Global stability of a commensal symbiosis model with Holling II functional response and feedback controls

FENGDE CHEN, YANBO CHONG, SHANGMING CHEN
College of Mathematics and Statistics
Fuzhou University
No. 2, wulongjiang Avenue, Minhou County, Fuzhou
CHINA

Abstract: - A commensal symbiosis model with Holling II functional response and feedback controls is proposed and studied in this paper. The system admits four equilibria, and three boundary equilibria are unstable, only positive equilibrium is locally asymptotically stable. By applying the comparison theorem of differential equation, we show that the unique positive equilibrium is globally attractive. Numeric simulations show the feasibility of the main result.

Key-Words: -Commensalism; Feedback controls; Holling II functional response; Comparison theorem; Global attractivity

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1 Introduction

The aim of this paper is to investigate the global stability property of the following commensal symbiosis model with Holling type functional response and feedback controls :

$$\begin{aligned}\dot{x} &= x\Big(b_1 - a_{11}x + \frac{a_{12}y}{a_{13}+a_{14}y} - \alpha_1 u_1\Big),\\ \dot{y} &= y(b_2 - a_{22}y - \alpha_2 u_2),\\ \dot{u}_1 &= -\eta_1 u_1 + a_1 x,\\ \dot{u}_2 &= -\eta_2 u_2 + a_2 y,\end{aligned} \tag{1}$$

where $x(t)$ and $y(t)$ denote the density of the first and second species at time t. u_1 and u_2 are feedback control variables. All parameters used in this model are positive.

During the lase decade, many scholars investigated the dynamic behaviors of the mutualism model or commensalism model ([1]-[30]). Also, due to it is importance, many scholars ([31]-[41]) investigated the dynamic behaviors of the ecosystem with feedback controls. however, it is very strange that to this day, only one paper[20] considered the influence of feedback controls to the commensalism models. In [20], Han and Chen proposed and studied the following Lotka-volterra commensal symbiosis model with feedback controls:

$$\begin{aligned}\dot{x} &= x(b_1 - a_{11}x + a_{12}y - \alpha_1 u_1),\\ \dot{y} &= y(b_2 - a_{22}y - \alpha_2 u_2),\\ \dot{u}_1 &= -\eta_1 u_1 + a_1 x,\\ \dot{u}_2 &= -\eta_2 u_2 + a_2 y.\end{aligned} \tag{2}$$

By constructing a suitable Lyapunov function, the authors showed that the positive equilibrium of the system is globally stable.

On the other hand, there were also several scholars ([7], [16], [17],[22], [23],[26]) argued that the the relationship of two commensalism model should be described by the suitable functional response, for example, Li, Lin and Chen[17] studied the positive periodic solution of a discrete commensalism model with Holling II functional response. The system takes the form

$$\begin{aligned}x_1(k+1) &= x_1(k)\exp\Big\{a_1(k) - b_1(k)x_1(k)\\ &\quad + \frac{c_1(k)x_2(k)}{e_1(k)+f_1(k)x_2(k)}\Big\},\\ x_2(k+1) &= x_2(k)\exp\{a_2(k) - b_2(k)x_2(k)\},\end{aligned} \tag{3}$$

Wu [7] argued that between two species nonlinear type of relationship between two species is more feasible, and she established the following two species commensal symbiosis model

$$\begin{aligned}\frac{dx}{dt} &= x\Big(a_1 - b_1 x + \frac{c_1 y^p}{1+y^p}\Big),\\ \frac{dy}{dt} &= y(a_2 - b_2 y),\end{aligned} \tag{4}$$

where $a_i, b_i, i = 1, 2$ p and c_1 are all positive constants, $p \geq 1$. The results of [7] is then generalized by Lei [23] and Wu, Li and Lin[16] to the commensalism model with Allee effect.

