4\} = 3 \end{aligned}$$

Therefore, the number of required points is $N = (k_1+1)(k_2+1) = 16$.
Step 2: (Evaluation Part)
Step 2a: We determine 16 random fixed points

$$S^{(3,3)} = \{(x_i, y_j) \mid x_i = i, y_j = j \text{ and } i = 0,\ldots,3,\ j = 0,\ldots,3\}$$

Step 2b: For each of those points we evaluate the numerical matrix $A(x_i, y_i)$ and compute their determinant $det(A(x_i, y_i))$. For example:

$$A(x_0,y_0) = A(0,0) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
and $\det(A(0,0)) = 0$

$$A(x_0,y_1) = A(0,1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$
and $\det(A(0,1)) = 0$

$$\vdots$$

$$A(x_3, y_3) = A(3,3) = \begin{bmatrix} 27 & 1 & 18 \\ -9 & -1 & 9 \\ 6 & -6 & 9 \end{bmatrix}$$

and $\det(A(3,3)) = 2430$

Step 3: (Interpolation Part)
Step 3a: We construct the zero order table ($k = 0$) of required points (interpolation set), with the determinants we computed in the previous step.

$$P^{(0)} = \begin{bmatrix} P^{(0)}_{0,0} & P^{(0)}_{0,1} & P^{(0)}_{0,2} & P^{(0)}_{0,3} \\ P^{(0)}_{1,0} & P^{(0)}_{1,1} & P^{(0)}_{1,2} & P^{(0)}_{1,3} \\ P^{(0)}_{2,0} & P^{(0)}_{2,1} & P^{(0)}_{2,2} & P^{(0)}_{2,3} \\ P^{(0)}_{3,0} & P^{(0)}_{3,1} & P^{(0)}_{3,2} & P^{(0)}_{3,3} \end{bmatrix} =$$
$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 20 & 64 & 126 \\ 0 & 100 & 360 & 756 \\ 0 & 300 & 1128 & 2430 \end{bmatrix}$$

where $P^{(0)}_{i,j} = det(A(x_i, y_j))$ for $i = 0, \dots, 3,\ j = 0, \dots, 3.$
Step 3b: For $k = 1$ to n, where $n = \max\{k_1, k_2\} = \max\{3, 3\} = 3$ we compute the k-th order table of divided differences according to [12].
For $k = 3$ we have

$$P^{(3)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 20 & 12 & -1 \\ 0 & 30 & 28 & -1 \\ 0 & 10 & 10 & 0 \end{bmatrix}$$

Step 3c: The Newton polynomial is

$$p(x, y) = 10x^3y^2 - x^2y^3 + x^2y^2 + 4xy^2 + 6xy$$

It occurs from the following

$$p(x, y) = X^T \cdot P \cdot Y$$

where

$$X = \begin{bmatrix} 1 \\ x \\ x(x-1) \\ x(x-1)(x-2) \end{bmatrix} \quad P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 20 & 12 & -1 \\ 0 & 30 & 28 & -1 \\ 0 & 10 & 10 & 0 \end{bmatrix}$$

and

$$Y = \begin{bmatrix} 1 \\ y \\ y(y-1) \\ y(y-1)(y-2) \end{bmatrix}$$

The calculated Newton polynomial is the determinant of the two-variable polynomial matrix, namely, $p(x, y) = \det(A(x, y))$.

4.2 Example in Complex basis

Construction of the determinant of a two-variable polynomial matrix, using Newton interpolation technique.
Let the two-variable polynomial matrix

$$A(x, y) = \begin{bmatrix} xy^2 & 1 & 2xy \\ -xy & -1 & 3x \\ 2y & -2x & xy \end{bmatrix}$$

Step 1: (Calculation of matrice's degree in each variable)
According to example in Real basis we have

$$k_1 = \min\{dc^x_s, dr^x_s\} = \min\{3, 3\} = 3$$
$$k_2 = \min\{dc^y_s, dr^y_s\} = \min\{3, 4\} = 3$$

