Automatization in Vague Database Relations via Lukasiewicz Fuzzy Implication Operator

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Abstract: The needs of the modern world often require automatization of certain aspects of mankind activities. Science is no exception to this. In this paper we pay attention to vague functional dependencies as generalized functional dependencies. These dependencies are considered as fuzzy formulas. We give strict proof of the equivalence: any two-element vague relation instance on given scheme (which satisfies some set of vague functional dependencies) satisfies given vague functional dependency if and only if the attached fuzzy formula is a logical consequence of the corresponding set of fuzzy formulas. Thanks to this result, we put ourselves into position to automatically verify if some vague functional dependency follows from some set of vague functional dependencies. An appropriate example which supports this claim is also provided.

Key–Words: Automatization, vague functional dependencies, fuzzy formulas

1 Introduction

Let U be the universe of discourse.

Suppose that V is a vague set in U.

Now, there exist functions $t_V : U \to [0,1]$, $f_V : U \to [0,1]$, such that $t_V(u) + f_V(u) \le 1$ for $u \in U$.

We shall write

$$V = \{\langle u, [t_V(u), 1 - f_V(u)]\rangle : u \in U\},$$

where $[t_V(u), 1 - f_V(u)] \subseteq [0,1]$ is the vague value joined to $u \in U$.

Recall that the vague value $[t_V(u), 1 - f_V(u)]$ reduces to the fuzzy value $t_V(u) = 1 - f_V(u) \in [0,1]$ if it happens that $t_V(u) = 1 - f_V(u) \in (0,1)$.

If it happens that $t_V(u) = 1 - f_V(u) = 1$, then the vague value $[t_V(u), 1 - f_V(u)]$ reduces to the ordinary value $t_V(u) = 1 - f_V(u) = 1 \in [0,1]$.

The ordinary case $t_V(u) = 1 - f_V(u) = 0$, we read as: the element u does not belong to the vague set V. In such scenario, we write

$$V = \{\langle u_1, [0.2, 0.7]\rangle, \langle u_2, [1,1]\rangle\}$$

instead of

$$V = \{\langle u_1, [0.2, 0.7]\rangle, \langle u_2, [1,1]\rangle, \langle u_3, [0,0]\rangle\},$$

where $U = \{u_1, u_2, u_3\}$ is some universe of discourse, and V is some vague set in U.

Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in \{1, 2, ..., n\} = I$.

Suppose that $V(U_i)$ is the family of all vague sets in U_i, $i \in I$.

A vague relation r on $R(A_1, A_2, ..., A_n)$ is a subset of the cross product $V(U_1) \times V(U_2) \times ... \times V(U_n)$.

A tuple t of r is then of the form

$$(t[A_1], t[A_2], ..., t[A_n]),$$

where $t[A_i]$ is a vague set in U_i, $i \in I$.

Note that we may (more freely speaking) consider $t[A_i]$ the value of the attribute A_i on t.

A vague relation r on $R(A_1, A_2, ..., A_n)$ can be visibly represented as a two-dimensional table with n columns and the table headings A_1, A_2,..., A_n, where each horizontal row of the table is a tuple of r, and each column of the table contains the attribute values under the corresponding heading.

Let $R(Name, Int, Succ)$ be a relation scheme on domains $U_1 =$ $\{Emy, Ted, Jim, Katie, Sara, Tina, Joe, John\}$, $U_2 = \{115, 130, 145\}$, $U_3 = \{5, 10\}$, where Int (as intelligence) and $Succ$ (as success) are vague attributes on universes U_2 and U_3, respectively, and $Name$ is ordinary attribute on the universe of discourse U_1.

Let r be the vague relation instance on $R\,(Name, Int, Succ)$ given by Table 1.

Table 1:

$Name$	Int	$Succ$
$\{Ted\}$	$\{\langle 115, [1,1]\rangle\}$	$\{\langle 10, [1,1]\rangle\}$
$\{Sara\}$	$\{\langle 115, [.7,.9]\rangle, \langle 130, [.9,.95]\rangle\}$	$\{\langle 5, [.6,.9]\rangle, \langle 10, [.8,.95]\rangle\}$
$\{Jim\}$	$\{\langle 130, [.8,.9]\rangle, \langle 145, [.85,.95]\rangle\}$	$\{\langle 10, [1,1]\rangle\}$
$\{Katie\}$	$\{\langle 145, [1,1]\rangle\}$	$\{\langle 5, [.9,.95]\rangle\}$

The vague sets $\{\langle 115, [1,1]\rangle\}$ and $\{\langle 10, [1,1]\rangle\}$, given in the first row of the Table 1, mean that the knowledge about Ted's intelligence and success is very accurate. More precisely, one knows that his intelligence and success are exactly 115 and 10, respectively. Having in mind that the ranges of person's intelligence and success are determined by the sets $\{115, 130, 145\}$ and $\{5, 10\}$, we may say that Ted is very successful person with regard to his intelligence. Sara's intelligence is determined by the vague set $\{\langle 115, [0.7, 0.9]\rangle, \langle 130, [0.9, 0.95]\rangle\}$. Since the truth value 0.7 is quite high, the false value $0.1 = 1 - 0.9$ is pretty small, and the difference $0.9 - 0.7 = 0.2$ is also very small, we conclude that Sara's intelligence must be close to 115. However, $0.9 > 0.7$, $0.05 = 1 - 0.95 < 0.1$, and $0.95 - 0.9 = 0.05 < 0.2 = 0.9 - 0.7$, so Sara's intelligence is definitively closer to 130 (from bellow) than to 115 (note that $115 \notin \{\langle 115, [0.7, 0.9]\rangle, \langle 130, [0.9, 0.95]\rangle\}$). Reasoning in the same way, we conclude that Sara's success is between 5 and 10, and it is closer to 10 than to 5. The data about Katie are quite precise. As opposed to Ted, however, she is a very intelligent person who is not so successful. Compared to Ted and Katie, Sara is a relatively intelligent person who is relatively successful. Finally, Jim is a pretty intelligent person who is also very successful.

For the basic relational concepts, see, e.g., [19].

Let r_1 be the fuzzy relation instance on $R\,(Name, Int, Succ)$ given by Table 2 (now, we assume that Int and $Succ$ are fuzzy attributes on U_2 and U_3, respectively).

Table 2:

	$Name$	Int	$Succ$
$t_1^{'}$	$\{Emy\}$	$\{\langle 130, a_1\rangle, \langle 145, a_2\rangle\}$	$\{\langle 5, a_3\rangle, \langle 10, a_4\rangle\}$
$t_2^{'}$	$\{John\}$	$\{\langle 115, a_5\rangle\}$	$\{\langle 5, 1\rangle\}$

In Table 2, $a_1 \in (0,1)$ denotes the membership value of the element $130 \in U_2$ to the fuzzy set $\{\langle 130, a_1\rangle, \langle 145, a_2\rangle\}$, etc., $a_5 \in (0,1)$ denotes the membership value of the element $115 \in U_2$ to the fuzzy set $\{\langle 115, a_2\rangle\}$.

The authors in [12] and [4], for example, apply fuzzy membership values to incorporate fuzzy data into relational database theory.

Note that the fuzzy relation instance r_1 may be represented as the vague relation instance given by Table 3.

Table 3:

	$Name$	Int	$Succ$
$t_1^{'}$	$\{Emy\}$	$\{\langle 130, [a_1, a_1]\rangle, \langle 145, [a_2, a_2]\rangle\}$	$\{\langle 5, [a_3, a_3]\rangle, \langle 10, [a_4, a_4]\rangle\}$
$t_2^{'}$	$\{John\}$	$\{\langle 115, [a_5, a_5]\rangle\}$	$\{\langle 5, [1,1]\rangle\}$

Similarly, the relation instance r_2 on $R\,(Name, Int, Succ)$ given by Table 4 (now, we assume that the attributes Int and $Succ$ are ordinary attributes on U_2 and U_3, respectively), may be represented as the vague relation instance given by Table 5.

Table 4:

	$Name$	Int	$Succ$
$t_1^{''}$	$\{Joe\}$	$\{130\}$	$\{5\}$
$t_2^{''}$	$\{Tina\}$	$\{145\}$	$\{10\}$

Table 5:

	$Name$	Int	$Succ$
$t_1^{''}$	$\{Joe\}$	$\{\langle 130, [1,1]\}$	$\{\langle 5, [1,1]\}$
$t_2^{''}$	$\{Tina\}$	$\{\langle 145, [1,1]\}$	$\{\langle 10, [1,1]\}$

For the ordinary relational database theory, see [31].

The aforementioned examples show clearly that the vague relation concept represents a natural generalization of the ordinary relation concept and the fuzzy relation concept. While the relation theory is not able to handle imprecise data almost at all, and the knowledge about fuzzy data has its own limitations, the quality of the information about vague data is obviously much more refined.

