On Some Classical and Weighted Estimates for SL4

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Abstract: The purpose of this paper is two-sided. First, we obtain the correct estimate of the error term in the classical prime geodesic theorem for compact symmetric space SL4. As it turns out, the corrected error term depends on the degree of a certain polynomial appearing in the functional equation of the attached zeta function. This is in line with the known result in the case of compact Riemann surface, or more generally, with the corresponding result in the case of compact locally symmetric spaces of real rank one. Second, we derive a weighted form of the theorem. In particular, we prove that the aforementioned error term can be significantly improved when the classicalapproachisreplacedbyitshigherlevelanalogue.

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1 Introduction

In [3] and [16], the authors derived two main results: a length spectrum for compact symmetric spaces represented as quotients of the Lie group $SL_4(\mathbb{R})$, and its application in totally quartic fields with no real quadratic subfield.

Length spectrum (prime geodesic theorem) is given as the sum of the explicit part $2\,\mathrm{li}(x)$ and the remainder $O\left(x^{\frac{3}{4}}(\log x)^{-1}\right)$.

The Selberg zeta function (the Ruelle zeta function) is usually applied in the proof of the prime geodesic theorem (see, e.g., [20], [9], [10], [17], [15], [4]-[7], etc.)

In fact, such functions are applied in a way analogous to the way the Riemann zeta function is applied in the proof of the prime number theorem (see, e.g., [1], [14], [22], etc.)

The prime geodesic theorem stated above is then applied in order to prove an asymptotic formula for class numbers of orders in totally complex quartic fields with no real quadratic subfield.

More precisely, it is proved that $\pi_S(x)$ behaves like $\frac{e^{4x}}{8x}$ during the process $x \to +\infty$, where $\pi_S(x)$ is defined by $\sum\limits_{\substack{\mathcal{O}\in O^c(S)\\ R(\mathcal{O})\leq x}} \lambda_S(\mathcal{O})\,h(\mathcal{O})$.

Here, S is a finite, non-empty set of prime numbers containing an even number of elements, $O^c(S) \subset O(S)$ is the subset of isomorphy classes of orders in fields in $C^c(S)$, where $C^c(S) \subset C(S)$ is the subset of fields with no real quadratic subfield. Furthermore, $R(\mathcal{O})$ resp. $h(\mathcal{O})$ denote the regulator resp. the class number of the order $\mathcal{O}$.

For a field $F \in C(S)$ and an order $\mathcal{O} \in O_F(S)$, the constant $\lambda_S(\mathcal{O})\ (= \lambda_S(F))$ is given as $\prod\limits_{p\in S} f_p(F)$, where $f_p(F)$ is the inertia degree of p in F.

$C(S)$ is the set of all totally complex quartic fields F such that all primes $p \in S$ are nondecomposed in F.

Finally, $O(S)$ is the union of all $O_F(S)$, where F ranges over $C(S)$, and $O_F(S)$ is the set of all isomorphism classes of orders in F which are maximal at all $p \in S$.

Note that for long time it was not possible to separate the class number and the regulator in the summation (see, e.g., [8], [21]).

However, in [19], the author proved that such a separation is actually possible.

In this paper we pay attention to the error term $O\left(x^{\frac{3}{4}}(\log x)^{-1}\right)$ in the corresponding prime geodesic therem. We prove that this error term should actually be replaced by the error term $O\left(x^{1-\frac{1}{2D}}(\log x)^{-1}\right)$, where D is the degree of the polynomial that appears in the functional equation of the Selberg zeta function in the case at hand.

2 Preliminaries

The counting functions $\psi_0(x)$ and $\psi_j(x)$ that will be used in this paper, are adopted from [13, p. 44].

Thus, $j \in \mathbb{N}$, where $\mathcal{E}_P(\Gamma)$ is the set of all conjugacy classes $[\gamma]$ in Γ, and $\chi_1(\Gamma_\gamma)$ is the first higher Euler characteristics of the symmetric space $X_{\Gamma_\gamma} = \Gamma_\gamma \setminus G_\gamma / K_\gamma$.

Namely, our object of research is the symmetric space $X_\Gamma = \Gamma \setminus G / K$, where $G = SL_4(\mathbb{R})$, $K = SO(4)$, and Γ is a discrete and co-compact subgroup of G.

More precisely, it is initially required K to be the maximal compact subgroup of G. Therefore, $K = SO(4)$.

In particular, G_γ and Γ_γ are the centralizers of γ in G and Γ, respectively, and $K_\gamma = K \cap G_\gamma$.

P is a parabolic with Langlands decomposition $P = MAN$ (for M, A and N, see [13, pp. 43]).

We use γ_0 to denote primitive elements.

If it happens that γ and γ_0 appear together in the same formula, we shall mean that γ_0 is the primitive element corresponding to γ.

It is assumed that for $[\gamma] \in \mathcal{E}_P(\Gamma)$, γ is conjugate in G to an element $a_\gamma b_\gamma \in A^- B$, where A^- and B are introduced in [13, p. 42].