Stimulated by the above works, we propose the system (1). As far as system (1) is concerned, since it seems that the system is similar to system (2), only with the cooperation term $a_{12}xy$ in system (2) changed to the term with Holling type functional response $\dfrac{a_{12}xy}{a_{13}+a_{14}y}$ in system (1). One may expect the analysis method used in Han and Chen[20] could be applied to system (1), however, this is impossible. In their paper, Han and Chen could deal with the stability property of the system (1) by constructing suitable Lyapunov function, by using this method, one could always obtain some interesting result about the linear system. When it come to the nonlinear case, it is very difficult to deal with the nonlinear term to ensure the negative definite of the Lyapunov function; We mention here that in [7], Wu investigated the global stability of the equilibrium of system (4) by using the Dulac criterion, which could only be applied to the two dimensional system, and could not be applied to the higher dimensional system.

The aim of this paper, is to investigated the stability property of the system (1). To deal with this, we need to develop some new analysis technique, more precisely, we will combine the analysis technique of Han and Chen [20], Wu[7] and Yue [42], to overcome the difficulty of nonlinearity.

The paper is arranged as follows. We will investigate the existence and locally stability property of the equilibria of system (1) in section 2. In section 3, we first establish a global stability result of single species feedback control system via the Lyapunov function, after that, by developing the analysis technique of Yue[32], more precisely, by using the differential inequality theory and the comparison theorem, we investigate the global attractivity property of the positive equilibrium of system (1). In section 4, we present some numerical simulations to show the feasibility of the main result. We end this paper by a briefly discussion.

2 Existence and local stability of Equilibria

This section we will focus our attention to investigate the existence and local stability property of the system (1).

The equilibria of system (1) is determined by the following system

$$
\begin{aligned}
&x\Big(b_1 - a_{11}x + \frac{a_{12}y}{a_{13}+a_{14}y} - \alpha_1 u_1\Big) = 0,\\
&y(b_2 - a_{22}y - \alpha_2 u_2) = 0,\\
&-\eta_1 u_1 + a_1 x = 0,\\
&-\eta_2 u_2 + a_2 y = 0.
\end{aligned} \tag{5}
$$

The system always admits three boundary equilibria:

$$A_1(0,0,0,0),$$

$$A_2\Big(\frac{\beta_1 b_1}{a_1\alpha_1 + a_{11}\eta_1}, 0, \frac{a_1 b_1}{a_1\alpha_1 + a_{11}\eta_1}, 0\Big),$$

$$A_3\Big(0, \frac{b_2\eta_2}{a_2\alpha_2 + a_{22}\eta_2}, 0, \frac{a_2 b_2}{a_2\alpha_2 + a_{22}\eta_2}\Big).$$

Also, system (1) admits a unique positive equilibrium $A_4\Big(x^*, y^*, u_1^*, u_2^*\Big)$, where

$$
\begin{aligned}
x^* &= \frac{b_1 + \dfrac{a_{12}y^*}{a_{13}+a_{14}y^*}}{a_{11} + \dfrac{\alpha_1 a_1}{\eta_1}},\\
y^* &= \frac{b_2}{a_{22} + \dfrac{\alpha_2 a_2}{\eta_2}},\\
u_1^* &= \frac{a_1 x^*}{\eta_1},\\
u_2^* &= \frac{a_2 y^*}{\eta_2}.
\end{aligned} \tag{6}
$$

Obviously, x^*, y^*, u_1^* and u_2^* satisfy the equations

$$
\begin{aligned}
&b_1 - a_{11}x^* + \frac{a_{12}y^*}{a_{13}+a_{14}y^*} - \alpha_1 u_1^* = 0,\\
&b_2 - a_{22}y^* - \alpha_2 u_2^* = 0,\\
&-\eta_1 u_1^* + a_1 x^* = 0,\\
&-\eta_2 u_2^* + a_2 y^* = 0.
\end{aligned} \tag{7}
$$

We shall now investigate the local stability property of the above equilibria.

