Pole Assignment by Proportional-Plus-Derivative State Feedback for Multivariable Linear Time-Invariant Systems

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Abstract: - In this paper the pole assignment problem by proportional-plus-derivative state feedback for multivariable linear-time invariant systems is studied. In particular, explicit necessary and sufficient conditions are established for a given polynomial with real coefficients to be characteristic polynomial of closed-loop system obtained by proportional-plus-derivative state feedback from the given multivariable linear time-invariant system. A procedure is given for the calculation of proportional-plus-derivative state feedback which places the poles of closed-loop system at desired locations. Our approach is based on properties of polynomial matrices.

Key-Words: - pole assignment, proportional-plus-derivative state feedback, linear time-invariant systems.

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1 Introduction

The problem of pole assignment by proportional-plus-derivative output feedback or equivalently by incomplete proportional-plus-derivative state feedback for multivariable linear time-invariant systems was introduced in [1]. In particular in [1] a method was given for placing up to max(2*m*, 2*p*), poles of closed-loop system, where *m* and *p* are the numbers of inputs and outputs respectively of closed-loop system. A method for assigning up to max(2*m*+*p*-1, 2*p*+*m*-1) poles of closed-loop system by proportional-plus-derivative output feedback(or equivalently by incomplete proportional-plus derivative state feedback) is presented in [2]. In [3] was introduced a new class of multivariable output feedback controllers consisting of proportional plus multiple derivative terms. It is shown that all poles of closed-loop system, can be placed at desired positions, provided a sufficient number of derivative terms. The pole placement equations for the proportional-plus-derivative output feedback compensator are derived in [4]. In [5] is proven that controllability is sufficient condition for the solution of pole assignment problem by full proportional-plus-derivative feedback for single input single output linear time-invariant systems.

In this paper, are established explicit necessary and sufficient conditions for the solution of pole assignment problem by proportional-plus- derivative state feedback for multivariable linear time-invariant systems. Furthermore a procedure is given for the computation of proportional-plus- derivative state feedback which assigns the poles of closed-loop system to any desired positions.

Problem Formulation

Consider a multivariable linear time-invariant system described by the following state-space equations

$$\dot{\mathbf{x}}(t)=\mathbf{M}\mathbf{x}(t) + \mathbf{N}\mathbf{u}(t) \qquad (1)$$

where **M** and **N** are real matrices of size $n \times n$, $n \times m$, respectively, $\mathbf{x}(t)$ is the state vector of dimensions $n \times 1$ and $\mathbf{u}(t)$ is the vector of inputs of dimensions $m\times 1$. Without any loss of generality we assume that

$$\text{rank}[\mathbf{N}]=m \qquad (2)$$

Let **T** be a non-singular matrix of size $n \times n$ such that

$$\mathbf{TN} = \begin{bmatrix} 0 \\ \mathbf{I}_m \end{bmatrix} \qquad (3)$$

where $\mathbf{I}_m$ is the identity matrix of size $m \times m$. Using the following similarity transformation

$$\mathbf{x}(t) = \mathbf{T}^{-1}\mathbf{z}(t) \qquad (4)$$

and the relationship (3), the state–space equations of system (1) can be rewritten as follows

$$\dot{\mathbf{z}}(t) = \mathbf{A}\mathbf{z}(t) + \mathbf{B}\mathbf{u}(t) \qquad (5)$$

The real matrices **A** and **B** of appropriate dimensions are given by

$$\mathbf{A}=\mathbf{TMT}^{-1}\ ,\ \mathbf{B}=\begin{bmatrix}0\\ \mathbf{I}_{\mathrm{m}}\end{bmatrix} \qquad (6)$$

Consider the control law

$$\mathbf{u}(t) = -\mathbf{Fz}(t) + \mathbf{D}\dot{\mathbf{z}}(t) + \mathbf{v}(t) \qquad (7)$$

where **F** and **D** are real matrices of size , $m \times n$ and $m \times n$ respectively and **v**(t) is the reference input vector of size *mx1*. By applying the proportional-plus-derivative state feedback (7) to the system (5) the state-space equations of closed–loop system are

$$[\mathbf{I}-\mathbf{BD}]\dot{\mathbf{z}}(t) = (\mathbf{A}-\mathbf{BF})\,\mathbf{z}(t) + \mathbf{Bv}(t) \qquad (8)$$