Thus, a_γ is a matrix in A^-.

Besides this notation, we write a_γ also for the top left entry in the matrix a_γ itself.

Consequently, we define the length l_γ of γ to be $8 \log a_\gamma$.

Finally, we define the counting function $\pi(x)$ in the same was as in [13, p. 43].

Thus, $\mathcal{E}_P^p(\Gamma)$ is the set of primitive classes in $\mathcal{E}_P(\Gamma)$.

The Ruelle zeta function attached to X_Γ will be denoted by $R_{\Gamma,1}(s)$, and the corresponding Selberg zeta function will be denoted by $Z_{P,\bigwedge^q \bar{\mathfrak{n}}}(s)$, $q \in \{0,1,...,4\}$, where $\bigwedge^*$ denotes the exterior product, and $\bar{\mathfrak{n}}$ is the complexified Lie algebra of $\bar{N}$ (see, [12, p. 22] for $\bar{N}$).

As it is usual for this kind of research, we apply the higher order differential operator $\Delta_k^+ f(x)$ (and its properties), where $k \in \mathbb{Z}$.

h will be an arbitrary constant.

For $t > 0$, let $N(t)$ denote the number of poles and zeros of $Z_{P,\bigwedge^q \bar{\mathfrak{n}}}(s)$, $q \in \{0,1,...,4\}$ at points $\frac{1}{2} + \mathrm{i}\,x$, where $0 < x < t$.

By Lemma 3.1.2 in [16], $N(t) = O\left(t^D\right)$.

3 Main result

The following theorem represents the main result of our research.

Theorem 1. *Let X_Γ be as above. Then,*

$$\pi(x) = 2\,\mathrm{li}(x) + O\left(x^{1-\frac{1}{2D}} (\log x)^{-1}\right)$$

as $x \to +\infty$.

Proof. By [16, (12)], $\psi_k(x)$ may be written as

$$\sum_{q=0}^{4} (-1)^q \sum_{\alpha \in S_{k,q}} c_\alpha(q,k),$$

where $S_{k,q}$ denotes the set of poles of the corresponding function, and $c_\alpha(q,k)$ is the corresponding residue.

As it is known, the Selberg zeta function $Z_{P,\bigwedge^q \bar{\mathfrak{n}}}\left(s + \frac{q}{4}\right)$ has a double zero at $1 - \frac{q}{4}$, while the remaining poles and zeros of $Z_{P,\bigwedge^q \bar{\mathfrak{n}}}\left(s + \frac{q}{4}\right)$ lie in $\left[-\frac{q}{4}, 1 - \frac{q}{4}\right] \cup \left(\frac{1}{2} - \frac{q}{4} + \mathrm{i}\,\mathbb{R}\right)$.

Note that the values 0, -1,..., $-k$ are single poles of the corresponding function.

Also note that 0 may appear as a simple pole of $\frac{Z'_{P,\bigwedge^q \bar{\mathfrak{n}}}\left(s+\frac{q}{4}\right)}{Z_{P,\bigwedge^q \bar{\mathfrak{n}}}\left(s+\frac{q}{4}\right)}$, $q \in \{0,1,...,4\}$, i.e., as a singularity of $Z_{P,\bigwedge^q \bar{\mathfrak{n}}}\left(s + \frac{q}{4}\right)$, $q \in \{0,1,...,4\}$. Finally, -1 may appear as a simple pole $\frac{Z'_{P,\bigwedge^4 \bar{\mathfrak{n}}}(s+1)}{Z_{P,\bigwedge^4 \bar{\mathfrak{n}}}(s+1)}$, i.e., as a singularity of $Z_{P,\bigwedge^4 \bar{\mathfrak{n}}}(s+1)$.

Denote by I_q the set of values $j \in \{0,-1,...,-k\}$ such that j is a singularity of $Z_{P,\bigwedge^q \bar{\mathfrak{n}}}\left(s + \frac{q}{4}\right)$.

Put $I'_q = I_k \setminus I_q$, where $I_k = \{0,-1,...,-k\}$.

Obviously, 0 may appear as an element of I_q, $q \in \{0,1,...,4\}$. Moreover, -1 can appear as an element I_4. Note that $\{-2,-3,...,-k\} \subseteq I'_q$ for $q \in \{0,1,...,4\}$.

Now, $I_k = I_q \cup I'_q$.

If $j \in I_q$, then j is a pole of order two of the corresponding function.

Otherwise, if $j \in I'_q$, then j is a simple pole.

Besides the set I_q of singularities of $Z_{P,\bigwedge^q \bar{\mathfrak{n}}}\left(s + \frac{q}{4}\right)$, the set of the remaining singularities s^q of $Z_{P,\bigwedge^q \bar{\mathfrak{n}}}\left(s + \frac{q}{4}\right)$ will be denoted by S^q.