The variational matrix of system (1) is

$$
\begin{aligned}
&J(x,y,u_1,u_2)\\
&= \begin{pmatrix} A_{11} & A_{12} & -\alpha_1 x & 0\\ 0 & A_{22} & 0 & -y\alpha_2\\ a_1 & 0 & -\eta_1 & 0\\ 0 & a_2 & 0 & -\eta_2 \end{pmatrix},
\end{aligned} \tag{8}
$$

where

$$
\begin{aligned}
A_{11} &= b_1 - 2a_{11}x + \frac{a_{12}y}{a_{14}y + a_{13}} - \alpha_1 u_1,\\
A_{12} &= x\left(\frac{a_{12}}{a_{14}y+a_{13}} - \frac{a_{12}ya_{14}}{(a_{14}y+a_{13})^2}\right),\\
A_{22} &= -2a_{22}y - \alpha_2 u_2 + b_2.
\end{aligned}
$$

Theorem 2.1 $A_1(0,0,0,0)$ *is unstable.*

Proof. From (8) we could see that the Jacobian matrix of the system about the equilibrium point $A_1(0,0,0,0)$ is given by

$$\begin{pmatrix} b_1 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 \\ a_1 & 0 & -\eta_1 & 0 \\ 0 & a_2 & 0 & -eta_2 \end{pmatrix}. \tag{9}$$

The characteristic equation of above matrix is

$$(\lambda - b_1)(\lambda - b_2)(\lambda + \eta_1)(\lambda + \eta_2) = 0. \tag{10}$$

Hence, it has two positive characteristic root $\lambda_1 = b_1, \lambda_2 = b_2$, consequently, $A_1(0,0,0,0)$ is unstable. This ends the proof of Theorem 2.1.

Theorem 2.2 $A_2\Big(\frac{\beta_1 b_1}{a_1\alpha_1 + a_{11}\eta_1}, 0, \frac{a_1 b_1}{a_1\alpha_1 + a_{11}\eta_1}, 0\Big)$ *is unstable.*

Proof. From (8) we could see that the Jacobian matrix of the system about the equilibrium point $A_2\Big(\frac{\beta_1 b_1}{a_1\alpha_1 + a_{11}\eta_1}, 0, \frac{a_1 b_1}{a_1\alpha_1 + a_{11}\eta_1}, 0\Big)$ is given by

$$\begin{pmatrix} B_{11} & B_{12} & B_{13} & 0 \\ 0 & b_2 & 0 & 0 \\ a_1 & 0 & -\eta_1 & 0 \\ 0 & a_2 & 0 & -eta_2 \end{pmatrix}. \tag{11}$$

where

$$\begin{aligned} B_{11} &= b_1 - \frac{2a_{11}\eta_1 b_1}{a_1\alpha_1 + a_{11}\eta_1} - \frac{\alpha_1 a_1 b_1}{a_1\alpha_1 + a_{11}\eta_1}, \\ B_{12} &= \frac{\eta_1 b_1 a_{12}}{(a_1\alpha_1 + a_{11}\eta_1)a_{13}}, \\ B_{13} &= -\frac{\eta_1 b_1 \alpha_1}{a_1\alpha_1 + a_{11}\eta_1}. \end{aligned}$$

The characteristic equation of above matrix is

$$(\lambda - b_2)(\lambda + \eta_2)\Big(C_1\lambda^2 + C_2\lambda + C_3\Big) = 0, \tag{12}$$

where

$$\begin{aligned} C_1 &= a_1\alpha_1 + a_{11}\eta_1, \\ C_2 &= a_1\alpha_1\eta_1 + a_{11}\eta_1 b_1 + a_{11}\eta_1^2, \\ C_3 &= a_1\eta_1 b_1\alpha_1 + a_{11}b_1\eta_1^2. \end{aligned}$$

Hence, it has a positive characteristic root $\lambda_1 = b_2$, consequently, A_2 is unstable. This ends the proof of Theorem 2.2.

Theorem 2.3 $A_3\Big(0, \frac{b_2\eta_2}{a_2\alpha_2 + a_{22}\eta_2}, 0, \frac{a_2 b_2}{a_2\alpha_2 + a_{22}\eta_2}\Big)$ *is unstable.*