Let $\mathcal{R}$ be the field of real numbers. Also let $\mathcal{R}$[s] be the ring of polynomials with coefficients in $\mathcal{R}$. Let c(s) be a given arbitrary monic polynomial over $\mathcal{R}$[s] of degree n. Further let μ is finite nonzero real number. The pole assignment problem considered in this paper can be stated as follows: Does there exist a proportional-plus-derivative state feedback (7) such that

$$\det[\,(\mathbf{I}-\mathbf{BD})s-\mathbf{A}\ +\mathbf{BF}\,] = \mu c(s) \qquad (9)$$

if so, give conditions for existence and a procedure for the computation of matrices **F** and **D**.

3 Basic concepts and preliminary results

Let us first introduce some notations that are used throughout the paper. Let **D**(s) be a non-singular polynomial matrix over $\mathcal{R}$[s] of dimensions $m \times m$, write $\deg_{ci}$ for the degree of column *i* of **D(**s). If

$$\deg_{ci}\mathbf{D}(s) \geq \deg_{cj}\mathbf{D}(s),\ i<j \qquad (10)$$

the polynomial matrix **D**(s) is said to be column degree ordered. Denote $\mathbf{D}_h$ the highest column degree coefficient matrix of polynomial matrix **D**(s). The polynomial matrix **D**(s) is said to be column reduced if the real matrix $\mathbf{D}_h$ is non-singular. The matrix **D**(s) is said to be column monic if its highest column degree coefficient matrix is the identity matrix. A polynomial matrix **U**(s) over $\mathcal{R}$[s] of dimensions $k \times k$ is said to be unimodular if and only if

$$\det[\mathbf{U}(s)] = \lambda \qquad (11)$$

where λ is finite nonzero real number; therefore every unimodular polynomial matrix has polynomial inverse. Let **D**(s) be a non-singular polynomial matrix over $\mathcal{R}$[s], then there exist unimodular matrices **U**(s) and **V**(s) over $\mathcal{R}$[s] such that

$$\mathbf{D}(s)=\mathbf{U}(s)\ \mathrm{diag}\,[a_1(s),\ a_2(s),\ \ldots a_m(s)]\mathbf{V}(s) \qquad (12)$$

where the polynomials $a_i(s)$ for *i=1,2, .., m* are termed invariant polynomials of **D**(s) and have the following property

$$a_i(s)\ \text{divides}\ a_{i+1}(s),\ \text{for}\ i=1,2,\ldots m-1 \qquad (13)$$

Furthermore we have that

$$a_i(s) = \frac{d_i(s)}{d_{i-1}(s)},\ \text{for}\ i = 1,2,\ldots\ldots m \qquad (14)$$

where $d_o(s)$ =1 by definition and $d_i(s)$ is the monic greatest common divisor of all minors of order *i* in **D**(s), for *i=1,2,….., m*. Let **A**(s), be a polynomial matrix over $\mathcal{R}$[s] if there are polynomial matrices **P**(s) and **Q**(s) such that

$$\mathbf{A}(s) = \mathbf{P}(s)\ \mathbf{Q}(s) \qquad (15)$$

Then, the polynomial matrix **P**(s) over $\mathcal{R}$[s] is termed left divisor of **A**(s). Let **A**(s) and **B(s)**, be polynomial matrices over $\mathcal{R}$[s] if

$$\mathbf{A}(s) = \mathbf{D}(s)\ \mathbf{M}(s) \qquad (16)$$

$$\mathbf{B}(s) = \mathbf{D}(s)\ \mathbf{N}(s) \qquad (17)$$

for polynomial matrices **M**(s), **N**(s) and **D**(s) over $\mathcal{R}$[s], then **D**(s) is termed common left divisor of polynomial matrices **A**(s) and **B**(s).
A greatest common left divisor of two polynomial matrices $\mathbf{A}(s)$ and $\mathbf{B}(s)$ is a common left divisor which is a right multiple of every common left divisor. Let $\mathbf{V}(s)$ be a greatest common left divisor of two polynomial matrices $\mathbf{A}(s)$ and $\mathbf{B}(s)$, then there is a unimodular matrix $\mathbf{U}(s)$ over $\mathcal{R}$[s], such that

$$[\mathbf{A}(s),\ \mathbf{B}(s)] = [\mathbf{V}(s,\ \mathbf{0}]\ \mathbf{U}(s) \qquad (18)$$

The system (5) is controllable if and only if

$$\mathrm{rank}\,[\mathbf{I}s-\mathbf{A},\ \mathbf{B}] = n \qquad (19)$$

for all complex numbers s.
The material on polynomial matrices and their properties presented in this section was obtained primarily from references [6], [7], [8] and [9].