Hence, the elements of S^q are also simple poles of the corresponding function.

Now, we calculate the residues given above.

In any neighborhood of the singularity z of $Z_{P,\bigwedge^q \bar{\mathfrak{n}}}\left(s + \frac{q}{4}\right)$, we write the logarithmic derivative $\frac{Z'_{P,\bigwedge^q \bar{\mathfrak{n}}}\left(s+\frac{q}{4}\right)}{Z_{P,\bigwedge^q \bar{\mathfrak{n}}}\left(s+\frac{q}{4}\right)}$, as the series with o_z^q's as the orders of z, and $a_{i,z}^q$'s as the corresponding coefficients.

Thus, if $s^q \in S^q$, then $c_{s^q}(q,k) = o^q_{s^q}(s^q)^{-1}(s^q+1)^{-1}\dots(s^q+k)^{-1}x^{s^q+k}$.

If $-j \in I_q$, then $c_{-j}(q,k)$ is given as the difference between $o^q_{-j}\prod_{\substack{l=0\\l\neq j}}^{k}(-j+l)^{-1}x^{-j+k}\log x$ and

$o^q_{-j}\prod_{\substack{l=0\\l\neq j}}^{k}(-j+l)^{-1}\times$

$\times\left(-\sum_{\substack{l=0\\l\neq j}}^{k}(-j+l)^{-1}+a^q_{1,-j}\right)x^{-j+k}$.

Finally, if $-j \in I'_q$, then $c_{-j}(q,k)$ is given by $\frac{Z'_{P,\wedge^q\bar{\mathfrak{n}}}\left(-j+\frac{q}{4}\right)}{Z_{P,\wedge^q\bar{\mathfrak{n}}}\left(-j+\frac{q}{4}\right)}\prod_{\substack{l=0\\l\neq j}}^{k}(-j+l)^{-1}x^{-j+k}$.

Put $S^q_{\mathbb{R}} = S^q \cap \mathbb{R}$, and $S^q_{\frac{1}{2}-\frac{q}{4}} = S^q \setminus S^q_{\mathbb{R}}$.

Let $z \in S^q_{\frac{1}{2}-\frac{q}{4}}$.

Since $S^q_{\frac{1}{2}-\frac{q}{4}} \subset S^q$, it follows that $h^{-k}\Delta^+_k c_z(q,k)$ is $O\left(h^{-k}|z|^{-k-1}x^{\frac{1}{2}+k}\right)$.

Moreover, the definition of the operator Δ^+_k in the form of the iterated integral yields that $h^{-k}\Delta^+_k c_z(q,k)$ is $O\left(|z|^{-1}x^{\frac{1}{2}}\right)$.

The sum of the elements $h^{-k}\Delta^+_k c_z(q,k)$ over $z \in S_{\frac{1}{2}-\frac{q}{4}}$ may be written as the sum over $z \in S_{\frac{1}{2}-\frac{q}{4}}$, $\left|\frac{1}{2}-\frac{q}{4}\right| < |z| \leq M$ plus the sum over $z \in S_{\frac{1}{2}-\frac{q}{4}}$, $|z| > M$, where M is a constant which will be fixed later.

Thus, it easily follows that the sum is $O\left(x^{\frac{1}{2}}M^{D-1}\right)+O\left(h^{-k}x^{\frac{1}{2}+k}M^{D-k-1}\right)$.

Now, we estimate $h^{-k}\ \Delta^+_k\ c_1(0,k)$.

By previous calculations, we know that $c_1(0,k)$ is given by $2\left((k+1)!\right)^{-1}x^{1+k}$.

Thus, our assumption that h is $O(x)$, yields that $h^{-k}\Delta^+_k c_1(0,k)$ is $Px+Q$ for some P and Q.

It is not so hard to determine P and Q explicitly.

Namely, since $\Delta^+_k c_1(0,k)$ is given by

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}2\left((k+1)!\right)^{-1}$$
$$\sum_{j=0}^{1+k}\binom{1+k}{j}x^{1+k-j}\left((k-i)h\right)^j,$$

it follows that P is

$$2\sum_{i=0}^{k}(-1)^i\frac{1}{(k-i)!i!}(k-i)^k,$$

i.e., P is 2.

Furthermore, Q is

$$2h(k+1)^{-1}\sum_{i=0}^{k}(-1)^i\frac{1}{(k-i)!i!}(k-i)^{1+k},$$

i.e., Q is hk. Hence, $h^{-mD}\Delta^+_{mD}c_1(0,mD)$ can be written as $2x+O(h)$.

By our previous calculations, we conclude that the sum of $h^{-k}\Delta^+_k c_z(q,k)$ along $q \in \{0,1,...,4\}$ and $z \in S^q_{\frac{1}{2}-\frac{q}{4}}$ is dominated by the sum $O\left(x^{\frac{1}{2}}M^{D-1}\right) + O\left(h^{-k}x^{\frac{1}{2}+k}M^{D-k-1}\right)$.