Let $a_1 = [t_{V_1}(u_1), 1 - f_{V_1}(u_1)] \subseteq [0,1]$ and $a_2 = [t_{V_2}(u_2), 1 - f_{V_2}(u_2)] \subseteq [0,1]$ be the vague values joined to $u_1 \in U_1$ and $u_2 \in U_2$, respectively, where

$$V_i = \{\langle u_i, [t_{V_i}(u_i), 1 - f_{V_i}(u_i)]\rangle \,:\, u_i \in U_i\}$$

is a vague set in the universe of discourse U_i, $i \in \{1, 2\}$.

We define the similarity measure $SE(a_1, a_2)$ between the vague values a_1 and a_2 following Lu-Ng [21].

Note that $SE(a_1, a_2) \in [0, 1]$.

Moreover, $SE(a_1, a_2) = SE(a_2, a_1)$, $SE(a_1, a_2) = 1$ if and only if $a_1 = a_2$, and $SE(a_1, a_2) = 0$ if and only if $a_1 = [0, 0]$, $a_2 = [1, 1]$ or $a_1 = [0, 1]$, $a_2 = [a, a]$, $a \in [0, 1]$ ($a_1 = [1, 1]$, $a_2 = [0, 0]$ or $a_1 = [a, a]$, $a_2 = [0, 1]$, $a \in [0, 1]$).

Note that several authors, including Chen [8], [9], Hong-Kim [18], Li-Xu [20], Szmidt-Kacprzyk [29], Grzegorzewski [13], proposed various definitions of similarity measures between vague sets and distances between intuitionistic fuzzy sets. According to Lu-Ng [21], however, the similarity measure given above, reflects reality in a more appropriate manner when it comes to more general cases.

Let

$$A = \{\langle u, [t_A(u), 1 - f_A(u)]\rangle : u \in U\}$$

and

$$B = \{\langle u, [t_B(u), 1 - f_B(u)]\rangle : u \in U\}$$

be two vague sets in some universe of discourse U.

We define the similarity measure $SE(A, B)$ between the vague sets A and B accordingly.

As it is usual, we write $A \subseteq B$ (and say that the vague set A is contained in the vague set B), if $t_A(u) \le t_B(u)$ and $1 - f_A(u) \le 1 - f_B(u)$ for all $u \in U$.

Hence, $A \subseteq B$ if and only if $t_A(u) \le t_B(u)$, $f_A(u) \ge f_B(u)$ for $u \in U$.

Since $A = B$ if $A \subseteq B$ and $B \subseteq A$, we obtain that $A = B$ if and only if $t_A(u) \le t_B(u)$, $1 - f_A(u) \le 1 - f_B(u)$ and $t_B(u) \le t_A(u)$, $1 - f_B(u) \le 1 - f_A(u)$ for $u \in U$, i.e., if and only if $t_A(u) = t_B(u)$, $1 - f_A(u) = 1 - f_B(u)$ for $u \in U$, i.e., if and only if $t_A(u) = t_B(u)$, $f_A(u) = f_B(u)$ for $u \in U$.

Note that $SE(A, B) \in [0, 1]$.

Furthermore, $SE(A, B) = SE(B, A)$, $SE(A, B) = 1$ if and only if $A = B$, and $SE(A, B) = 0$ if and only if $[t_A(u), 1 - f_A(u)] = [0, 0]$, $[t_B(u), 1 - f_B(u)] = [1, 1]$ for all $u \in U$ or $[t_A(u), 1 - f_A(u)] = [0, 1]$, $[t_B(u), 1 - f_B(u)] = [a, a]$, $a \in [0, 1]$, for all $u \in U$.

Now, we are able to calculate the similarity measures $SE(t_i[Int], t_j[Int])$ and the similarity measures $SE(t_i[Succ], t_j[Succ])$ for $i, j \in \{1, 2, 3, 4\}$, where t_i, $i \in \{1, 2, 3, 4\}$ are tuples of the vague relation instance r on $R(Name, Int, Succ)$ given by Table 1.

We obtain the following results:

$$I = \begin{bmatrix} 1 & 0.69 & 0.22 & 0.33 \\ 0.69 & 1 & 0.78 & 0.22 \\ 0.22 & 0.78 & 1 & 0.46 \\ 0.33 & 0.22 & 0.46 & 1 \end{bmatrix},$$

$$S = \begin{bmatrix} 1 & 0.64 & 1 & 0.13 \\ 0.64 & 1 & 0.64 & 0.55 \\ 1 & 0.64 & 1 & 0.13 \\ 0.13 & 0.55 & 0.13 & 1 \end{bmatrix},$$

where, for example, 0.78 means that

$$\begin{aligned} 0.78 =& SE(t_3[Int], t_2[Int]) \\ =& SE(t_2[Int], t_3[Int]) \\ =& \frac{1}{3} SE([0.7, 0.9], [0, 0]) + \\ & \frac{1}{3} SE([0.9, 0.95], [0.8, 0.9]) + \\ & \frac{1}{3} SE([0, 0], [0.85, 0.95]). \end{aligned}$$

Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$. Suppose that r is a vague relation instance on $R(A_1, A_2, ..., A_n)$. Let t_1 and t_2 be any two (vague) tuples in r. Finally, let $X \subseteq \{A_1, A_2, ..., A_n\}$ be some set of attributes.

We define the similarity measure $SE_X(t_1, t_2)$ between the tuples t_1 and t_2 on the attribute set X as

$$SE_X(t_1, t_2) = \min_{A \in X} \{SE(t_1[A], t_2[A])\}.$$

The following auxiliary results hold true.

Lemma 1. *Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$. Let r be a vague relation instance on $R(A_1, A_2, ..., A_n)$. If $Y \subseteq X \subseteq \{A_1, A_2, ..., A_n\}$, then*

$$SE_Y(t_1, t_2) \ge SE_X(t_1, t_2)$$

for any t_1 and t_2 in r.

Lemma 2. *Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$. Let r be a vague relation instance on $R(A_1, A_2, ..., A_n)$. Suppose that $X = \{A_{i_1}, A_{i_2}, ..., A_{i_k}\}$, where X is a subset of $\{A_1, A_2, ..., A_n\}$. If $SE\left(t_1\left[A_{i_j}\right], t_2\left[A_{i_j}\right]\right) \ge \theta$ for all $j \in \{1, 2, ..., k\}$, where t_1 and t_2 are some two tuples in r, then $SE_X(t_1, t_2) \ge \theta$.*

Lemma 3. *Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$. Let r be a vague relation instance on $R(A_1, A_2, ..., A_n)$. If $SE_X(t_1, t_2) \geq \theta$ and $SE_X(t_2, t_3) \geq \theta$, where t_1, t_2 and t_3 are some three, mutually distinct tuples in r, and X is a subset of $\{A_1, A_2, ..., A_n\}$, then the inequality $SE_X(t_1, t_3) \geq \theta$ does not necessarily hold true.*

Note that the fact that $SE(t_1[A], t_2[A]) \in [0, 1]$ for all $A \in X$, yields that $SE_X(t_1, t_2) \in [0, 1]$.

Moreover,

$$\begin{aligned} SE_X(t_1, t_2) &= \min_{A \in X} \{SE(t_1[A], t_2[A])\} \\ &= \min_{A \in X} \{SE(t_2[A], t_1[A])\} \\ &= SE_X(t_2, t_1). \end{aligned}$$

Furthermore, $SE_X(t_1, t_2) = 1$ if and only if $\min_{A \in X} \{SE(t_1[A], t_2[A])\} = 1$ if and only if $SE(t_1[A], t_2[A]) = 1$ for all $A \in X$ if and only if $t_1[A] = t_2[A]$ for all $A \in X$.

Finally, $SE_X(t_1, t_2) = 0$ if and only if $\min_{A \in X} \{SE(t_1[A], t_2[A])\} = 0$ if and only if there exists $A \in X \subseteq \{A_1, A_2, ..., A_n\}$, such that $SE(t_1[A], t_2[A]) = 0$ if and only if there exists $A \in X$ such that $\left[t_{t_1[A]}(u), 1 - f_{t_1[A]}(u)\right] = [0, 0]$, $\left[t_{t_2[A]}(u), 1 - f_{t_2[A]}(u)\right] = [1, 1]$ for all $u \in U_A$ or $\left[t_{t_1[A]}(u), 1 - f_{t_1[A]}(u)\right] = [0, 1]$, $\left[t_{t_2[A]}(u), 1 - f_{t_2[A]}(u)\right] = [a, a]$, $a \in [0, 1]$ for all $u \in U_A$, where $U_A \in \{U_1, U_2, ..., U_n\}$ is the universe of discourse that corresponds to the attribute $A \in X$.

2 Vague functional dependencies

Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$. Suppose that r is a relation instance on $R(A_1, A_2, ..., A_n)$. Furthermore, let X and Y be subsets of $\{A_1, A_2, ..., A_n\}$.