Definiton 1. Relatively right prime polynomials matrices **D**(s) and **N**(s) of dimensions $m \times m$ and $n \times m$ respectively with **D**(s) to be column reduced and column degree ordered such that

$$(\mathbf{I}s-\mathbf{A})^{-1}\mathbf{B} = \mathbf{N}(s)\mathbf{D}^{-1}(s) \qquad (20)$$

are said to form a standard right matrix fraction description of system (5). The column degrees of

the matrix $\mathbf{D}$(s) are the controllability indices of system (5).
The following Lemmas are needed to prove the main theorem of this paper.

Lemma 1. Let $\mathbf{A}$(s) and $\mathbf{B}$(s) be matrices over $\mathcal{R}$[s] of size $m \times k$ and $m \times p$ respectively. The following are equivalent:

(a) The polynomial matrices $\mathbf{A}$(s) and $\mathbf{B}$(s) are relatively left prime over $\mathcal{R}$[s].
(b) The greatest common left divisor of polynomial matrices $\mathbf{A}$(s) and $\mathbf{B}$(s) over $\mathcal{R}$[s] is unimodular matrix.
(c) rank[$\mathbf{A}$(s), $\mathbf{B}$(s)] = m, for all complex numbers s.

Proof. See [7, p. 538, Theorem 2.4].

Lemma 2. Let the matrices $\mathbf{D}$(s) and $\mathbf{N}$(s) of dimensions $m \times m$ and $n \times m$ over $\mathcal{R}$[s] be a standard matrix fraction description of controllable system (5). Then

(a) The rows of polynomial matrix $\mathbf{N}$(s) form a basis for the linear space over $\mathcal{R}$ of all polynomial vectors $\mathbf{v}$(s) of size $1 \times m$ such that $\mathbf{v}(s)\mathbf{D}^{-1}(s)$ is strictly proper.

Proof. See [8, pp. 73-74, Theorem 2.17].

The following lemma was first proved in [9, pp.184-185].
Lemma 3. Let $\mathbf{C}$(s) be a column monic polynomial matrix over $\mathcal{R}$[s] of size $m \times m$. Suppose that $\deg_{cp}\mathbf{C}$(s< $\deg_{cq}\mathbf{C}$(s) for some p and q. Then $\mathbf{C}$(s) can be transformed by unimodular transformations to a column reduced polynomial matrix $\mathbf{C}_1$(s) with column degrees given by

$$\deg_{ci}\mathbf{C}_1(s) = \deg_{ci}\mathbf{C}(s) \quad \text{for } i \neq p,q$$

$$\deg_{ci}\mathbf{C}_1(s) = \deg_{cp}\mathbf{C}(s) + 1 \quad \text{for } i = p$$

$$\deg_{ci}\mathbf{C}_1(s) = \deg_{cq}\mathbf{C}(s) - 1 \quad \text{for } i \neq q$$

Proof. Add s times row p to row q in $\mathbf{C}$(s). This leaves the degree of each column but p and q unchanged. It also leaves unchanged the degrees of the elements in position (i, p), $i \neq q$ and (i, q), $i \neq q$; places a monic polynomial of degree $\deg_{cp}\mathbf{C}$(s)+1 in position (p, q) and does not increase the degree of the element in position (q, q). Let $\beta = \deg_{cq}\mathbf{C}$(s). Further let α be the coefficient of s^{β} in the element in position (q, q). Put $d = \deg_{cp}\mathbf{C}$(s) – 1. Substract αs^d times column p from column q. This reduces the degree of column q below $\deg_{cq}\mathbf{C}$(s). The resulting matrix $\mathbf{C}_1$(s) then satisfies the conditions of the lemma. This completes the proof of the Lemma.
The following lemma was first proved in [10].

Lemma 4. Let (5) be a controllable system. Further let $\mathbf{D}$(s), $\mathbf{N}$(s) be a standard right matrix fraction description of system (5). Then for every $m \times n$ real matrix $\mathbf{F}$ the polynomial matrices [$\mathbf{I}$s – $\mathbf{A}$+$\mathbf{BF}$] and [$\mathbf{D}$(s)+$\mathbf{FN}$(s)] have the same non-unit invariant polynomials.