Since the left sides of the estimates derived above, are obviously summands of $h^{-k}\ \Delta^+_k\ \psi_k(x)$, and $\psi_0(x)$ is not larger than $h^{-k}\ \Delta^+_k\ \psi_k(x)$, it is clear that the error terms on the right hand sides of the mentioned estimates, play the key role in determining the error term of $\psi_0(x)$. Thus, they play the key role in determining the error term in the prime geodesic theorem in the case at hand.

We want to determine h and M such that

$$h = x^{\frac{1}{2}}M^{D-1} = h^{-mD}x^{\frac{1}{2}+mD}M^{D-mD-1},$$

where $k = mD$ for some even m.

Put $h = x^\alpha$, $M = x^\beta$.

Hence,

$$\begin{aligned}h =& x^\alpha,\\ x^{\frac{1}{2}}M^{D-1} =& x^{\frac{1}{2}+\beta D-\beta},\\ &h^{-mD}x^{\frac{1}{2}+mD}M^{D-mD-1}\\ =& x^{-\alpha mD+\frac{1}{2}+mD+\beta D-\beta mD-\beta}.\end{aligned}$$

We require that

$$\begin{aligned}\alpha =& \frac{1}{2}+\beta D-\beta\\ =& -\alpha mD+\frac{1}{2}+mD+\beta D-\beta mD-\beta.\end{aligned}$$

We obtain, $\beta = \frac{1}{2D}$. Then, $\alpha = \frac{1}{2}+\beta D-\beta = 1 - \frac{1}{2D}$.

Thus, $h = x^{1-\frac{1}{2D}} = O(x)$, $M = x^{\frac{1}{2D}}$.

In this scenario, the expected error term is determined uniquely, i.e., it is given by $O\left(x^{1-\frac{1}{2D}}\right)$.

Since $h^{-k}\,\Delta_k^+\,c_{-j}\,(q,k)=0$ for $-j\in\{-2,-3,...,-k\}$, it follows that the sum of $h^{-k}\Delta_k^+c_{-j}\,(q,k)$, along $q\in\{0,1,...,4\}$ and $j\in\{2,3,...,k\}$ is 0.

Now, we estimate the sum of $h^{-k}\Delta_k^+c_{-1}\,(q,k)$ over $q\in\{0,1,...,4\}$. Obviously, we shall represent this sum in the form

$$\sum_{q=0}^{3}(-1)^q\,h^{-k}\Delta_k^+c_{-1}\,(q,k)+h^{-k}\Delta_k^+c_{-1}\,(4,k)\,.$$

Since $h^{-k}\,\Delta_k^+\,c_{-1}\,(q,k)=0$ for $q\in\{0,1,...,3\}$, it follows that the sum is actually $h^{-k}\Delta_k^+c_{-1}\,(4,k)$.

If $-1\in I_4'$, then $h^{-k}\,\Delta_k^+\,c_{-1}\,(4,k)=0$.

Suppose that $-1\in I_4$.

Now,

$$h^{-k}\Delta_k^+c_{-1}\,(4,k)=o_{-1}^4\,(-1)^{-1}\,\frac{1}{\tilde{x}_{-1,4,k}}$$

for some $\tilde{x}_{-1,4,k}\in[x,x+kh]$.

We conclude,

$$\sum_{q=0}^{4}(-1)^q\,h^{-k}\Delta_k^+c_{-1}\,(q,k)=O\left(x^{-1}\right).$$

Finally, we estimate the sum of the elements $h^{-k}\Delta_k^+c_0\,(q,k)$ over $q\in\{0,1,...,4\}$.

If $0\in I_q'$, then $h^{-k}\Delta_k^+c_0\,(q,k)$ is given by $\frac{Z'_{P,\bigwedge^q\bar{\mathfrak{n}}}\left(\frac{q}{4}\right)}{Z_{P,\bigwedge^q\bar{\mathfrak{n}}}\left(\frac{q}{4}\right)}$.

Suppose that $0\in I_q$.

Now, $c_0\,(q,k)$ is the difference between $o_0^q\prod\limits_{\substack{l=0\\l\neq 0}}^{k}l^{-1}x^k\log x$ and

$o_0^q\prod\limits_{\substack{l=0\\l\neq 0}}^{k}l^{-1}\left(-\sum\limits_{\substack{l=0\\l\neq 0}}^{k}l^{-1}+a_{1,0}^q\right)x^k.$

Hence, $h^{-k}\Delta_k^+c_0\,(q,k)$ is $o_0^q\log\tilde{x}_{0,q,k}+o_0^q a_{1,0}^q$ for some $\tilde{x}_{0,q,k}\in[x,x+kh]$.

It immediately follows that

$$\sum_{q=0}^{4}(-1)^q\,h^{-k}\Delta_k^+c_0\,(q,k)=O\left(\log x\right).$$

It remains to estimate the sum of the elements $h^{-k}\Delta_k^+c_{s^q}\,(q,k)$, where $q\in\{0,1,...,4\}$, and $s^q\in S_{\mathbb{R}}^q$.