Proof. From (8) we could see that the Jacobian matrix of the system about the equilibrium point $A_3\Big(0, \frac{b_2\eta_2}{a_2\alpha_2 + a_{22}\eta_2}, 0, \frac{a_2 b_2}{a_2\alpha_2 + a_{22}\eta_2}\Big)$ is given by

$$\begin{pmatrix} D_{11} & 0 & 0 & 0 \\ 0 & D_{22} & 0 & D_{24} \\ a_1 & 0 & -\eta_1 & 0 \\ 0 & a_2 & 0 & -eta_2 \end{pmatrix}, \tag{13}$$

where

$$\begin{aligned} D_{11} &= b_1 + \frac{a_{12}b_2\eta_2}{(a_2\alpha_2 + a_{22}\eta_2)\,\Delta_1}, \\ D_{22} &= -\frac{2a_{22}b_2\eta_2}{a_2\alpha_2 + a_{22}\eta_2} \\ &\quad -\frac{\alpha_2 a_2 b_2}{a_2\alpha_2 + a_{22}\eta_2} + b_2, \\ D_{24} &= -\frac{b_2\eta_2\alpha_2}{a_2\alpha_2 + a_{22}\eta_2}, \\ \Delta_1 &= \frac{a_{14}b_2\eta_2}{a_2\alpha_2 + a_{22}\eta_2} + a_{13}. \end{aligned}$$

The characteristic equation of above matrix is

$$(\lambda - D_{11})(\lambda + \eta_1)\Big(E_1\lambda^2 + E_2\lambda + E_3\Big) = 0, \tag{14}$$

where

$$\begin{aligned} E_1 &= a_2\alpha_2 + a_{22}\eta_2, \\ E_2 &= a_2\alpha_2\eta_2 + a_{22}\eta_2 b_2 + a_{22}\eta_2^2, \\ E_3 &= a_2\eta_2 b_2\alpha_2 + a_{22}b_2\eta_2^2. \end{aligned}$$

Hence, it has a positive characteristic root $\lambda_1 = D_{11}$, consequently, A_3 is unstable. This ends the proof of Theorem 2.3.

Theorem 2.4 $A_4\Big(x^*, y^*, u_1^*, u_2^*\Big)$ *is locally asymptotically stable.*

Proof. From (8) we could see that the Jacobian matrix of the system about the equilibrium point $A_4\Big(x^*, y^*, u_1^*, u_2^*\Big)$ is given by

$$\begin{pmatrix} -a_{11}x^* & m & -\alpha_1 x^* & 0 \\ 0 & -a_{22}y^* & 0 & -\alpha_2 y^* \\ a_1 & 0 & -\eta_1 & 0 \\ 0 & a_2 & 0 & -eta_2 \end{pmatrix}, \tag{15}$$

where

$$m = x^* \left(\frac{a_{12}}{a_{14}y^* + a_{13}} - \frac{a_{12}y^* a_{14}}{(a_{14}y^* + a_{13})^2} \right).$$

The characteristic equation of above matrix is

$$(\lambda^2 + F_1\lambda + F_2)\Big(\lambda^2 + G_1\lambda + G_2\Big) = 0, \qquad (16)$$

where

$$\begin{aligned} F_1 &= a_{22}y^* + \eta_2, \\ F_2 &= a_2\alpha_2 y^* + a_{22}\eta_2 y^*, \\ G_1 &= a_{11}x^* + \eta_1, \\ G_2 &= a_1\alpha_1 x^* + a_{11}\eta_1 x^*. \end{aligned}$$

From F_1, F_2, G_1, G_2 are all positive constants, one could easily see that four roots of equation (16) is negative. Consequently, $A_4\Big(x^*, y^*, u_1^*, u_2^*\Big)$ is locally asymptotically stable. This ends the proof of Theorem 2.4.

3 Global attractivity

We had showed in the previous section system (1) admits four equilibria, however, only the positive equilibrium $A_4\Big(x^*, y^*, u_1^*, u_2^*\Big)$ is locally asymptotically stable, while the other three equilibria are all unstable. Now, one interesting issue proposed: Is it possible for us to find out the suitable conditions to ensure the positive equilibrium $A_4\Big(x^*, y^*, u_1^*, u_2^*\Big)$ be globally asymptotically stable?

We will give the affirm answer to this issue, more precisely, we will prove the following result.

Theorem 3.1 $A_4\Big(x^*, y^*, u_1^*, u_2^*\Big)$ *is globally attractive.*

To prove this result, we need the following Lemma. The result of the Lemma seems simple, however, sine our proof of Theorem 3.1 deeply depend on the Lemma, for the sake of completeness, we also give a detail proof of the Lemma.