Relation instance r is said to satisfy the functional dependency $X \to Y$, if for every pair of tuples t_1 and t_2 in r, $t_1[X] = t_2[X]$ implies that $t_1[Y] = t_2[Y]$. Here, $t_1[X] = t_2[X]$ means that $t_1[A] = t_2[A]$ for every $A \in X$.

As it is known, the relational model restricts the attribute values to be atomic (if the attribute value is precise and crisp, then the value is atomic), i.e., $t[A_i] \in U_i$, $i \in I$ for every $t \in r$. Moreover, each U_j, $j \in I$ is equipped with the identity relation $i_j : U_j \times U_j \to \{0, 1\}$, such that $i_j(x, y) = 1$ if and only if $x = y$, and $i_j(x, y) = 0$ if and only if $x \neq y$. In other words, the crisp relational model compares two attribute values by checking whether or not the two values are equal. Thus, $i_l(t_j[A_l], t_k[A_l]) = 1$ if and only if $t_j[A_l] = t_k[A_l]$, and $i_l(t_j[A_l], t_k[A_l]) = 0$ if and only if $t_j[A_l] \neq t_k[A_l]$, where t_j, $t_k \in r$, and $A_l \in \{A_1, A_2, ..., A_n\}$.

Unfortunately, the ordinary relational database model is far from being enough to capture all of the information about the real-world facts. Namely, the attribute values are usually imprecise ones, i.e., fuzzy. In order to be able to store such fuzzy attribute value, one stores a set of crisp values in place of the fuzzy value, where the crisp values are some, mutually distinct elements from the attribute domain, and are similar to the fuzzy value. Therefore, the following definition is more than justified.

Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$. A fuzzy relation instance r on $R(A_1, A_2, ..., A_n)$ is a subset of the cross product $2^{U_1} \times 2^{U_2} \times ... \times 2^{U_n}$ of the power sets of the domains of the attributes. A tuple t of r is then of the form $(t[A_1], t[A_2], ..., t[A_n])$, where $t[A_i] \subseteq U_i$ for $i \in I$, and where we assume that $t[A_i] \neq \emptyset$ for $i \in I$.

As we already noted, any fuzzy attribute value is described by some set of crisp values, where each of the crisp values is similar to the fuzzy value. More precisely, each attribute domain U_j, $j \in I$ is equipped with some similarity relation $s_j : U_j \times U_j \to [0, 1]$, where $s_j : U_j \times U_j \to [0, 1]$ is said to be a similarity relation on U_j, if for every x, y, $z \in U_j$, the conditions: $s_j(x, x) = 1$, $s_j(x, y) = s_j(y, x)$, and $s_j(x, z) \geq \max_{y \in U_j} (\min(s_j(x, y), s_j(y, z)))$ hold true. Thus, while in the case of relational database model we were able to check if $t_j[A_l] = t_k[A_l]$, now, in the case of fuzzy relational database model, we are able to define how conformant $t_j[A_l]$ and $t_k[A_l]$ are. In particular (see, [28]), if $s_l : U_l \times U_l \to [0, 1]$ is a similarity relation on U_l, and $t_j[A_l] = d_j$, $t_k[A_l] = d_k$, then the conformance $\varphi(A_l[t_j, t_k])$ of the attribute A_l on tuples t_j and t_k is defined by

$$\begin{aligned} &\varphi(A_l[t_j, t_k]) \\ &= \min\left\{ \min_{x \in d_j} \left\{ \max_{y \in d_k} \{s_l(x, y)\} \right\}, \right. \\ &\left. \min_{x \in d_k} \left\{ \max_{y \in d_j} \{s_l(x, y)\} \right\} \right\}. \end{aligned}$$

Hence, we calculate $\varphi(A_l[t_j, t_k])$ instead of calculating $i_l(t_j[A_l], t_k[A_l])$, i.e., instead of checking whether or not $t_j[A_l] = t_k[A_l]$. Consequently, we calculate $\varphi(X[t_j, t_k])$ instead of checking whether or not $t_j[X] = t_k[X]$, where X is a subset of $\{A_1, A_2, ..., A_n\}$, and $\varphi(X[t_j, t_k])$ is the conformance of the attribute set X on tuples t_j and t_k, defined by

$$\varphi(X[t_j, t_k]) = \min_{A \in X} \{\varphi(A[t_j, t_k])\}.$$

For the similarity-based fuzzy relational database approach, we refer to [5]-[7].

Now, the condition: $t_1[X] = t_2[X]$ implies that $t_1[Y] = t_2[Y]$, could be read as: if $\varphi(X[t_1, t_2])$ resp. $\varphi(Y[t_1, t_2])$ is the conformance of the attribute set X resp. Y on tuples t_1 and t_2, then $\varphi(Y[t_1, t_2]) \geq \varphi(X[t_1, t_2])$. More precisely, we could say that some fuzzy relation instance r on $R(A_1, A_2, ..., A_n)$ satisfies the fuzzy functional dependency $X \to_F Y$, if for every pair of tuples t_1 and t_2 in r, $\varphi(Y[t_1, t_2]) \geq \varphi(X[t_1, t_2])$.

Consider the following example.

Let $R(Tea, Exp, Sal)$ be a relation scheme on domains $U_1 = \{Grace, Harry, Oscar\}$, $U_2 = \{low, high\}$, $U_2 = \{3800USD, 4500USD\}$, where Exp (as experience) and Sal (as salary) are fuzzy attributes on universes U_2 and U_3, respectively, and Tea (as teachers) is ordinary attribute on the universe of discourse U_1.

Let $s_2 : U_2 \times U_2 \to [0,1]$ be the similarity relation on U_2 defined by $s_2(low, high) = 0.3$, and $s_3 : U_3 \times U_3 \to [0,1]$ be the similarity relation on U_3 defined by $s_3(3800USD, 4500USD) = 0.5$. Let r_3 be the fuzzy relation instance on $R(Tea, Exp, Sal)$ given by Table 6.

Table 6:

Tea	Exp	$Sall$
$\{Grace\}$	$\{low, high\}$	$\{3800\$, 4500\$\}$
$\{Oscar\}$	$\{low\}$	$\{3800\$\}$

Consider the dependency: *teachers with similar experiences should have similar salaries*. Note that the values of the attributes: *experience* and *salary* my be imprecise. This fact, as well as the fact that the word *similar* is applied within dependency, imply that this dependency can be taken as an example of fuzzy functional dependency. It can be written in the form $X \to_F Y$, where $X = \{Exp\}$ and $Y = \{Sal\}$.

Let's check if the fuzzy relation instance r_3 satisfies $X \to_F Y$. We obtain,

$$\begin{aligned} &\varphi(X[t_1, t_2]) \\ &= \min_{A \in X} \{\varphi(A[t_1, t_2])\} = \varphi(Exp[t_1, t_2]) \\ &= \min\{\min\{1, 0.3\}, \min\{1\}\} \\ &= \min\{0.3, 1\} = 0.3, \end{aligned}$$

$$\begin{aligned} &\varphi(Y[t_1, t_2]) \\ &= \min_{A \in Y} \{\varphi(A[t_1, t_2])\} = \varphi(Sal[t_1, t_2]) \\ &= \min\Big\{ \min\Big\{ \max\{s_3(3800USD, 3800USD)\}, \\ &\quad \max\{s_3(4500USD, 3800USD)\}\Big\}, \\ &\quad \min\Big\{ \max\Big\{s_3(3800USD, 3800USD), \\ &\quad s_3(3800USD, 4500USD)\Big\}\Big\}\Big\} \\ &= \min\{\min\{1, 0.5\}, \min\{1\}\} \\ &= \min\{0.5, 1\} = 0.5. \end{aligned}$$

The condition $\varphi(Y[t_1, t_2]) \geq \varphi(X[t_1, t_2])$ is satisfied. This means that r_3 satisfies $X \to_F Y$.

Assume for a moment that $t_2[Exp] = \{low, high\}$. Now,

$$\begin{aligned} &\varphi(X[t_1, t_2]) \\ &= \varphi(Exp[t_1, t_2]) \\ &= \min\Big\{ \min\Big\{ \max\Big\{s_2(low, low), \\ &\quad s_2(low, high)\Big\}, \\ &\quad \max\{s_2(high, low), s_2(high, high)\}\Big\}, \\ &\quad \min\Big\{ \max\{s_2(low, low), s_2(low, high)\}, \\ &\quad \max\{s_2(high, low), s_2(high, high)\}\Big\}\Big\} \\ &= \min\Big\{ \min\{\max\{1, 0.3\}, \max\{0.3, 1\}\}, \\ &\quad \min\{\max\{1, 0.3\}, \max\{0.3, 1\}\}\Big\} \\ &= \min\{\min\{1, 1\}, \min\{1, 1\}\} \\ &= \min\{1, 1\} = 1. \end{aligned}$$

In this case, $\varphi(Y[t_1, t_2]) = 0.5 < 1 = \varphi(X[t_1, t_2])$, i.e., r_3 violates $X \to_F Y$.