Since $h^{-k}\Delta_k^+c_{s^q}\,(q,k)$ is $o_{s^q}^q\,(s^q)^{-1}\,\tilde{x}_{s^q,q,k}^{s^q}$ for some $\tilde{x}_{s^q,q,k}\in[x,x+kh]$, and $s^4<0$ for $s^4\in S_{\mathbb{R}}^4$, $s^3\leq\frac{1}{4}$ for $s^3\in S_{\mathbb{R}}^3$, $s^2\leq\frac{1}{2}$ for $s^2\in S_{\mathbb{R}}^2$, $s^1\leq\frac{3}{4}$ for $s^1\in S_{\mathbb{R}}^1$, $s^0\leq\frac{3}{4}$ for $s^0\in S_{\mathbb{R}}^1\setminus\{1\}$, it follows that

$$\sum_{q=0}^{4}(-1)^q\sum_{s^q\in S_{\mathbb{R}}^q}h^{-k}\Delta_k^+c_{s^q}\,(q,k)=O\left(x^{\frac{3}{4}}\right).$$

Now, taking $k=mD$, m even, $h=x^{1-\frac{1}{2D}}$, $M=x^{\frac{1}{2D}}$, combining the estimates derived above, and having in mind that $\psi_0\,(x)$ is not larger than $h^{-mD}\,\Delta_{mD}^+\,\psi_{mD}\,(x)$, we obtain that $\psi_0\,(x)$ is not larger than $2x+O\left(x^{1-\frac{1}{2D}}\right)$.

Similarly, $2x+O\left(x^{1-\frac{1}{2D}}\right)$ is not larger than $\psi_0\,(x)$.

Hence, $\psi_0\,(x)$ is $2x+O\left(x^{1-\frac{1}{2D}}\right)$.

As it is known, this equality yields that

$$\pi\,(x)=2\,\mathrm{li}\,(x)+O\left(x^{1-\frac{1}{2D}}\,(\log x)^{-1}\right)$$

as $x\to+\infty$.

This completes the proof. □

4 Weighted form $\psi_2\,(x)$

In this section we are interested in the second level analogue of the result derived in the previous section.

Suppose that $z\in S_{\frac{1}{2}-\frac{q}{4}}^q$.

We may apply the definition of the operator Δ_{k-2}^+ to conclude that $h^{-k+2}\Delta_{k-2}^+c_z\,(q,k)$ is dominated by the bound $O\left(h^{-k+2}\,|z|^{-k-1}\,x^{\frac{1}{2}+k}\right)$.

Besides this estimate, we are also able to apply the property that the operator Δ_{k-2}^+ has the integral representation, to conclude that $h^{-k+2}\Delta_{k-2}^+c_z\,(q,k)$ is bounded by $O\left(|z|^{-3}\,x^{\frac{5}{2}}\right)$.

Note that these two estimates depend on assumption that h is bounded by $O\,(h)$.

Having in mind these two estimates, we may estimate the sum of the elements $h^{-k+2}\Delta_{k-2}^+c_z\,(q,k)$, where z runs over $S_{\frac{1}{2}-\frac{q}{4}}^q$.

First, we distinguish between two cases: $z\in S_{\frac{1}{2}-\frac{q}{4}}^q$, $\left|\frac{1}{2}-\frac{q}{4}\right|<|z|\leq M$, and $z\in S_{\frac{1}{2}-\frac{q}{4}}^q$, $|z|>$

M, where M is some constant (note that this constant does not have to be bounded by $O(x)$ and will be fixed in the sequel).

It immediately follows that the sum of the elements $h^{-k+2}\Delta^{+}_{k-2}c_z(q,k)$ over $z \in S^{q}_{\frac{1}{2}-\frac{q}{4}}$ is bounded by the sum $O\left(x^{\frac{5}{2}}M^{D-3}\right) + O\left(h^{-k+2}x^{\frac{1}{2}+k}M^{D-k-1}\right)$.

In order to determine the error term which will dominate in this form of the prime geodesic theorem, we calculate $h^{-k+2}\Delta^{+}_{k-2}c_1(0,k)$.

It is equal to $\frac{1}{3}\tilde{x}^3$, for some $x \le \tilde{x} \le x + (k-2)h$.

Writing, $\tilde{x} = x + \varepsilon$ for some $0 \le \varepsilon \le (k-2)h$, we easily obtain that $\frac{1}{3}\tilde{x}^3$ is equal to $\frac{1}{3}x^3 + O\left(x^2h\right) + O\left(xh^2\right) + O\left(h^3\right)$.

Note that the assumption that h is bounded by $O(x)$ will lead us to conclusion that $h^{-k+2}\Delta^{+}_{k-2}c_1(0,k)$ is estimated by $\frac{1}{3}x^3 + O\left(x^2h\right)$ (h will be determined explicitly in the sequel).