Proof. Let $\mathbf{D}$(s) and $\mathbf{N}$(s) are relatively right prime polynomial matrices over $\mathcal{R}$[s] of respective dimensions $m \times m$ and $n \times m$ respectively such that

$$(\mathbf{I}s - \mathbf{A})^{-1}\mathbf{B} = \mathbf{N}(s)\mathbf{D}^{-1}(s) \tag{21}$$

We have that

$$[\mathbf{I}s - \mathbf{A}]\mathbf{N}(s) = \mathbf{BD}(s) \tag{22}$$

We add $\mathbf{BFN}(s)$ to both sides of the above identity and rearrange to get

$$[\mathbf{I}s - \mathbf{A} + \mathbf{BF}]^{-1}\mathbf{B} = \mathbf{N}(s)[\mathbf{D}(s) + \mathbf{FN}(s)]^{-1} \tag{23}$$

Since $[\mathbf{I}s - \mathbf{A}]$and $\mathbf{B}$ are relatively left prime over $\mathcal{R}$[s] by controllability of (5) and since

$$[\mathbf{I}s - \mathbf{A} + \mathbf{BF}, \mathbf{B}] = [\mathbf{I}s - \mathbf{A}, \mathbf{B}]\begin{bmatrix}\mathbf{I}_n & \mathbf{0} \\ \mathbf{F} & \mathbf{I}_m\end{bmatrix} \tag{24}$$

it follows that $[\mathbf{I}s - \mathbf{A} + \mathbf{BF}]$ and $\mathbf{B}$ are relatively left prime over $\mathcal{R}$[s]. On the other hand $\mathbf{D}$(s) and $\mathbf{N}$(s) are relatively right prime over $\mathcal{R}$[s] and

$$\begin{bmatrix}\mathbf{N}(s) \\ \mathbf{D}(s) + \mathbf{FN}(s)\end{bmatrix} = \\ = \begin{bmatrix}\mathbf{I}_n & \mathbf{0} \\ \mathbf{F} & \mathbf{I}_m\end{bmatrix}\begin{bmatrix}\mathbf{N}(s) \\ \mathbf{D}(s)\end{bmatrix} \tag{25}$$

Hence $[\mathbf{D}(s) + \mathbf{FN}(s)]$and $\mathbf{N}(s)$ are relatively right prime over $\mathcal{R}$[s]. It follows that the matrices $[\mathbf{I}s - \mathbf{A} + \mathbf{BF}]$ and $[\mathbf{D}(s) + \mathbf{FN}(s)]$ must share the same non-unit invariant polynomials. This completes the proof of the Lemma.

Lemma 5. Let (5) be a controllable system. Further let $\mathbf{D}$ be an arbitrary matrix over $\mathcal{R}$ of size $m \times n$. Then the following condition holds:

(a) rank [$(\mathbf{I} - \mathbf{BD})$s – $\mathbf{A}$, $\mathbf{B}$] = n for all complex numbers s.

Proof. Suppose that the system (5) is controllable. From (19) it follows

$$\text{rank}\,[\mathbf{I}s - \mathbf{A}, \mathbf{B}] = n \tag{26}$$

we rewrite the polynomial matrix [$(\mathbf{I} - \mathbf{BD})$s – $\mathbf{A}$, $\mathbf{B}$] as

$$[(\mathbf{I} - \mathbf{BD})s - \mathbf{A}, \mathbf{B}] = [\mathbf{I}s - \mathbf{A}, \mathbf{B}]\begin{bmatrix}\mathbf{I}_n & \mathbf{0} \\ -\mathbf{D}s & \mathbf{I}_m\end{bmatrix} \tag{27}$$

Since the following polynomial matrix over $\mathcal{R}$[s]

$$\begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ -\mathbf{D}s & \mathbf{I}_m \end{bmatrix}$$

is unimodular, condition (a) of the Lemma 5 follows from (26) and (27) and the proof is complete.

Lemma 6. Let (5) be a controllable system. Further let **D** be a matrix over $\mathcal{R}$ of size $m \times n$ such that $\det[\mathbf{I}-\mathbf{BD}] \neq 0$. Then the following condition holds:

(a) The pair $[(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{A},\ (\mathbf{I}-\mathbf{BD})^{-1}\mathbf{B}]$ is controllable.