Note that the dependency: *teachers with similar experiences should have similar salaries*, tells the truth about the real-world. Obviously, both scenarios $t_2[Exp] = \{low\}$ and $t_2[Exp] = \{low, high\}$ are possible in reality. In the first case, Grace and Oscar have similar experiences and similar salaries, where the salaries are more similar than the experiences are. In the second case, their salaries are similar, but not identical, although their experiences are identical. This discussion shows that the dependency: *teachers with similar experiences should have similar salaries*, makes sense by itself, and that the instance r_3 should satisfy this dependency in both cases, $t_2 [Exp] = \{low\}$ and $t_2 [Exp] = \{low, high\}$. The inequalities $\varphi (Y [t_1, t_2]) \geq \varphi (X [t_1, t_2])$, $\varphi (Y [t_1, t_2]) < \varphi (X [t_1, t_2])$, where $t_2 [Exp] = \{low\}$, $t_2 [Exp] = \{low, high\}$, respectively, tell us, however, that the condition $\varphi (Y [t_1, t_2]) \geq \varphi (X [t_1, t_2])$ is not adequate for determining whether or not the fuzzy relation instance r satisfies the fuzzy functional dependency $X \to_F Y$. If this condition is satisfied, the instance r satisfies the dependency $X \to_F Y$ for sure. Otherwise, if the condition fails, the instance r may or may not satisfy $X \to_F Y$.

In order to overcome these difficulties and correct the irregularities, Sozat and Yazici [28] introduced the following definition.

Let $R (A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$. Suppose that r is a fuzzy relation instance on $R (A_1, A_2, ..., A_n)$. Furthermore, let X and Y be subsets of $\{A_1, A_2, ..., A_n\}$, and $\theta \in [0, 1]$. Fuzzy relation instance r is said to satisfy the fuzzy functional dependency $X \overset{\theta}{\to}_F Y$, if for every pair of tuples t_1 and t_2 in r, $\varphi (Y [t_1, t_2]) \geq \min \{\theta, \varphi (X [t_1, t_2])\}$.

Thus, if it happens that $\varphi (Y [t_1, t_2]) \geq \varphi (X [t_1, t_2])$ for t_1, $t_2 \in r$, then

$$\begin{aligned}\varphi (Y [t_1, t_2]) \geq& \varphi (X [t_1, t_2]) \\ \geq& \min \{\theta, \varphi (X [t_1, t_2])\}\end{aligned}$$

for every t_1, $t_2 \in r$, and $\theta \in [0, 1]$, i.e., r satisfies $X \overset{\theta}{\to}_F Y$ for every $\theta \in [0, 1]$.

More generally, if it happens that for every t_1, $t_2 \in r$, either $\varphi (Y [t_1, t_2]) \geq \varphi (X [t_1, t_2])$ or $\varphi (Y [t_1, t_2]) \geq \theta$, then

$$\varphi (Y [t_1, t_2]) \geq \min \{\theta, \varphi (X [t_1, t_2])\}$$

for every $t_1, t_2 \in r$, i.e., r satisfies $X \overset{\theta}{\to}_F Y$.

Consequently, if it happens that for some t_1, $t_2 \in r$, $\varphi (Y [t_1, t_2]) < \varphi (X [t_1, t_2])$, and $\varphi (Y [t_1, t_2]) < \theta$, then the instance r violates the dependency $X \overset{\theta}{\to}_F Y$.

In particular, the instance r_3 satisfies the dependency $X \overset{\theta}{\to}_F Y$ for every $\theta \in [0, 1]$ (see, Table 6). Furthermore, if $t_2 [Exp] = \{low, high\}$, then the instance r_3 satisfies resp. violates the dependency $X \overset{\theta}{\to}_F Y$ if $\theta \in [0, 0.5]$ resp. $\theta \in (0.5, 1]$.

The value $\theta \in [0, 1]$ that appears in the notation $X \overset{\theta}{\to}_F Y$ is called the linguistic strength of the fuzzy functional dependency. If $\theta = 1$, the fuzzy functional dependency $X \overset{\theta}{\to}_F Y$ becomes $X \to_F Y$.

Now, one could try to say that some vague relation instance r on $R (A_1, A_2, ..., A_n)$ satisfies the vague functional dependency $X \to_V Y$, if for every pair of tuples t_1 and t_2 in r, $SE_Y (t_1, t_2) \geq SE_X (t_1, t_2)$ (see, e.g., [21], [32]).

Recall the vague relation instance r given by Table 1.

Consider the dependency: *the intelligence level of a person more or less determines the degree of success*.

Since the values of the attributes *intelligence* and *success* may be imprecise, we may consider this dependency as a vague functional dependency. We can write it in the form $X \to_V Y$, where $X = \{Int\}$ and $Y = \{Succ\}$ (see, Table 1).

Let's check if the vague relation instance r satisfies $X \to_V Y$.

Since (see, matrices I and S),

$$\begin{aligned}SE_Y (t_3, t_4) =& \min_{A \in Y} \{SE (t_3 [A], t_4 [A])\} \\ =& SE (t_3 [Succ], t_4 [Succ]) = 0.13\end{aligned}$$

and

$$\begin{aligned}SE_X (t_3, t_4) =& \min_{A \in X} \{SE (t_3 [A], t_4 [A])\} \\ =& SE (t_3 [Int], t_4 [Int]) = 0.46,\end{aligned}$$

it follows that the instance r violates $X \to_V Y$. The vague functional dependency: *the intelligence level of a person more or less determines the degree of success*, however, tells the truth about the real-world in the same way the fuzzy functional dependency: *teachers with similar experiences should have similar salaries* does. Moreover, the scenario presented by the vague relation instance r (Table 1), makes sense in reality. This actually means that the instance r should somehow satisfy the dependency $X \to_V Y$.

Reasoning as in the case of fuzzy functional dependencies, we conclude that the condition $SE_Y (t_1, t_2) \geq SE_X (t_1, t_2)$, t_1, $t_2 \in r$, must be adapted. We introduce the following definition.

Let $R (A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$. Suppose that r is a vague relation instance on $R (A_1, A_2, ..., A_n)$. Furthermore, let X and Y be subsets of $\{A_1, A_2, ..., A_n\}$, and $\theta \in [0, 1]$. Vague relation instance r is said to satisfy the vague functional dependency $X \overset{\theta}{\to}_V Y$, if for every pair of tuples t_1 and t_2 in r, $SE_Y(t_1, t_2) \geq \min\{\theta, SE_X(t_1, t_2)\}$.

Thus, if $SE_Y(t_1, t_2) \geq SE_X(t_1, t_2)$ for t_1, $t_2 \in r$, the instance r satisfies $X \overset{\theta}{\to}_V Y$ for $\theta \in [0, 1]$. If for every t_1, $t_2 \in r$, either $SE_Y(t_1, t_2) \geq SE_X(t_1, t_2)$ or $SE_Y(t_1, t_2) \geq \theta$, the instance r satisfies $X \overset{\theta}{\to}_V Y$. Finally, if for some t_1, $t_2 \in r$, the conditions $SE_Y(t_1, t_2) < SE_X(t_1, t_2)$ and $SE_Y(t_1, t_2) < \theta$ hold true, the instance r violates $X \overset{\theta}{\to}_V Y$.

Now, the vague relation instance r given by Table 1, satisfies resp. violates the vague functional dependency $X \overset{\theta}{\to}_V Y$ ($\{Int\} \overset{\theta}{\to}_V \{Succ\}$), if $\theta \in [0, 0.13]$ resp. $\theta \in (0.13, 1]$.

If $\theta = 1$, the vague functional dependency $X \overset{\theta}{\to}_V Y$ becomes $X \to_V Y$.

For yet another definition of vague functional dependency, called α-vague functional dependency, see [24].

3 Soundness of inference rules for vague functional dependencies

The following rules are the inference rules for vague functional dependencies (VFDs).

VF1 Inclusive rule for VFDs: If $X \overset{\theta_1}{\longrightarrow}_V Y$ holds, and $\theta_1 \geq \theta_2$, then $X \overset{\theta_2}{\longrightarrow}_V Y$ holds.

VF2 Reflexive rule for VFDs: If $X \supseteq Y$, then $X \to_V Y$ holds.

VF3 Augmentation rule for VFDs: If $X \overset{\theta}{\to}_V Y$ holds, then $XZ \overset{\theta}{\to}_V YZ$ holds.

VF4 Transitivity rule for VFDs: If $X \overset{\theta_1}{\longrightarrow}_V Y$ and $Y \overset{\theta_2}{\longrightarrow}_V Z$ hold true, then $X \overset{\min(\theta_1,\theta_2)}{\to}_V Z$ holds true.

Here, XZ means $X \cup Z$.