By our previous calculations we know that the sum $h^{-k+2}\Delta^{+}_{k-2}c_z(q,k)$ along $g \in \{0,1,...,4\}$ and $z \in S^{q}_{\frac{1}{2}-\frac{q}{4}}$ is equal to $O\left(x^{\frac{5}{2}}M^{D-3}\right) + O\left(h^{-k+2}x^{\frac{1}{2}+k}M^{D-k-1}\right)$.

Now, we determine h and M explicitly, by comparing $O\left(x^2h\right)$, $O\left(h^3\right)$ and $O\left(xh^2\right)$ by $O\left(x^{\frac{5}{2}}M^{D-3}\right)$, $O\left(h^{-k+2}x^{\frac{1}{2}+k}M^{D-k-1}\right)$.

In the first case, we obtain (by temporarily putting h and M to be some x^{α} and x^{β}), that

$$\begin{aligned} 2+\alpha &= \frac{5}{2} + \beta(D-3) \\ &= \alpha(-k+3) + \frac{1}{2} + k + \beta(D-k-1). \end{aligned}$$

Hence, $\alpha = \frac{1}{2} + \frac{1}{2}\frac{D-3}{D-2}$, $\beta = \frac{1}{2}\frac{1}{D-2}$.

Thus, $h = x^{\frac{1}{2}+\frac{1}{2}\frac{D-3}{D-2}} = O(x)$ (since $D-3 \le D-2$), $M = x^{\frac{1}{2}\frac{1}{D-2}}$.

The largest error term is obviously $O\left(x^2h\right)$ (since $O\left(h^3\right)$ and $O\left(xh^2\right)$ are contained in it), so the error term in this case is $O\left(x^{\frac{5}{2}+\frac{1}{2}\frac{D-3}{D-2}}\right)$.

If we compare $O\left(h^3\right)$ by $O\left(x^{\frac{5}{2}}M^{D-3}\right)$, $O\left(h^{-k+2}x^{\frac{1}{2}+k}M^{D-k-1}\right)$, we obtain that

$$\begin{aligned} 3\alpha &= \frac{5}{2} + \beta(D-3) \\ &= \alpha(-k+2) + \frac{1}{2} + k + \beta(D-k-1). \end{aligned}$$

Hence, $\alpha = \frac{5}{6} + \frac{1}{6}\frac{D-3}{D}$ and $\beta = \frac{1}{2D}$.

Thus, $h = x^{\frac{5}{6}+\frac{1}{6}\frac{D-3}{D}}$ ($= O(x)$, since $D-3 \le 3D$) and $M = x^{\frac{1}{2D}}$.

The error term $O\left(x^2h\right)$ dominates once again, and is equal $O\left(x^{\frac{17}{6}+\frac{1}{6}\frac{D-3}{D}}\right)$.

Note that $\frac{5}{2} + \frac{1}{2}\frac{D-3}{D-2} \le \frac{17}{6} + \frac{1}{6}\frac{D-3}{D}$ since $0 \le 6$.

Thus, for now, the optimal error term is $O\left(x^{\frac{5}{2}+\frac{1}{2}\frac{D-3}{D-2}}\right)$.

Finally, we compare $O\left(xh^2\right)$ by $O\left(x^{\frac{5}{2}}M^{D-3}\right)$, $O\left(h^{-k+2}x^{\frac{1}{2}+k}M^{D-k-1}\right)$.

It follows that

$$\begin{aligned} 1+2\alpha &= \frac{5}{2} + \beta(D-3) \\ &= \alpha(-k+2) + \frac{1}{2} + k + \beta(D-k-1). \end{aligned}$$

We obtain that $\alpha = \frac{3}{4} + \frac{1}{4}\frac{D-3}{D-1}$ and $\beta = \frac{1}{2(D-1)}$.

Consequently, $h = x^{\frac{3}{4}+\frac{1}{4}\frac{D-3}{D-1}} = O(x)$ (since $\frac{1}{4}\frac{D-3}{D-1} \le \frac{1}{4}$ if and only if $-3 \le -1$), and $M = x^{\frac{1}{2(D-1)}}$.

Also, the largest error term in this case is $O\left(x^2h\right)$. It is equal to $O\left(x^{\frac{11}{4}+\frac{1}{4}\frac{D-3}{D-1}}\right)$.

Since $\frac{5}{2} + \frac{1}{2}\frac{D-3}{D-2} \le \frac{11}{4} + \frac{1}{4}\frac{D-3}{D-1}$ if and only if $D^2 - 4D + 5 \ge 0$, and the last inequality holds true, we conclude that the optimal error term in all of three discussed cases is $O\left(x^{\frac{5}{2}+\frac{1}{2}\frac{D-3}{D-2}}\right)$, and is achieved for $h = x^{\frac{1}{2}+\frac{1}{2}\frac{D-3}{D-2}}$, $M = x^{\frac{1}{2}\frac{1}{D-2}}$.