Therefore, the number of required points is $N = (k_1 + 1)(k_2 + 1) = 16.$
Step 2: (Evaluation Part)
Step 2a: We define the set

$$S^{(3,3)} = \{(x_{\ell_1} \cdot i, y_{\ell_2}) \mid x_{\ell_1} = \{-3, -1, 1, 3\}, y_{\ell_2} = \{-3, -1, 1, 3\} \text{ and } \ell_1 = 0, \dots, 3,\ \ell_2 = 0, \dots, 3\}$$

where i is the imaginary unit.
Step 2b: (Evaluation part) From 2 we have that

$$\det(A(-ai, b)) = \det(A(\overline{ai, b})) = \overline{\det(A(ai, b))}$$

Thus, we evaluate only $\lceil \frac{N}{2} \rceil = 8$ complex interpolation points. Specifically, we determine the following interpolation points $(-3i, -3)$, $(-3i, -1)$, $(-3i, 1)$, $(-3i, 3)$, $(-i, -3)$, $(-i, -1)$, $(-i, 1)$, $(-i, 3)$, and we compute the determinant of the matrix that occurs from the evaluation of each one of those points.

$$\begin{aligned} \det(A(x_0 i, y_0)) &= \det(A(-3i, -3)) \\ &= \det(\begin{bmatrix} -27i & 1 & 18i \\ -9i & -1 & -9i \\ -6 & 6i & 9i \end{bmatrix}) \\ &= -324 + 2376i \end{aligned}$$

$$\begin{aligned} \det(A(x_0 i, y_1)) &= \det(A(-3i, -1)) = \\ &= \det(\begin{bmatrix} -3i & 1 & 6i \\ -3i & -1 & -9i \\ -2 & 6i & 3i \end{bmatrix}) \\ &= -18 + 276i \end{aligned}$$

$$\begin{aligned} \det(A(x_0 i, y_2)) &= \det(A(-3i, 1)) = \\ &= \det(\begin{bmatrix} -3i & 1 & -6i \\ 3i & -1 & -9i \\ 2 & 6i & -3i \end{bmatrix}) \\ &= 240i \end{aligned}$$

$$\begin{aligned}\det(A(x_0 i, y_3)) &= \det(A(-3i,3)) = \\ &= \det(\begin{bmatrix} -27i & 1 & -18i \\ 9i & -1 & -9i \\ 6 & 6i & -9i \end{bmatrix}) \\ &= 162 + 2268i\end{aligned}$$

$$\begin{aligned}\det(A(x_1 i, y_0)) &= \det(A(-i,-3)) = \\ &= \det(\begin{bmatrix} -9i & 1 & 6i \\ -3i & -1 & -3i \\ -6 & 2i & 3i \end{bmatrix}) \\ &= -36 + 72i\end{aligned}$$

$$\begin{aligned}\det(A(x_1 i, y_1)) &= \det(A(-i,-1)) = \\ &= \det(\begin{bmatrix} -i & 1 & 2i \\ -i & -1 & -3i \\ -2 & 2i & i \end{bmatrix}) \\ &= -2 + 12i\end{aligned}$$

$$\begin{aligned}\det(A(x_1 i, y_2)) &= \det(A(-i,1)) = \\ &= \det(\begin{bmatrix} -i & 1 & -2i \\ i & -1 & -3i \\ 2 & 2i & -i \end{bmatrix}) \\ &= 0\end{aligned}$$

$$\begin{aligned}\det(A(x_1 i, y_3)) &= \det(A(-i,3)) = \\ &= \det(\begin{bmatrix} -9i & 1 & -6i \\ 3i & -1 & -3i \\ 6 & 2i & -3i \end{bmatrix}) \\ &= 18 + 36i\end{aligned}$$

For the rest interpolation points, according to,

$$\det(A(-ai,b)) = \det(A(\overline{ai,b})) = \overline{\det(A(ai,b))}$$

we have

$$\begin{aligned}\det(A(x_3 i, y_0)) &= \det(A(3i,-3)) \\ &= det(A(\overline{-3i,-3})) \\ &= \overline{det(A(-3i,-3))} \\ &= -324 - 2376i\end{aligned}$$

$$\begin{aligned}\det(A(x_3 i, y_1)) &= \det(A(3i,-1)) \\ &= det(A(\overline{-3i,-1})) \\ &= \overline{det(A(-3i,-1))} \\ &= -18 - 276i\end{aligned}$$