Theorem 4. *The inference rules: VF1, VF2, VF3 and VF4 are sound.*

4 Soundness of additional inference rules for vague functional dependencies

The following inference rules are additional inference rules for vague functional dependencies. We note that these rules follow from the rules: VF1, VF2, VF3 and VF4. This means that the vague functional dependencies obtained by the additional rules can certainly be obtained by successive application of the rules: VF1, VF2, VF3 and VF4. The additional inference rules, however, can make such an effort much shorter and easier.

VF5 Union rule for VFDs: If $X \overset{\theta_1}{\longrightarrow}_V Y$ and $X \overset{\theta_2}{\longrightarrow}_V Z$ hold true, then $X \overset{\min(\theta_1,\theta_2)}{\to}_V YZ$ holds also true.

VF6 Pseudo-transitivity rule for VFDs: If $X \overset{\theta_1}{\longrightarrow}_V Y$ and $WY \overset{\theta_2}{\longrightarrow}_V Z$ hold true, then $WX \overset{\min(\theta_1,\theta_2)}{\to}_V Z$ holds true.

VF7 Decomposition rule for VFDs: If $X \overset{\theta}{\to}_V Y$ holds, and $Z \subseteq Y$, then $X \overset{\theta}{\to}_V Z$ also holds.

Theorem 5. *The inference rules: VF5, VF6 and VF7 are sound.*

5 Completeness of inference rules for vague functional dependencies

Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$.

Suppose that $\mathcal{V}$ is some set of vague functional dependencies on $\{A_1, A_2, ..., A_n\}$. The closure $\mathcal{V}^+$ of $\mathcal{V}$ is the set of all vague functional dependencies that can be derived from $\mathcal{V}$ by repeated applications of the inference rules: VF1, VF2, VF3 and VF4.

Note that the set $\mathcal{V}^+$ is infinite one regardless of whether the set $\mathcal{V}$ is finite or not. Namely, if $X \overset{\theta}{\to}_V Y$ belongs to $\mathcal{V}$, then, by VF1, $X \overset{\theta_1}{\longrightarrow}_V Y$ belongs to $\mathcal{V}^+$ for all $\theta_1 \in [0, \theta)$.

Let $X \overset{\theta}{\to}_V Y$ be some vague functional dependency on $\{A_1, A_2, ..., A_n\}$. The dependency $X \overset{\theta}{\to}_V Y$ may or may not belong to $\mathcal{V}^+$. The limit strength of $X \overset{\theta}{\to}_V Y$ (with respect to $\mathcal{V}$) is the number $\theta_l(\mathcal{V}) \in [0, 1]$, such that $X \overset{\theta_l(\mathcal{V})}{\longrightarrow}_V Y$ belongs to $\mathcal{V}^+$, and $\theta' \leq \theta_l(\mathcal{V})$ for each $X \overset{\theta'}{\longrightarrow}_V Y$ that belongs to $\mathcal{V}^+$.

If $X \overset{\theta}{\to}_V Y$ belongs to $\mathcal{V}^+$, then the limit strength $\theta_l(\mathcal{V})$ obviously exists. Otherwise, if $X \overset{\theta}{\to}_V Y$ does not belong to $\mathcal{V}^+$, the limit strength $\theta_l(\mathcal{V})$ does not necessarily exist.

Let X be a subset of $\{A_1, A_2, ..., A_n\}$, and θ be a number in $[0, 1]$. The closure $X^+(\theta, \mathcal{V})$ of

X (with respect to $\mathcal{V}$) is the set of attributes $A \in \{A_1, A_2, ..., A_n\}$, such that $X \overset{\theta}{\to}_V A$ belongs to $\mathcal{V}^+$.

Now, if $A \in X$, then, by VF2, $X \to_V A$ belongs to $\mathcal{V}^+$. Hence, by VF1, $X \overset{\theta}{\to}_V A$ belongs to $\mathcal{V}^+$. Therefore, $A \in X^+(\theta, \mathcal{V})$. Since $A \in X$, we conclude that $X \subseteq X^+(\theta, \mathcal{V})$.

Theorem 6. *Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$. Let $\mathcal{V}^+$ be the closure of $\mathcal{V}$, where $\mathcal{V}$ is some set of vague functional dependencies on $\{A_1, A_2, ..., A_n\}$. Suppose that $X \overset{\theta}{\to}_V Y$ is some vague functional dependency on $\{A_1, A_2, ..., A_n\}$. Then, $X \overset{\theta}{\to}_V Y$ belongs to $\mathcal{V}^+$ if and only if $Y \subseteq X^+(\theta, \mathcal{V})$.*

Theorem 7. *The set $\{VF1, VF2, VF3, VF4\}$ is complete set.*

6 Main result

For various definitions of similarity measures, see, [21], [8], [9], [18] and [20].

Furthermore, for various definitions of vague functional dependencies, see, [21], [24] and [32].

By Theorem 7, the set $\{VF1, VF2, VF3, VF4\}$ is complete set.

This means that there exists a vague relation instance r^* on $R(A_1, A_2, ..., A_n)$ (r^* is denoted by r in [16]), such that r^* satisfies $A \overset{1\theta}{\to}_V B$ if $A \overset{1\theta}{\to}_V B$ belongs to $\mathcal{V}^+$, and violates $X \overset{\theta}{\to}_V Y$, where $X \overset{\theta}{\to}_V Y$ is some vague functional dependency on $\{A_1, A_2, ..., A_n\}$ which is not an element of the closure $\mathcal{V}^+$ of $\mathcal{V}$.

In this paper we shall apply the following operators (see, e.g., [27]):

$$T_M(x, y) = \min\{x, y\},$$
$$S_M(x, y) = \max\{x, y\},$$
$$I_L(x, y) = \min\{1 - x + y, 1\}.$$

T_M is the minimum t-norm, S_M is the maximum t-co-norm, and I_L is the Lukasiewicz fuzzy implication.

Lukasiewicz fuzzy implication I_L is pretty universal fuzzy implication since it is at the same time an S-implication, an R-implication, and a QL-implication.

For various works on S, R and QL-implications, see, [1], [2], [22], [30], [26], [23], [25].

For detailed study on fuzzy implications, we refer to [3].

Let $r = \{t_1, t_2\}$ be any two-element vague relation instance on $R(A_1, A_2, ..., A_n)$, and $\beta \in [0, 1]$.

If $SE(t_1[A_k], t_2[A_k]) \geq \beta$ resp. $SE(t_1[A_k], t_2[A_k]) < \beta$, we put $i_{r,\beta}(A_k)$ to be some value in the interval $\left(\frac{1}{2}, 1\right]$ resp. $\left[0, \frac{1}{2}\right]$.

Through the rest of the paper, we shall assume that each time some $r = \{t_1, t_2\}$ and some $\beta \in [0, 1]$ are given, the fuzzy formula

$$(\wedge_{A \in X} A) \Rightarrow (\wedge_{B \in Y} B)$$

with respect to $i_{r,\beta}$, is joined to $X \overset{\theta}{\to}_V Y$, where $X \overset{\theta}{\to}_V Y$ is a vague functional dependency on $\{A_1, A_2, ..., A_n\}$.

Theorem 8. *Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$. Let C be some set of vague functional dependencies on $\{A_1, A_2, ..., A_n\}$. Suppose that $X \overset{\theta}{\to}_V Y$ is some vague functional dependency on $\{A_1, A_2, ..., A_n\}$. The following two conditions are equivalent:*

(a) Any two-element vague relation instance on $R(A_1, A_2, ..., A_n)$ which satisfies all dependencies in C, satisfies the dependency $X \overset{\theta}{\to}_V Y$.

(b) Let r be any two-element vague relation instance on $R(A_1, A_2, ..., A_n)$, and $\beta \in [0, 1]$. Suppose that $i_{r,\beta}(\mathcal{K}) > \frac{1}{2}$ for all $\mathcal{K} \in C'$, where C' is the set of fuzzy formulas with respect to $i_{r,\beta}$, joined to the elements of C. Then,

$$i_{r,\beta}((\wedge_{A \in X} A) \Rightarrow (\wedge_{B \in Y} B)) > \frac{1}{2}.$$

Proof. For the sake of simplicity, we may assume that $U_1 = U_2 = ... = U_n = \{u\} = U$.

Let $\theta' = \min\{\theta, \theta_C\}$, where

$$\theta_C = \min_{K \overset{1\theta}{\to}_V L \in C} \{_1\theta\}.$$

We may assume that $\theta' < 1$.

Otherwise, if $\theta' = 1$, then $\theta = 1$ and $\theta_C = 1$, i.e., $\theta = 1$ and $_1\theta = 1$ for all $K \overset{1\theta}{\to}_V L \in C$.

This case, where the linguistic strength of f_1 is 1 if $f_1 \in C \cup \left\{X \overset{\theta}{\to}_V Y\right\}$ is not interesting, however.

Fix some $\theta'' < \theta'$.