Now, we consider the sum of the elements $h^{-k+2}\Delta^{+}_{k-2}c_{s^q}(q,k)$ for $q \in \{0,1,...,4\}$ and $s^q \in S^{q}_{\mathbb{R}}$, $0 < s^q \le \frac{3}{4}$.

By the singularity pattern of the Ruelle zeta function in this setting, we know that the last sum is actually the sum of the same elements over $q \in \{0,1,...,3\}$ and $s^q \in S^{q}_{\mathbb{R}}$, $0 < s^q \le \frac{3}{4}$.

Now, the properties of the operator Δ^{+}_{k-2}, yield that the sum is

$$\begin{aligned} &\sum_{q=0}^{3}(-1)^q \sum_{\substack{s^q \in S^{q}_{\mathbb{R}} \\ 0<s^q\le\frac{3}{4}}} o^{q}_{s^q}\left(s^q\right)^{-1}\left(s^q+1\right)^{-1} \times \\ &\times\left(s^q+2\right)^{-1}x^{s^q+2} + O\left(x^{\frac{7}{4}}h\right). \end{aligned}$$

Note that $h^{-k+2}\Delta_{k-2}^{+}c_{-j}(q,k)=0$ for $-j\in\{-3,-4,...,-k\}$.

Hence, the sum of $h^{-k+2}\Delta_{k-2}^{+}c_{-j}(q,k)$ over $q\in\{0,1,...,4\}$ and $-j\in\{-3,-4,...,-k\}$ is 0.

Since $h^{-k+2}\Delta_{k-2}^{+}c_{-2}(q,k)$ is $\frac{1}{2}\frac{Z'_{P,\bigwedge^{q}\bar{\mathfrak{n}}}\left(-2+\frac{q}{4}\right)}{Z_{P,\bigwedge^{q}\bar{\mathfrak{n}}}\left(-2+\frac{q}{4}\right)}$, it follows that the sum of $h^{-k+2}\Delta_{k-2}^{+}c_{-2}(q,k)$ along $q\in\{0,1,...,4\}$ is

$$\frac{1}{2}\sum_{q=0}^{4}(-1)^{q}\frac{Z'_{P,\bigwedge^{q}\bar{\mathfrak{n}}}\left(-2+\frac{q}{4}\right)}{Z_{P,\bigwedge^{q}\bar{\mathfrak{n}}}\left(-2+\frac{q}{4}\right)}.$$

Note that the sum of the elements $h^{-k+2}\Delta_{k-2}^{+}c_{-1}(q,k)$ over $q\in\{0,1,...,4\}$ is

$$\sum_{q=0}^{3}(-1)^{q}h^{-k+2}\Delta_{k-2}^{+}c_{-1}(q,k)$$
$$+h^{-k+2}\Delta_{k-2}^{+}c_{-1}(4,k).$$

Since $-1\in I'_q$ for $q\in\{0,1,...,3\}$, it follows that the sum of $h^{-k+2}\Delta_{k-2}^{+}c_{-1}(q,k)$ over $q\in\{0,1,...,3\}$ is $O(x)$.

Next, we determine $h^{-k+2}\Delta_{k-2}^{+}c_{-1}(4,k)$.

It is clear that we have two possibilities: $-1\in I'_4$ or $-1\in I_4$.

Suppose that $-1\in I'_4$.

Reasoning as in the previous case, we obtain that $h^{-k+2}\Delta_{k-2}^{+}c_{-1}(4,k)$ is $O(x)$.

Now, suppose that $-1\in I_4$.

$c_{-1}(4,k)$ is the difference between $o_{-1}^{4}\prod_{\substack{l=0\\l\neq1}}^{k}(-1+l)^{-1}x^{k-1}\log x$ and $o_{-1}^{4}\prod_{\substack{l=0\\l\neq1}}^{k}(-1+l)^{-1}\times$
$\times\left(-\sum_{\substack{l=0\\l\neq1}}^{k}(-1+l)^{-1}+a_{1,-1}^{4}\right)x^{k-1}$.

Therefore, in this case, $h^{-k+2}\Delta_{k-2}^{+}c_{-1}(4,k)$ is $O(x\log x)$.

Consequently, the sum of the elements $h^{-k+2}\Delta_{k-2}^{+}c_{-1}(q,k)$ along $q\in\{0,1,...,4\}$ is $O(x\log x)$.

Now, we consider the sum of $h^{-k+2}\Delta_{k-2}^{+}c_{0}(q,k)$ along $q\in\{0,1,...,4\}$.

As in the previous case, it can happen that either $0\in I'_q$ or $0\in I_q$.

If $0\in I'_q$, then, the fact that $h=O(x)$ immediately yields that $h^{-k+2}\Delta_{k-2}^{+}c_{0}(q,k)$ is $O\left(x^{2}\right)$.