Let us consider the following single species feedback control ecosystem.

$$\begin{aligned} \frac{dx}{dt} &= x(a - bx - cu), \\ \frac{du}{dt} &= -eu + fx, \end{aligned} \qquad (17)$$

where a, b, c, d, e are all positive constants. The system (17) admits a unique positive equilibrium $A(x^*, u^*)$, where

$$x^* = \frac{a}{b + \dfrac{cf}{e}}, \quad u^* = \frac{f}{e}x^*.$$

Concerned with the stability property of this equilibrium, we have the following result.

Lemma 3.1 $A(x^*, u^*)$ *is globally stable.*

Proof. Obviously, $A(x^*, u^*)$ satisfies the equation

$$\begin{aligned} a - bx^* - cu^* = 0, \\ -eu^* + fx^* = 0, \end{aligned} \qquad (18)$$

Now let's construct a Lyapunov function

$$V = \Big(x - x^* - x^* \ln \frac{x}{x^*}\Big) + \frac{c}{2f}(u - u^*)^2, \qquad (19)$$

Calculating the derivative along the solution of system (17), we have

$$\begin{aligned} &\frac{dV}{dt} \\ = \ & (x - x^*)(a - bx - cu) \\ & + \frac{c}{f}(u - u^*)(-eu + fx) \\ = \ & (x - x^*)(bx^* + cu^* - bx - cu) \\ & + \frac{c}{f}(u - u^*)(-eu + fx + eu^* - fx^*) \\ = \ & (x - x^*)\Big(b(x^* - x) + c(u^* - u)\Big) \\ & + \frac{c}{f}(u - u^*)\Big(-e(u - u^*) + f(x - x^*)\Big) \\ = \ & -b(x - x^*)^2 - \frac{ce}{f}(u - u^*)^2. \end{aligned} \qquad (20)$$

Thus $\frac{dV}{dt} < 0$ strictly for all $x > 0, u > 0$ except the positive equilibrium $P(x^*, u^*)$, where $\frac{dV}{dt} = 0$. Thus, $V(t)$ satisfies Lyapunov's asymptotic stability theorem, and the positive equilibrium $P(x^*, u^*)$ of system (17) is globally stable. This ends the proof of Lemma 3.1.

Proof of Theorem 3.1. Noting that in (1), the second and forth equations are independent of the variable x and u_1, hence, we could consider the following subsystem previously.

$$\begin{aligned} \dot{y} &= y(b_2 - a_{22}y - \alpha_2 u_2), \\ \dot{u}_2 &= -\eta_2 u_2 + a_2 y, \end{aligned} \qquad (21)$$

Form Lemma 3.1, the unique positive equilibrium (y^*, u^*) of system (20) is globally stable, where

$$\begin{aligned} y^* &= \frac{b_2}{a_{22} + \frac{\alpha_2 a_2}{\eta_2}}, \\ u_2^* &= \frac{a_2 y^*}{\eta_2}. \end{aligned} \tag{22}$$

Hence,

$$\begin{aligned} &\lim_{t\to+\infty} y(t) = y^*, \\ &\lim_{t\to+\infty} u_2(t) = u_2^*. \end{aligned} \tag{23}$$

For $\varepsilon > 0$ enough small, (23) implies that there exists a enough T such that

$$y^* - \varepsilon < y(t) < y^* + \varepsilon. \tag{24}$$

For $t > T$, from the first and third equations in (1) and (23), we have

$$\begin{aligned} \dot{x} &= x\Big(b_1 - a_{11}x + \frac{a_{12}y}{a_{13} + a_{14}y} - \alpha_1 u_1\Big) \\ &\le x\Big(b_1 - a_{11}x + \frac{a_{12}(y^* + \varepsilon)}{a_{13} + a_{14}(y^* + \varepsilon)} \\ &\quad -\alpha_1 u_1\Big), \\ \dot{u}_1 &= -\eta_1 u_1 + a_1 x, \end{aligned} \tag{25}$$

Now let us consider the system

$$\begin{aligned} \dot{w}_1 &= w_1\Big(b_1 - a_{11}w_1 + \frac{a_{12}(y^* + \varepsilon)}{a_{13} + a_{14}(y^* + \varepsilon)} \\ &\quad -\alpha_1 v_1\Big), \\ \dot{v}_1 &= -\eta_1 v_1 + a_1 w_1. \end{aligned} \tag{26}$$