Let

$$V_1 = \{\langle u, [t_{V_1}(u), 1 - f_{V_1}(u)]\rangle \ : \ u \in U\}$$
$$= \{\langle u, [t_{V_1}(u), 1 - f_{V_1}(u)]\rangle\} = \{\langle u, a\rangle\},$$
$$V_2 = \{\langle u, [t_{V_2}(u), 1 - f_{V_2}(u)]\rangle \ : \ u \in U\}$$
$$= \{\langle u, [t_{V_2}(u), 1 - f_{V_2}(u)]\rangle\} = \{\langle u, b\rangle\}$$

be two vague sets in U, where

$$SE_U(a, b) = \theta''$$

for $SE_U : Vag(U) \times Vag(U) \to [0,1]$, a similarity measure on $Vag(U) = \{a, b\}$.

We obtain,

$$SE(V_1, V_2)$$
$$= \min\Big\{ \min_{\langle u,a\rangle \in V_1} \Big\{ \max_{\langle u,b\rangle \in V_2} \Big\{ SE_U\Big(a, b\Big)\Big\}\Big\},$$
$$\min_{\langle u,b\rangle \in V_2} \Big\{ \max_{\langle u,a\rangle \in V_1} \Big\{ SE_U\Big(b, a\Big)\Big\}\Big\}\Big\}$$
$$= \theta''.$$

Obviously, $SE(V_1, V_1) = SE(V_2, V_2) = 1$.
Furthermore,

$$SE(A, B)$$
$$= \min\Big\{ \min_{\langle u,x\rangle \in A} \Big\{ \max_{\langle u,y\rangle \in B} \Big\{ SE_U\Big(x, y\Big)\Big\}\Big\},$$
$$\min_{\langle u,y\rangle \in B} \Big\{ \max_{\langle u,x\rangle \in A} \Big\{ SE_U\Big(y, x\Big)\Big\}\Big\}\Big\}$$
$$\geq \min\Big\{\theta'', \theta''\Big\} = \theta''$$

for any two vague sets $A = \{\langle u, x\rangle\}$ and $B = \{\langle u, y\rangle\}$ in U.

$(a) \Rightarrow (b)$ Suppose that the condition (a) is satisfied.

Moreover, suppose that the condition (b) is not satisfied.

It follows that there is a two-element vague relation instance r on $R(A_1, A_2, ..., A_n)$, and $\beta \in [0,1]$, such that

$$i_{r,\beta}((\wedge_{A\in K} A) \Rightarrow (\wedge_{B\in L} B)) > \frac{1}{2}.$$

for all $(\wedge_{A\in K} A) \Rightarrow (\wedge_{B\in L} B) \in C'$, and

$$i_{r,\beta}((\wedge_{A\in X} A) \Rightarrow (\wedge_{B\in Y} B)) \leq \frac{1}{2}.$$

Define

$$W = \left\{ A \in \{A_1, A_2, ..., A_n\} \ : \ i_{r,\beta}(A) > \frac{1}{2} \right\}.$$

Suppose that $W = \emptyset$.

It follows that $i_{r,\beta}(A) \leq \frac{1}{2}$ for all $A \in \{A_1, A_2, ..., A_n\}$.

Hence,

$$i_{r,\beta}(\wedge_{A\in M} A)$$
$$= \min\{i_{r,\beta}(A) \ : \ A \in M\} \leq \frac{1}{2}$$

for all $M \subseteq \{A_1, A_2, ..., A_n\}$.

Since

$$i_{r,\beta}((\wedge_{A\in X} A) \Rightarrow (\wedge_{B\in Y} B)) \leq \frac{1}{2},$$

it follows that

$$\min\{1 - i_{r,\beta}(\wedge_{A\in X} A) + i_{r,\beta}(\wedge_{B\in Y} B), 1\}$$
$$= i_{r,\beta}((\wedge_{A\in X} A) \Rightarrow (\wedge_{B\in Y} B)) \leq \frac{1}{2}.$$

If

$$\min\{1 - i_{r,\beta}(\wedge_{A\in X} A) + i_{r,\beta}(\wedge_{B\in Y} B), 1\} = 1,$$

then, $1 \leq \frac{1}{2}$. This is a contradiction.

Hence,

$$\min\{1 - i_{r,\beta}(\wedge_{A\in X} A) + i_{r,\beta}(\wedge_{B\in Y} B), 1\}$$
$$= 1 - i_{r,\beta}(\wedge_{A\in X} A) + i_{r,\beta}(\wedge_{B\in Y} B).$$

We obtain,

$$1 - i_{r,\beta}(\wedge_{A\in X} A) + i_{r,\beta}(\wedge_{B\in Y} B) \leq \frac{1}{2},$$

i.e.,

$$\frac{1}{2} + i_{r,\beta}(\wedge_{B\in Y} B) \leq i_{r,\beta}(\wedge_{A\in X} A).$$

Since $i_{r,\beta}(\wedge_{B\in Y}B)\geq 0$, it follows that $i_{r,\beta}(\wedge_{A\in X}A)\geq \frac{1}{2}$.

This contradicts the fact that $i_{r,\beta}(\wedge_{A\in X}A)\leq \frac{1}{2}$.

We conclude, $W\neq\emptyset$.

Suppose that $W=\{A_1,A_2,...,A_n\}$.

Now, $i_{r,\beta}(A)>\frac{1}{2}$ for all $A\in\{A_1,A_2,...,A_n\}$.

Consequently,

$$\begin{aligned}&i_{r,\beta}(\wedge_{A\in M}A)\\&=\min\{i_{r,\beta}(A)\ :\ A\in M\}>\frac{1}{2}\end{aligned}$$

for all $M\subseteq\{A_1,A_2,...,A_n\}$.

Since

$$i_{r,\beta}((\wedge_{A\in X}A)\Rightarrow(\wedge_{B\in Y}B))\leq\frac{1}{2},$$

it follows that

$$\frac{1}{2}+i_{r,\beta}(\wedge_{B\in Y}B)\leq i_{r,\beta}(\wedge_{A\in X}A).$$

Now, $i_{r,\beta}(\wedge_{B\in Y}B)>\frac{1}{2}$ yields that $i_{r,\beta}(\wedge_{A\in X}A)>1$.

This is a contradiction.

We conclude, $W\neq\{A_1,A_2,...,A_n\}$.

Let $r'=\left\{t',t''\right\}$ be the vague relation instance on $R(A_1,A_2,...,A_n)$ given by Table 7.

Table 7:

	attributes of W	other attributes
t'	$V_1,V_1,...,V_1$	$V_1,V_1,...,V_1$
t''	$V_1,V_1,...,V_1$	$V_2,V_2,...,V_2$

We shall prove that the instance r' satisfies $K\xrightarrow{1\theta}_V L$ if $K\xrightarrow{1\theta}_V L$ belongs to C, and violates $X\xrightarrow{\theta}_V Y$.

Suppose that $K\xrightarrow{1\theta}_V L$ belongs to C.

Assume that $i_{r,\beta}(\wedge_{A\in K}A)\leq\frac{1}{2}$.

Since,

$$\begin{aligned}&i_{r,\beta}(\wedge_{A\in K}A)\\&=\min\{i_{r,\beta}(A)\ :\ A\in K\},\end{aligned}$$

it follows that there exists $A_0\in K$ such that $i_{r,\beta}(A_0)\leq\frac{1}{2}$.

Hence, $A_0\notin W$.

It follows that

$$SE\left(t'[A_0],t''[A_0]\right)=SE(V_1,V_2)=\theta''.$$

Consequently,

$$SE_K\left(t',t''\right)=\min_{A\in K}\left\{SE\left(t'[A],t''[A]\right)\right\}=\theta''$$

since

$$SE\left(t'[A],t''[A]\right)=SE(V_1,V_1)=1$$

if $A\in W$, and

$$SE\left(t'[A],t''[A]\right)=SE(V_1,V_2)=\theta''$$

if $A\notin W$.

Note that $SE_M\left(t',t''\right)\geq\theta''$ for all $M\subseteq\{A_1,A_2,...,A_n\}$.

In particular, $SE_L\left(t',t''\right)\geq\theta''$.

We obtain,

$$\begin{aligned}SE_L\left(t',t''\right)\geq&\theta''=\min\left\{{}_1\theta,\theta''\right\}\\&=\min\left\{{}_1\theta,SE_K\left(t',t''\right)\right\}.\end{aligned}$$

This means that r' satisfies $K\xrightarrow{1\theta}_V L$.

Suppose that $i_{r,\beta}(\wedge_{A\in K}A)>\frac{1}{2}$.

Since

$$i_{r,\beta}((\wedge_{A\in K}A)\Rightarrow(\wedge_{B\in L}B))>\frac{1}{2},$$

it follows that

$$\min\{1-i_{r,\beta}(\wedge_{A\in K}A)+i_{r,\beta}(\wedge_{B\in L}B),1\}>\frac{1}{2}.$$

Suppose that $i_{r,\beta}(\wedge_{B\in L}B)\leq\frac{1}{2}$.