Suppose that $0\in I_q$.

As we noted in the previous section, $c_0(q,k)$ is the difference between $o_0^{q}(k!)^{-1}x^{k}\log x$ and $o_0^{q}(k!)^{-1}\left(-\sum_{l=1}^{k}\frac{1}{l}+a_{1,0}^{q}\right)x^{k}$.

Hence, in this case, $h^{-k+2}\Delta_{k-2}^{+}c_{0}(q,k)$ is $O\left(x^{2}\log x\right)$.

In other words, the sum of $h^{-k+2}\Delta_{k-2}^{+}c_{0}(q,k)$ along $q\in\{0,1,...,4\}$ is $O\left(x^{2}\log x\right)$.

Finally, one easily finds that the sum of the elements $h^{-k+2}\Delta_{k-2}^{+}c_{s^q}(q,k)$ over $q\in\{0,1,...,4\}$ and $s^{q}\in S_{\mathbb{R}}^{q}$, $-1<s^{q}<0$ is $O\left(x^{2}\right)$.

Combining the estimates derived above, and taking into account that $\psi_2(x)$ is not larger than $h^{-k+2}\Delta_{k-2}^{+}\psi_k(x)$, we conclude that $\psi_2(x)$ is not larger than

$$\frac{1}{3}x^{3}+\sum_{q=0}^{3}(-1)^{q}\sum_{\substack{s^{q}\in S_{\mathbb{R}}^{q}\\0<s^{q}\leq\frac{3}{4}}}o_{s^q}^{q}\left(s^{q}\right)^{-1}\times$$
$$\times\left(s^{q}+1\right)^{-1}\left(s^{q}+2\right)^{-1}x^{s^{q}+2}+$$
$$O\left(x^{2}h\right)+O\left(x^{\frac{5}{2}}M^{D-3}\right)+$$
$$O\left(h^{-k+2}x^{\frac{1}{2}+k}M^{D-k-1}\right)+O\left(x^{2}\log x\right).$$

Putting $h=x^{\frac{1}{2}+\frac{1}{2}\frac{D-3}{D-2}}$, $M=x^{\frac{1}{2}\frac{1}{D-2}}$, we obtain that $\psi_2(x)$ is not larger than

$$\frac{1}{3}x^{3}+\sum_{q=0}^{3}(-1)^{q}\sum_{\substack{s^{q}\in S_{\mathbb{R}}^{q}\\0<s^{q}\leq\frac{3}{4}}}o_{s^q}^{q}\left(s^{q}\right)^{-1}\times$$
$$\times\left(s^{q}+1\right)^{-1}\left(s^{q}+2\right)^{-1}x^{s^{q}+2}+$$
$$O\left(x^{\frac{5}{2}+\frac{1}{2}\frac{D-3}{D-2}}\right).$$

Reasoning in an analogous way, we also conclude that the last sum is not larger than $\psi_2(x)$.

Thus, we have proved the following theorem.

Theorem 2. *Let X_Γ be as above. Then, $\frac{\psi_2(x)}{x^2}$ is*

$$\frac{1}{3}x+O\left(x^{\frac{1}{2}+\frac{1}{2}\frac{D-3}{D-2}}\right)$$

as $x\rightarrow+\infty$.

5 Remarks

The author in [16, p. 64], derived that

$$\Delta\left(\frac{2}{(2D+1)!}x^{2D+1}\right) = ax + b$$

for some $a, b \in \mathbb{R}$.

Then, it was not so hard to calculate a and b explicitly.

While it was done for the a, the b was considered as a constant, and hence as a non-important term in further calculations. This approach let to the conclusion that the error term $O\left(x^{\frac{3}{4}}\right)$ could be achieved via remaining two error terms $O\left(K^{D-1}x^{\frac{1}{2}}\right)$ and $O\left(K^{-D-1}x^{2D+\frac{1}{2}}d^{-2D}\right)$.

Recently [13], we have shown that this is really possible. Actually, we have deduced that $O\left(x^{\frac{3}{4}}\right)$ can be achieved if we take $K = x^{\frac{1}{4(D-1)}}$ and $d = x^{\frac{4D-5}{4D-4}}$.

However, as it can be seen from the proof of our main result in this paper, the b (Q in our case) must be taken into account in calculations since it does not represent an arbitrary constant. More precisely, it represents the error term $O(h)$.

Thus, the error terms $O(h)$, $O\left(x^{\frac{1}{2}}M^{D-1}\right)$ and $O\left(h^{-mD}x^{\frac{1}{2}+mD}M^{D-mD-1}\right)$ are responsible for achieving our $O\left(x^{1-\frac{1}{2D}}\right)$.

Regarding the corresponding results in [16], [3] and [13], it is enough to replace $\frac{3}{4}$ by $1 - \frac{1}{2D}$ in the final form of the prime geodesic theorem.

Also, note that some important ideas that the author applied in this research are adopted from [2], [5], [11], [18] and [23].

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