Proof. We rewrite the polynomial matrix $[(\mathbf{I}-\mathbf{BD})s-\mathbf{A}, \mathbf{B}]$ as

$$[(\mathbf{I}-\mathbf{BD})s-\mathbf{A}, \mathbf{B}] = [\mathbf{I}-\mathbf{BD}][\mathbf{I}s-(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{A}, ,(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{B}] \quad (28)$$

Since by assumption the matrix $[\mathbf{I}-\mathbf{BD}]$ is non-singular, from (28) and Lemma 5 it follows that

$$\text{rank}[\mathbf{I}s-(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{A}, (\mathbf{I}-\mathbf{BD})^{-1}\mathbf{B}] = n \quad (29)$$

for all complex numbers s .

Condition (a) of the Lemma 6 follows from (29) and (19) and the proof is complete.

Lemma 7. Let **D**(s) and **N**(s) be a standard right matrix fraction description of a controllable system (5) and **F** is a matrix over $\mathcal{R}$ of size $m \times n$. Further, let $c_1(s)$, $c_2(s)$, …., $c_r(s)$ be the non-unit invariant polynomials of polynomial matrix [**D**(s)+**FN**(s) of size $m \times m$, $r \leq m$. Then

(a) $\det[\mathbf{I}s-\mathbf{A}+\mathbf{BF}[] = \Pi_{i=1}^{r} c_i(s)$

Proof. Let $c_i(s)$ for $i=1,2,...,r$ be the non-unit invariant polynomials of the matrix det[**D**(s)+**FN**(s)] By Lemma 4, the polynomials $c_1(s)$, $c_2(s)$, …., $c_r(s)$ are the non-unit invariant polynomials of [**I**s – **A**+**BF**] and remaining invariant polynomials are $c_{r+1}(s),=c_2(s)=….=c_n(s)=1$. The characteristic polynomial of the matrix [**I**s – **A**+**BF**] is given by [8, p. 47]

$$\det[\mathbf{I}s-\mathbf{A}+\mathbf{BF}] = \Pi_{i=1}^{r} c_i(s) \quad (30)$$

condition (a) of the Lemma 7 follows from (30) and the proof is complete.

4 Problem Solution

The following theorem is the main result of this paper and gives explicit necessary and sufficient conditions for the solution of the pole assignment problem by proportional-plus-derivative state feedback for multivariable linear time-invariant systems.

Theorem 1. The pole assignment problem by proportional-plus-derivative state feedback for multivariable linear time-invariant systems has solution over $\mathcal{R}$ if and only if the following condition holds:

(a) The open-loop system (5) is controllable.

Proof. Let c(s) be an arbitrary monic polynomial over $\mathcal{R}$[s] of degree n. Suppose that the pole assignment problem by proportional-plus-derivative state feedback has a solution over $\mathcal{R}$. From (9) it follows that

$$\det[(\mathbf{I}-\mathbf{BD})s-\mathbf{A}+\mathbf{BF}] = \mu c(s) \quad (31)$$

where μ is finite nonzero real number Let **V**(s) be the greatest common left divisor of polynomial matrices [**I**s-**A**] and **B**. Then, from (16) and (17) it follows that

$$[\mathbf{I}s-\mathbf{A}] = \mathbf{V}(s)\,\mathbf{X}(s) \quad (32)$$

$$\mathbf{B} = \mathbf{V}(s)\,\mathbf{Y}(s) \quad (33)$$

for polynomial matrices **X**(s) and **Y**(s) over $\mathcal{R}$[s] of appropriate dimensions. We rewrite the polynomial matrix $[(\mathbf{I}-\mathbf{BD})s-\mathbf{A}+\mathbf{BF}]$ as

$$[(\mathbf{I}-\mathbf{BD})s-\mathbf{A}+\mathbf{BF}]=[\mathbf{I}s-\mathbf{A}, \mathbf{B}]\begin{bmatrix} \mathbf{I} \\ -\mathbf{D}s+\mathbf{F} \end{bmatrix} \quad (34)$$

Using (32), (33) and (34) and after simple algebraic manipulations, the relationship (31) can be rewritten as

$$\det[\mathbf{V}(s)]\det[[\mathbf{X}(s), \mathbf{Y}(s)]\begin{bmatrix} \mathbf{I} \\ -\mathbf{D}s+\mathbf{F} \end{bmatrix}] = \mu c(s) \quad (35)$$