We obtain,

$$\begin{aligned}&1-i_{r,\beta}(\wedge_{A\in K}A)+i_{r,\beta}(\wedge_{B\in L}B)\\&<1-\frac{1}{2}+\frac{1}{2}=1.\end{aligned}$$

Hence,

$$1 - i_{r,\beta}\left(\wedge_{A\in K}A\right) + i_{r,\beta}\left(\wedge_{B\in L}B\right)$$
$$= \min\left\{1 - i_{r,\beta}\left(\wedge_{A\in K}A\right) + i_{r,\beta}\left(\wedge_{B\in L}B\right), 1\right\} > \frac{1}{2},$$

i.e.,

$$\frac{1}{2} + i_{r,\beta}\left(\wedge_{B\in L}B\right) > i_{r,\beta}\left(\wedge_{A\in K}A\right).$$

Since the last inequality holds always true, we conclude that $i_{r,\beta}\left(\wedge_{A\in K}A\right) < \frac{1}{2}$.

This is a contradiction.

We conclude, $i_{r,\beta}\left(\wedge_{B\in L}B\right) > \frac{1}{2}$.

Since

$$i_{r,\beta}\left(\wedge_{B\in L}B\right)$$
$$= \min\left\{i_{r,\beta}\left(B\right) \ : \ B \in L\right\},$$

it follows that $i_{r,\beta}\left(B\right) > \frac{1}{2}$ for all $B \in L$.

Hence, $B \in W$ for all $B \in L$, i.e., $L \subseteq W$.

Consequently,

$$SE_L\left(t^{'}, t^{''}\right) = \min_{A\in L}\left\{SE\left(t^{'}\left[A\right], t^{''}\left[A\right]\right)\right\} = 1.$$

Therefore,

$$SE_L\left(t^{'}, t^{''}\right) = 1 \geq \min\left\{{}_1\theta, SE_K\left(t^{'}, t^{''}\right)\right\}.$$

This means that $r^{'}$ satisfies $K \xrightarrow{1\theta}_V L$.

It remains to prove that $r^{'}$ violates $X \xrightarrow{\theta}_V Y$.

Since

$$i_{r,\beta}\left(\left(\wedge_{A\in X}A\right) \Rightarrow \left(\wedge_{B\in Y}B\right)\right) \leq \frac{1}{2},$$

we have that

$$\frac{1}{2} + i_{r,\beta}\left(\wedge_{B\in Y}B\right) \leq i_{r,\beta}\left(\wedge_{A\in X}A\right).$$

If $i_{r,\beta}\left(\wedge_{B\in Y}B\right) > \frac{1}{2}$, then $i_{r,\beta}\left(\wedge_{A\in X}A\right) > 1$.

This is a contradiction.

Hence, $i_{r,\beta}\left(\wedge_{B\in Y}B\right) \leq \frac{1}{2}$.

Since

$$\frac{1}{2} + i_{r,\beta}\left(\wedge_{B\in Y}B\right) \leq i_{r,\beta}\left(\wedge_{A\in X}A\right)$$

holds always true, it follows that $i_{r,\beta}\left(\wedge_{A\in X}A\right) = 1$.

Now, reasoning in the same way as earlier, we obtain that $SE_Y\left(t^{'}, t^{''}\right) = \theta^{''}$ resp. $SE_X\left(t^{'}, t^{''}\right) = 1$ follows from $i_{r,\beta}\left(\wedge_{B\in Y}B\right) \leq \frac{1}{2}$ resp. $i_{r,\beta}\left(\wedge_{A\in X}A\right) = 1$.

We have,

$$SE_Y\left(t^{'}, t^{''}\right) = \theta^{''} < \theta^{'} \leq \theta$$
$$= \min\left\{\theta, 1\right\}$$
$$= \min\left\{\theta, SE_X\left(t^{'}, t^{''}\right)\right\}.$$

This means that $r^{'}$ satisfies $X \xrightarrow{\theta}_V Y$.

Thus, $r^{'}$ is a two-element vague relation instance on $R\left(A_1, A_2, ..., A_n\right)$ which satisfies all dependencies in C, and violates the dependency $X \xrightarrow{\theta}_V Y$.

This contradicts the fact that the condition (a) is satisfied.

Consequently, the condition (b) is satisfied.

$(b) \Rightarrow (a)$ Suppose that the condition (b) is satisfied.

Moreover, suppose that the condition (a) is not satisfied.

It follows that there is a two-element vague relation instance $r^{'} = \left\{t^{'}, t^{''}\right\}$ on $R\left(A_1, A_2, ..., A_n\right)$ which satisfies all dependencies in C, and violates the dependency $X \xrightarrow{\theta}_V Y$.

Define

$$W = \Big\{A \in \left\{A_1, A_2, ..., A_n\right\} \ : \ SE\left(t^{'}\left[A\right], t^{''}\left[A\right]\right) = 1\Big\}.$$

Suppose that $W = \emptyset$.

It follows that $SE\left(t^{'}\left[A\right], t^{''}\left[A\right]\right) = \theta^{''}$ for all $A \in \left\{A_1, A_2, ..., A_n\right\}$.

Hence, $SE_M\left(t^{'}, t^{''}\right) = \theta^{''}$ for all $M \subseteq \left\{A_1, A_2, ..., A_n\right\}$.

Since $r^{'}$ violates $X \xrightarrow{\theta}_V Y$, we have that

$$SE_Y\left(t^{'}, t^{''}\right) < \min\left\{\theta, SE_X\left(t^{'}, t^{''}\right)\right\}.$$

Now, $SE_Y\left(t^{'}, t^{''}\right) = SE_X\left(t^{'}, t^{''}\right) = \theta^{''}$ yields that

$$\theta'' < \min\left\{\theta, \theta''\right\} = \theta''.$$

This is a contradiction.

We conclude, $W \neq \emptyset$.

Suppose that $W = \{A_1, A_2, ..., A_n\}$.

It follows that $SE\left(t'[A], t''[A]\right) = 1$ for all $A \in \{A_1, A_2, ..., A_n\}$.

Consequently, $SE_M\left(t', t''\right) = 1$ for all $M \subseteq \{A_1, A_2, ..., A_n\}$.

Now, $SE_Y\left(t', t''\right) = SE_X\left(t', t''\right) = 1$, and the fact that r' violates $X \xrightarrow{\theta}_V Y$, yield that

$$1 < \min\{\theta, 1\} = \theta.$$

This is a contradiction.

We obtain, $W \neq \{A_1, A_2, ..., A_n\}$.

Since r' is a two-element vague relation instance on $R(A_1, A_2, ..., A_n)$, and $1 \in [0, 1]$ is a number, we are able to define $i_{r',1}$.

We have,

$$i_{r',1}(A_k) > \frac{1}{2} \text{ if } SE\left(t'[A_k], t''[A_k]\right) \geq 1,$$
$$i_{r',1}(A_k) \leq \frac{1}{2} \text{ if } SE\left(t'[A_k], t''[A_k]\right) < 1,$$

$k \in \{1, 2, ..., n\}$, i.e.,

$$i_{r',1}(A_k) > \frac{1}{2} \text{ if } SE\left(t'[A_k], t''[A_k]\right) = 1,$$
$$i_{r',1}(A_k) \leq \frac{1}{2} \text{ if } SE\left(t'[A_k], t''[A_k]\right) = \theta'',$$

$k \in \{1, 2, ..., n\}$, i.e.,

$$i_{r',1}(A_k) > \frac{1}{2} \text{ if } A_k \in W,$$
$$i_{r',1}(A_k) \leq \frac{1}{2} \text{ if } A_k \notin W,$$

$k \in \{1, 2, ..., n\}$.

Thus,

$$i_{r',1}(A) > \frac{1}{2} \text{ if } A \in W,$$
$$i_{r',1}(A) \leq \frac{1}{2} \text{ if } A \notin W.$$

We shall prove that

$$i_{r',1}\left((\wedge_{A\in K}A) \Rightarrow (\wedge_{B\in L}B)\right) > \frac{1}{2}$$

for all $(\wedge_{A\in K}A) \Rightarrow (\wedge_{B\in L}B) \in C'$, and

$$i_{r',1}\left((\wedge_{A\in X}A) \Rightarrow (\wedge_{B\in Y}B)\right) \leq \frac{1}{2}.$$

We may assume that $(\wedge_{A\in K}A) \Rightarrow (\wedge_{B\in L}B) \in C'$ corresponds to $K \xrightarrow{1\theta}_V L \in C$.

Suppose that $i_{r',1}\left((\wedge_{A\in K}A) \Rightarrow (\wedge_{B\in L}B)\right) \leq \frac{1}{2}$.

Reasoning as earlier, we obtain

$$\frac{1}{2} + i_{r',1}(\wedge_{B\in L}B) \leq i_{r',1}(\wedge_{A\in K}A).$$

It follows that $i_{r',1}(\wedge_{B\in L}B) \leq \frac{1}{2}$, and $i_{r',1}(\wedge_{A\in K}A) = 1$.