$$\begin{aligned}\det(A(x_3 i, y_2)) &= \det(A(3i,1)) \\ &= det(A(\overline{-3i,1})) \\ &= \overline{det(A(-3i,1))} \\ &= -240i\end{aligned}$$

$$\begin{aligned}\det(A(x_3 i, y_3)) &= \det(A(3i,3)) \\ &= det(A(\overline{-3i,3})) \\ &= \overline{det(A(-3i,3))} \\ &= 162 - 2268i\end{aligned}$$

$$\begin{aligned}\det(A(x_2 i, y_0)) &= \det(A(i,-3)) \\ &= det(A(\overline{-i,-3})) \\ &= \overline{det(A(-i,-3))} \\ &= -36 - 72i\end{aligned}$$

$$\begin{aligned}\det(A(x_2 i, y_1)) &= \det(A(i,-1)) \\ &= det(A(\overline{-i,-1})) \\ &= \overline{det(A(-i,-1))} \\ &= -2 - 12i\end{aligned}$$

$$\begin{aligned}\det(A(x_2 i, y_2)) &= \det(A(i,1)) \\ &= det(A(\overline{-i,1})) \\ &= \overline{det(A(-i,1))} \\ &= 0\end{aligned}$$

$$\begin{aligned}\det(A(x_2 i, y_3)) &= \det(A(i,3)) \\ &= det(A(\overline{-i,3})) \\ &= \overline{det(A(-i,3))} \\ &= 18 - 36i\end{aligned}$$

Step 3: (Interpolation Part)
Step 3a: We construct the zero order table ($k = 0$) of initial values (interpolation set).

$$P^{(0)} =$$
$$\begin{bmatrix} -324+2376i & -18+276i & 240i & 162+2268i \\ -36+72i & -2+12i & 0 & -2+12i \\ \mathbf{-36-72i} & \mathbf{-2-12i} & \mathbf{0} & \mathbf{-2-12i} \\ \mathbf{-324-2376i} & \mathbf{-18-276i} & \mathbf{-240i} & \mathbf{162-2268i} \end{bmatrix}$$

where $P^{(0)}_{\ell_1,\ell_2} = \det(A(x_{\ell_1} \cdot i, y_{\ell_2}))$ for $\ell_1 = 0,\dots,3$, $\ell_2 = 0,\dots,3$. The values in bold are indicating the conjugate numbers that we do not need to calculate.
Step 3b: For $k = 1$ to n, where $n = \max\{k_1,k_2\} = \max\{3,3\} = 3$ we compute the k-th order table of divided differences according to [12].
For $k = 3$ we have

$$P^{(3)} =$$
$$\begin{bmatrix} -324+2376i & 153-1050i & -36+258i & 9 \\ -1152-144i & 510+68i & -126-16i & 4i \\ 36-270i & -17+120i & 4-30i & -1 \\ 90 & -40 & 10 & 0 \end{bmatrix}$$

Step 3c: The Newton polynomial is

$$p(x, y) = 10x^3y^2 - x^2y^3 + x^2y^2 + 4xy^2 + 6xy$$

which is the determinant of the matrix $A(x, y)$. It occurs from the following

$$p(x, y) = X^T \cdot P \cdot Y$$

where

$$X = \begin{bmatrix} 1 \\ x+3i \\ (x+3i)(x+i) \\ (x+3i)(x+i)(x-i) \end{bmatrix}, \quad P = \begin{bmatrix} -324+2376i & 153-1050i & -36+258i & 9 \\ -1152-144i & 510+68i & -126-16i & 4i \\ 36-270i & -17+120i & 4-30i & -1 \\ 90 & -40 & 10 & 0 \end{bmatrix}$$

and

$$Y = \begin{bmatrix} 1 \\ y+3 \\ (y+3)(y+1) \\ (y+3)(y+1)(y-1) \end{bmatrix}$$

5 Conclusion

An optimized technique of calculating the determinant of a two - variable polynomial matrix has been proposed. This technique is based on Newton Evaluation - Interpolation method in rectangular basis, in complex domain.

As we can see from the above, step 1 as well as steps 3a, 3b and 3c, are the same regardless of the domain (Real or Complex) we choose every time. The novelty of the proposed technique is that if we choose conjugate numbers as required points, we can reduce numerical operations in evaluation part even up to the half.

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Example
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