From relationship (35) it follows that

$$\det[\mathbf{V}(s)] \text{ divides } (\mu c(s)) \quad (36)$$

Since by assumption c(s) is an arbitrary monic polynomial over $\mathcal{R}$[s] of degree n, relationship (36) is satisfied if and only if

$$\det[\mathbf{V}(s)] = \lambda \quad (37)$$

where λ is finite nonzero real number. From (37) and (11) it follows that the polynomial matrix **V**(s) is unimodular; therefore by Lemma 1 the polynomial matrices [**I**s-**A**] and **B** are relatively left prime over $\mathcal{R}$[s] or equivalently

$$\text{rank}[\mathbf{I}s-\mathbf{A}, \mathbf{B}] = n \quad (38)$$

for all complex numbers s.

From (38) and (19) it follows that the pair (**A**, B) is controllable. This is condition (a).

To prove sufficiency, we assume that condition (a) holds. We form the matrix

$$\mathbf{D} = [\mathbf{0}, (-2)\mathbf{I}_m] \quad (39)$$

From (6) and (39) it follows that the matrix

$$[\mathbf{I}-\mathbf{BD}] = \text{diag}[\mathbf{I}_{n-m}, -\mathbf{I}_m] \quad (40)$$

is non-singular. Let $\mathbf{A_1}$ and $\mathbf{B}_1$ be real matrices of appropriate dimensions given by

$$\mathbf{A}_1 = [(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{A} \quad (41)$$

$$\mathbf{B}_1 = [(\mathbf{I}-\mathbf{BD})^{-1}\mathbf{B} \quad (42)$$

By Lemma 6 the pair ($\mathbf{A}_1$, $\mathbf{B}_1$) with $[\mathbf{I}-\mathbf{BD}]$ given by (39), is controllable. Let $\mathbf{D_1}$(s) and $\mathbf{N_1}$(s) be relatively prime polynomial matrices over $\mathcal{R}$[s], with $\mathbf{D_1}$(s) column reduced and column degree ordered, which satisfy

$$(\mathbf{I}s - \mathbf{A}_1)^{-1}\mathbf{B}_1 = \mathbf{N_1}(s)\mathbf{D}_1^{-1}(s) \quad (43)$$

Let $\nu_1 \geq \nu_2 \geq ...... \geq \nu_m$ and $\mathbf{D}_h$ be the ordered list of column degrees and the highest column degree coefficient matrix of polynomial matrix $\mathbf{D_1}$(s) respectively. Further, let c(s) be an arbitrary monic polynomial over $\mathcal{R}$[s] of degree *n*. We form the polynomial matrix $\mathbf{C}$(s) of size *m x m*

$$\mathbf{C}(s) = \text{diag}[c(s), 1, \ldots, 1] \quad (44)$$

whose invariant polynomials are c(s), 1, …, 1. Then, Lemma 3 can be applied several times if necessary in order to make the polynomial matrix $\mathbf{C}$(s) column reduced with column degrees equal to those of $\mathbf{D_1}$(s), without changing its invariant polynomials. We call the resulting matrix $\mathbf{C_1}$(s). Let $\mathbf{C}_h$ be the highest column degree coefficient matrix of polynomial matrix $\mathbf{C_1}$(s) [10]. We form the polynomial matrix $\mathbf{C_2}$(s)

$$\mathbf{C_2}(s) = \mathbf{D}_h \mathbf{C}_h^{-1}\, \mathbf{C_1}(s) \quad (45)$$

From (45) it follows that the polynomial matrices $\mathbf{C_2}$(s) and $\mathbf{D_1}$(s) have the same highest column degree coefficient matrix [8, p.123], [10]; therefore

$$\deg_{ci}[\mathbf{C}_2(s) - \mathbf{D_1}(s)] \leq \nu_i - 1 \ \forall\ i = 1,2, ..., m \quad (46)$$

from (46) it follows directly that the following rational matrix

$$[\mathbf{C}_2(s) - \mathbf{D_1}(s)]\, \mathbf{D}_1^{-1}(s) \quad (47)$$

is strictly proper [8, p.39]. From (47) and Lemma 2 it follows directly that the equation