Form Lemma 3.1, the unique positive equilibrium $(w_1^*(\varepsilon), v_1^*(\varepsilon))$ of system (26) is globally stable, where

$$\begin{aligned} w_1^*(\varepsilon) &= \frac{b_1 + \frac{a_{12}(y^* + \varepsilon)}{a_{13} + a_{14}(y^* + \varepsilon)}}{a_{11} + \frac{\alpha_1 a_1}{\eta_1}}, \\ v_1^*(\varepsilon) &= \frac{a_1 w_1^*}{\eta_1}. \end{aligned} \tag{27}$$

From (25)-(27), we have

$$\begin{aligned} &\limsup_{t\to+\infty} x(t) \le \lim_{t\to+\infty} w_1(t) = w_1^*(\varepsilon), \\ &\limsup_{t\to+\infty} u_1(t) \le \lim_{t\to+\infty} v_1(t) = v_1^*(\varepsilon). \end{aligned} \tag{28}$$

For $t > T$, from the first and third equations in (1) and (24), we also have

$$\begin{aligned} \dot{x} &= x\Big(b_1 - a_{11}x + \frac{a_{12}y}{a_{13} + a_{14}y} - \alpha_1 u_1\Big) \\ &\ge x\Big(b_1 - a_{11}x + \frac{a_{12}(y^* - \varepsilon)}{a_{13} + a_{14}(y^* - \varepsilon)} \\ &\quad -\alpha_1 u_1\Big), \\ \dot{u}_1 &= -\eta_1 u_1 + a_1 x, \end{aligned} \tag{29}$$

Now let us consider the system

$$\begin{aligned} \dot{w}_2 &= w_2\Big(b_1 - a_{11}w_2 + \frac{a_{12}(y^* - \varepsilon)}{a_{13} + a_{14}(y^* - \varepsilon)} \\ &\quad -\alpha_1 v_2\Big), \\ \dot{v}_2 &= -\eta_1 v_2 + a_1 w_2. \end{aligned} \tag{30}$$

Form Lemma 3.1, the unique positive equilibrium $(w_2^*(\varepsilon), v_2^*(\varepsilon))$ of system (29) is globally stable, where

$$\begin{aligned} w_2^*(\varepsilon) &= \frac{b_1 + \frac{a_{12}(y^* - \varepsilon)}{a_{13} + a_{14}(y^* - \varepsilon)}}{a_{11} + \frac{\alpha_1 a_1}{\eta_1}}, \\ v_2^*(\varepsilon) &= \frac{a_1 w_2^*}{\eta_1}. \end{aligned} \tag{31}$$

From (29)-(31), we have

$$\begin{aligned} &\liminf_{t\to+\infty} x(t) \ge \lim_{t\to+\infty} w_2(t) = w_2^*(\varepsilon), \\ &\liminf_{t\to+\infty} u_1(t) \ge \lim_{t\to+\infty} v_2(t) = v_2^*(\varepsilon). \end{aligned} \tag{32}$$

From (28) and (32) we have

$$\begin{aligned} w_2^*(\varepsilon) &= \lim_{t\to+\infty} w_2(t) \le \liminf_{t\to+\infty} x(t) \\ &\le \limsup_{t\to+\infty} x(t) \le \lim_{t\to+\infty} w_1(t) = w_1^*(\varepsilon), \\ v_2^*(\varepsilon) &= \lim_{t\to+\infty} v_2(t) \le \liminf_{t\to+\infty} u_1(t) \\ &\le \limsup_{t\to+\infty} u_1(t) \le \lim_{t\to+\infty} v_1(t) = v_1^*(\varepsilon). \end{aligned} \tag{33}$$

Noting that

$$w_i(\varepsilon) \to x^*, \ \ v_i(\varepsilon) \to u_1^* \ \text{ as } \ \varepsilon \to 0, \ i = 1, 2. \tag{34}$$

Since ε is enough small positive constant, setting $\varepsilon \to 0$ in (33) leads to

$$\lim_{t\to+\infty} x(t) = x^* \ \ \lim_{t\to+\infty} u_1(t) = u_1^*. \tag{35}$$

(23) and (35) show that $A_4\left(x^*, y^*, u_1^*, u_2^*\right)$ is globally attractive. This ends the proof of Theorem 3.1.