Hence, there exists $B_0 \in L$ such that $i_{r',1}(B_0) \leq \frac{1}{2}$.

We obtain, $SE\left(t'[B_0], t''[B_0]\right) = \theta''$.

Consequently, $SE_L\left(t', t''\right) = \theta''$.

Similarly, $i_{r',1}(\wedge_{A\in K}A) = 1$ yields that $SE_K\left(t', t''\right) = 1$.

Therefore,

$$\begin{aligned} SE_L\left(t', t''\right) &= \theta'' < \theta' \leq_1 \theta \\ &= \min\{_1\theta, 1\} \\ &= \min\left\{_1\theta, SE_K\left(t', t''\right)\right\}. \end{aligned}$$

This contradicts the fact that r' satisfies $K \xrightarrow{1\theta}_V L$.

Hence, $i_{r',1}\left((\wedge_{A\in K}A) \Rightarrow (\wedge_{B\in L}B)\right) > \frac{1}{2}$.

Similarly, suppose that $i_{r',1}\left((\wedge_{A\in X}A) \Rightarrow (\wedge_{B\in Y}B)\right) > \frac{1}{2}$.

Let $i_{r',1}(\wedge_{A\in X}A) \leq \frac{1}{2}$.

Now, as earlier, $SE_X\left(t', t''\right) = \theta''$.

Since $SE_M\left(t', t''\right) \geq \theta''$ for all $M \subseteq \{A_1, A_2, ..., A_n\}$, it follows that

$$\begin{aligned} SE_Y\left(t', t''\right) \geq \theta'' &= \min\left\{\theta, \theta''\right\} \\ &= \min\left\{\theta, SE_X\left(t', t''\right)\right\}. \end{aligned}$$

This contradicts the fact that $r^{'}$ violates $X \overset{\theta}{\to}_V Y$.

Let $i_{r^{'},1}\left(\wedge_{A\in X}A\right) > \frac{1}{2}$.

Reasoning as before,
$i_{r^{'},1}\left(\left(\wedge_{A\in X}A\right) \Rightarrow \left(\wedge_{B\in Y}B\right)\right) > \frac{1}{2}$ yields that
$i_{r^{'},1}\left(\wedge_{B\in Y}B\right) > \frac{1}{2}$.

Hence, $SE_Y\left(t^{'}, t^{''}\right) = 1$.

Consequently,

$$SE_Y\left(t^{'}, t^{''}\right) = 1 \geq \min\left\{\theta, SE_X\left(t^{'}, t^{''}\right)\right\}.$$

This contradicts the fact $r^{'}$ violates $X \overset{\theta}{\to}_V Y$.

We obtain, $i_{r^{'},1}\left(\left(\wedge_{A\in X}A\right) \Rightarrow \left(\wedge_{B\in Y}B\right)\right) \leq \frac{1}{2}$.

Thus,

$$i_{r^{'},1}\left(\left(\wedge_{A\in K}A\right) \Rightarrow \left(\wedge_{B\in L}B\right)\right) > \frac{1}{2}$$

for all $\left(\wedge_{A\in K}A\right) \Rightarrow \left(\wedge_{B\in L}B\right) \in C^{'}$, and

$$i_{r^{'},1}\left(\left(\wedge_{A\in X}A\right) \Rightarrow \left(\wedge_{B\in Y}B\right)\right) \leq \frac{1}{2}.$$

This contradicts the fact that the condition (b) is satisfied.

Therefore, the condition (a) is satisfied.

This completes the proof. □

7 Applications

Example 1. Let $R(A, B, C, D, E)$ be a relation scheme on domains U_1, U_2, U_3, U_4, U_5, where A is an attribute on the universe of discourse U_1,..., E is an attribute on the universe of discourse U_5. Suppose that the following vague functional dependencies on $\{A, B, C, D, E\}$ hold true.

$$\{A, B\} \overset{\theta_1}{\to}_V C,$$
$$B \overset{\theta_2}{\to}_V D,$$
$$\{C, D\} \overset{\theta_3}{\to}_V E.$$

Then, the vague functional dependency $\{A, B\} \overset{\theta}{\to}_V E$ on $\{A, B, C, D, E\}$ holds also true. Here, $\theta = \min\{\theta_1, \theta_2, \theta_3\}$.

Proof. I We may apply the inference rules VF1-VF7. We obtain:

1) $B \overset{\theta_2}{\to}_V D$ (input)
2) $\{A, B\} \overset{\theta_2}{\to}_V \{A, D\}$ (from 1) and VF3)
3) $\{A, D\} \to_V D$ (from $\{A, D\} \supseteq D$ and VF2)
4) $\{A, D\} \overset{\theta}{\to}_V D$ (from 3) and VF1)
5) $\{A, B\} \overset{\theta}{\to}_V D$ (from 2), 4) and VF4)
6) $\{A, B\} \overset{\theta_1}{\to}_V C$ (input)
7) $\{A, B\} \overset{\theta}{\to}_V \{C, D\}$ (from 5), 6) and VF5)
8) $\{C, D\} \overset{\theta_3}{\to}_V E$ (input)
9) $\{A, B\} \overset{\theta}{\to}_V E$ (from 7), 8) and VF4)

□

Proof. II We may apply Theorem 4 in [17, p. 262].

Note that the condition (a) of Theorem 4 actually states that the dependency $\{A, B\} \overset{\theta}{\to}_V E$ follows from the set

$$C = \left\{\{A, B\} \overset{\theta_1}{\to}_V C, B \overset{\theta_2}{\to}_V D, \{C, D\} \overset{\theta_3}{\to}_V E\right\}$$

of given vague functional dependencies.

Since the conditions (a) and (b) of Theorem 4 are equivalent, it is enough to prove that the condition (b) is satisfied.

As it is usual, we apply the resolution principle.

Suppose that

$$i_{r,\beta}(\mathcal{K}_1) = i_{r,\beta}((A \wedge B) \Rightarrow C) > \frac{1}{2},$$
$$i_{r,\beta}(\mathcal{K}_2) = i_{r,\beta}(B \Rightarrow D) > \frac{1}{2},$$
$$i_{r,\beta}(\mathcal{K}_3) = i_{r,\beta}((C \wedge D) \Rightarrow E) > \frac{1}{2},$$

where r is a two-element vague relation instance on $R(A, B, C, D, E)$, and $\beta \in [0, 1]$ is a number.

Our goal is to prove that

$$i_{r,\beta}\left(c^{'}\right) = i_{r,\beta}\left((A \wedge B) \Rightarrow E\right) > \frac{1}{2}.$$

First, we find the conjunctive normal forms of the formulas $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$ and $\neg c^{'}$. This is in line with the resolution principle.

We obtain,

$$\mathcal{K}_1 \equiv \neg A \vee \neg B \vee C,$$
$$\mathcal{K}_2 \equiv \neg B \vee D,$$
$$\mathcal{K}_3 \equiv \neg C \vee \neg D \vee E,$$
$$\neg c^{'} \equiv A \wedge B \wedge \neg E.$$

The set M of all conjunctive terms that appear within conjunctive normal forms of the formulas $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$ and $\neg c^{'}$ is given by

$$M = \Big\{\neg A \vee \neg B \vee C, \neg B \vee D, \neg C \vee \neg D \vee E, A, B, \neg E\Big\}.$$

Applying the resolution principle to the elements of the set M, we obtain

1) $\neg A \vee \neg B \vee C$ (input)

2) A (input)

3) $\neg B \vee C$ (resolvent from 1) and 2))

4) B (input)

5) C (resolvent from 3) and 4))

6) $\neg B \vee D$ (input)

7) D (resolvent from 4) and 6))

8) $\neg C \vee \neg D \vee E$ (input)

9) $\neg D \vee E$ (resolvent from 5) and 8))

10) E (resolvent from 7) and 9))

11) $\neg E$ (input)

Resolving 10) and 11), we conclude that the inequalities: $i_{r,\beta}(\mathcal{K}_1) > \frac{1}{2}$, $i_{r,\beta}(\mathcal{K}_2) > \frac{1}{2}$, $i_{r,\beta}(\mathcal{K}_3) > \frac{1}{2}$ and $i_{r,\beta}\left(c^{'}\right) \leq \frac{1}{2}$ cannot be valid at the same time.

Since, $i_{r,\beta}(\mathcal{K}_1) > \frac{1}{2}$, $i_{r,\beta}(\mathcal{K}_2) > \frac{1}{2}$, $i_{r,\beta}(\mathcal{K}_3) > \frac{1}{2}$, it follows that must be $i_{r,\beta}\left(c^{'}\right) > \frac{1}{2}$.

Thus, the condition (b) of Theorem 4 is satisfied.

Hence, the condition (a) of the same theorem is satisfied.

Therefore, $\{A, B\} \xrightarrow{\theta}_V E$ follows. □

For analogous results in the case of fuzzy functional (and fuzzy multivalued) dependencies, see, [10], [11], [14], [15].

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