$$\mathbf{D_1}(s) + \mathbf{F}\, \mathbf{N_1}(s) = \mathbf{C_2}(s) \quad (48)$$

or equivalently the equation

$$\mathbf{F}\, \mathbf{N_1}(s) = \mathbf{C_2}(s) - \mathbf{D_1}(s) \quad (49)$$

has a solution for $\mathbf{F}$ over $\mathcal{R}$. From (44), (45) and (48) it follows that the invariant polynomials of the polynomial matrix $\mathbf{D_1}(s) + \mathbf{F}\,\mathbf{N_1}(s)$ of size *m x m* are

$$c_1(s) = c(s),\ c_2(s) = \ldots = c_m(s) = 1 \quad (50)$$

Since by Lemma 4 the polynomial matrices $[\mathbf{I}s - \mathbf{A}_1 + \mathbf{B}_1\mathbf{F}]$ and $[\mathbf{D}(s)+\mathbf{FN}(s)]$ have the same non-unit invariant polynomials, from (48) it follows that the invariant polynomials of the matrix $[\mathbf{I}s - \mathbf{A}_1 + \mathbf{B}_1\mathbf{F}]$ of size *n x n* are

$$c_1(s) = c(s),\ c_2(s) = \ldots = c_n(s) = 1 \quad (51)$$

Then by Lemma 7

$$\det[\mathbf{I}s - \mathbf{A}_1 + \mathbf{B}_1\mathbf{F}] = \Pi_{i=1}^{n} c_i(s) = c(s) \quad (52)$$

using (41) and (42) the relationship (52) can be rewritten as

$$\det[\mathbf{I}s - (\mathbf{I}-\mathbf{BD})^{-1}\mathbf{A} + (\mathbf{I}-\mathbf{BD})^{-1}\mathbf{BF}] = c(s) \quad (53)$$

since according to (40) the matrix $[\mathbf{I}-\mathbf{BD}]$ is non-singular, the relationship (53) after simple algebraic manipulations, can be rewritten as

$$\det[\mathbf{I}s - (\mathbf{I}-\mathbf{BD})^{-1}\mathbf{A} + (\mathbf{I}-\mathbf{BD})^{-1}\mathbf{BF}] =$$

$$= \det[(\mathbf{I}-\mathbf{BD})^{-1}]\det[\,(\mathbf{I}-\mathbf{BD})s - \mathbf{A} + \mathbf{BF}\,] =$$

$$= c(s) \quad (54)$$

From (54) it follows that

$$\det[\,(\mathbf{I}-\mathbf{BD})s - \mathbf{A} + \mathbf{BF}\,] = \mu c(s) \quad (55)$$

where μ is nonzero real number given by

$$\mu = 1/(\det[(\mathbf{I}-\mathbf{BD})^{-1}) \quad (56)$$

from (55) it follows that the polynomial $[\mu c(s)]$ over $\mathcal{R}$[s], is the characteristic polynomial of closed-loop system (8). This completes the proof.

Corollary 1. For every multivariable linear time-invariant controllable system (5) there exists a proportional-plus-derivative state feedback (7) such that the closed-loop system is stable.

Proof. Let c(s) be a monic polynomial over $\mathcal{R}$[s] of degree *n* whose roots lie in the open left half complex plane. From Theorem 1 it follows that there exists matrices $\mathbf{F}$ and $\mathbf{D}$ over $\mathcal{R}$ of appropriate dimensions such that

$$\det[\,(\mathbf{I}-\mathbf{BD})s - \mathbf{A} + \mathbf{BF}\,] = \mu c(s) \quad (57)$$

where μ is nonzero real number given by

$$\mu = 1/(\det[(\mathbf{I}-\mathbf{BD})^{-1}) \quad (58)$$

From (57) it follows that the polynomial $[\mu c(s)]$ over $\mathcal{R}$[s], is the characteristic polynomial of closed-loop system (8). Since by assumption the roots of polynomial c(s) over $\mathcal{R}$[s] of degree *n*, lie in the open left half complex plane, the closed-loop system (8) is stable. This completes the proof.

The sufficiency part of the proof of Theorem 1 suggests a simple procedure to compute the matrices $\mathbf{F}$ and $\mathbf{D}$ of proportional-plus-derivative state feedback (7) which assigns the poles of closed-loop system (8) to desired positions.