4 Numeric simulations

In this section, we provide an example to illustrate the theoretical result by numerical simulations.

Example 4.1.

$$\begin{aligned} \dot{x} &= x\Big(1 - x + \frac{3y}{2+2y} - 3u_1\Big), \\ \dot{y} &= y(3 - 2y - 2u_2), \\ \dot{u}_1 &= -2u_1 + 3x, \\ \dot{u}_2 &= -u_2 + 2y. \end{aligned} \tag{36}$$

Here, corresponding to system (1), we choose $b_1 = 1, a_{11} = 1, a_{12} = 3, a_{13} = a_{14} = 2, \alpha_1 = 3, b_2 = 3, a_{22} = 2, \alpha_2 = 2, \eta_1 = 2, a_1 = 3, \eta_2 = 1, a_2 = 2$. By simple computation, one could easily see that the system (36) admits a unique positive equilibrium $A_4(0.27, 0.5, 0.41, 1)$, from Theorem 3.1, A_4 is globally attractive. Numeric simulations (Fig.1 and 2) also support this assertion.

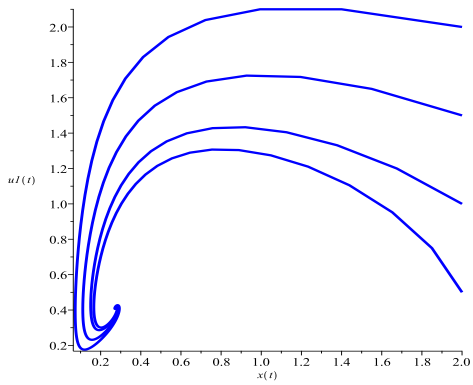

Figure 1: Phase portraits of the first component x and third component u_1 in system (36) with the initial condition $(x(0), y(0), u_1(0), u_2(0)) = (0.5, 2, 0.5, 0.5)$, $(1, 2, 1, 1)$, $(1.5, 2, 1.5, 1.5)$ and $(2, 2, 2, 2)$, respectively.

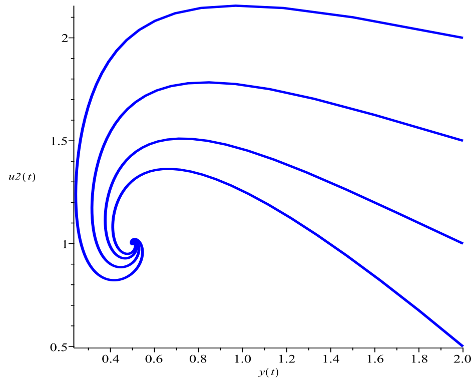

Figure 2: Phase portraits of the second and third component y and forth component u_2 in system (36) with the initial condition $(x(0), y(0), u_1(0), u_2(0)) = (0.5, 2, 0.5, 0.5)$, $(1, 2, 1, 1)$, $(1.5, 2, 1.5, 1.5)$ and $(2, 2, 2, 2)$, respectively.

5 Conclusion

In [20], Han and Chen proposed a Lotka-Volterra commensalism system with feedback controls (i.e., system (2), by constructing some suitable Lyapunov function, they showed that system admits a unique positive equilibrium which is globally stable. Stimulated by the works of Han and Chen[20], Wu, Li and Zhou[7] and Li, Lin and Chen[17], we propose a commensalism system with Holling II functional response and feedback controls.

With the introduction of the nonlinear functional response, the method used in [20] could not be applied to our case. However, we find that in system (1), the second and forth equations are independent of variable x and u_1, this stimulate us to investigate the dynamic behaviors of subsystem (1) firstly. By applying the differential inequality theory and comparison theorem of differential equation, we finally show that the unique positive equilibrium of system (1) is globally attractive.

It is well known that the system with Allee effect may have very complex dynamic behaviors, to this day, still no scholar propose the commensalism model with both Allee effect and feedback controls, whether the idea of this paper could be applied to that case is still unknown, we will try to do some work on this direction.

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