Given: $\mathbf{A}$, $\mathbf{B}$ and c(s)

Find: $\mathbf{F}$ and $\mathbf{D}$

Step 1: Form the polynomial matrix

$$[\mathbf{I}s - \mathbf{A}, \mathbf{B}]$$

and check whether has full row rank for all complex numbers s. If not, the open-loop system (5) is uncontrollable; therefore the pole assignment problem by proportional-plus-derivative state feedback is impossible.
Step 2: Set

$$\mathbf{D} = [\mathbf{0}, (-2)\mathbf{I}_m]$$

*Step 3:*Calculate the matrices

$$[\mathbf{I} - \mathbf{BD}] = \text{diag}[\mathbf{I}_{n-m}, -\mathbf{I}_m]$$

$$\mathbf{A}_1 = [(\mathbf{I} - \mathbf{BD})^{-1}\mathbf{A}$$

$$\mathbf{B}_1 = [(\mathbf{I} - \mathbf{BD})^{-1}\mathbf{B}$$

Step 4: Calculate relatively prime polynomial matrices $\mathbf{D_1}(s)$ and $\mathbf{N_1}(s)$ over $\mathcal{R}[s]$ [10], with $\mathbf{D_1}(s)$ column reduced and column degree ordered, which satisfy

$$(\mathbf{I}s - \mathbf{A}_1)^{-1}\mathbf{B}_1 = \mathbf{N_1}(s)\mathbf{D}_1^{-1}(s)$$

Step 5: Read out $\nu_1, \nu_2, \ldots\ldots, \nu_m$, the column degrees of polynomial matrix $\mathbf{D_1}(s)$.
Step 6: Form the matrix $\mathbf{C}(s)$ of size $m \times m$

$$\mathbf{C}(s) = \text{diag}[c(s), 1, \ldots, 1]$$

Step 7: Apply Lemma 3 several times if necessary in order to make the polynomial matrix $\mathbf{C}(s)$ column reduced with column degrees $\nu_1, \nu_2, \ldots\ldots, \nu_m$. Call the resulting matrix $\mathbf{C_1}(s)$.
Step 8: Set

$$\mathbf{C_2}(s) = \mathbf{D}_h\mathbf{C}_h^{-1}\ \mathbf{C_1}(s)$$

where $\mathbf{D}_h$ and $\mathbf{C}_h$ are the highest column degree coefficient matrices of polynomial matrices $\mathbf{D_1}(s)$ and $\mathbf{C_1}(s)$ respectively
Step 9: Calculate the solution for $\mathbf{F}$ over $\mathcal{R}$ of the equation

$$\mathbf{D_1}(s) + \mathbf{F}\ \mathbf{N_1}(s) = \mathbf{C_2}(s)$$

In [10] has been proposed a computationally efficient method for the calculation of solution for $\mathbf{F}$ over $\mathcal{R}$ of the above equation. In particular the proposed method in [10] reduces the solution of the equation $\mathbf{D_1}(s) + \mathbf{F}\ \mathbf{N_1}(s) = \mathbf{C_2}(s)$ to that of solving a system of linear equations of the form $\mathbf{FK}=\mathbf{L}$, where the real matrices $\mathbf{K}$ and $\mathbf{L}$ of appropriate dimensions, are constructed from polynomial matrices $\mathbf{N_1}(s)$, $\mathbf{D_1}(s)$ and $\mathbf{C_2}(s)$ over $\mathcal{R}[s]$. For more details the reader is referred to [10].

Remark. The pole assignment problem by proportional state feedback for multivariable linear time invariant systems has been completely solved in [11] and [12](according to [13]). As far as we know the pole assignment problem by proportional-plus-derivative state feedback for multivariable linear time invariant systems is an open problem yet. The main theorem of this paper gives explicit necessary and sufficient conditions for the solution of the pole assignment problem by proportional-plus-derivative state feedback for multivariable linear time invariant systems. This clearly demonstrates the originality of the contribution of the main theorem of this paper with respect to existing results.

5 Conclusions

In this paper is proven that the pole assignment problem by proportional-plus-derivative state feedback for multivariable linear time invariant systems has solution over the field of real numbers if and only if the given open-loop system is controllable. Furthermore is proven that every multivariable linear time-invariant controllable system is stabilizable by proportional-plus - derivative state feedback. The proof of the main result of this paper is constructive and furnishes a procedure for the computation of proportional-plus-derivative state feedback which assigns the poles of closed-loop system to any desired positions.

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