New Vague Dependencies as a Result of Automatization

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Abstract: Today, higher output and increased productivity are two of the biggest reasons in justifying the use of automatization. It is involved in each aspect of life and human activity. The same is true of science. In this paper we consider generalized functional and multivalued dependencies, that is, vague functional and vague multivalued dependencies. We consider both types as fuzzy formulas. We provide very strict proof of the equivalence: any two-element vague relation instance on given scheme (which satisfies some set of vague functional and vague multivalued dependencies) satisfies given vague functional or vague multivalued dependency if and only if the joined fuzzy formula is a logical consequence of the corresponding set of fuzzy formulas. This result represents natural continuation and a generalization of our recent study where we were particularly interested in vague functional dependencies. The key role of such results is to encourage automatically checking if some vague dependency (functional or multivalued) follows from some set of vague dependencies (functional and multivalued). An example which includes both kinds of vague dependencies is also given.

Key–Words: Vague dependencies, fuzzy formulas, automatization, Lukasiewicz fuzzy implication

1 Introduction

Let V be a vague set in U, where U is some universe of discourse. We have,

$$V = \{\langle u, [t_V(u), 1 - f_V(u)]\rangle : u \in U\},$$

where $t_V,\ f_V : U \to [0,1]$ are some functions such that $t_V(u) + f_V(u) \leq 1$ for all $u \in U$.

The interval $[t_V(u), 1 - f_V(u)] \subseteq [0,1]$ represents a vague value associated to $u \in U$.

Obviously, if $t_V(u) = 1 - f_V(u) \in (0,1)$, the vague value $[t_V(u), 1 - f_V(u)]$ becomes the fuzzy value $t_V(u)$. In particular, $[t_V(u), 1 - f_V(u)]$ becomes the crisp value 1 if $t_V(u) = 1 - f_V(u) = 1$. Finally, if $t_V(u) = 1 - f_V(u) = 0$, then we assume that u is not an element of the vague set V.

Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in \{1, 2, ..., n\} = I$. Suppose that $V(U_i)$ is the family of all vague sets in U_i, $i \in I$. A vague relation instance r on $R(A_1, A_2, ..., A_n)$ is a subset of the cross product

$$V(U_1) \times V(U_2) \times ... \times V(U_n).$$

Suppose that

$$t[A_i] = \left\{\langle u_i, a^t_{u_i}\rangle : u_i \in U_i\right\},$$

for all $i \in I$, and all tuples
$t = (t[A_1], t[A_2], ..., t[A_n]) \in r$, where r is some fuzzy relation instance on $R(A_1, A_2, ..., A_n)$. More precisely, suppose that $a^t_{u_i} \in [0,1]$ for all $u_i \in U_i$, $i \in I$, and all $t \in r$, where $a^t_{u_i}$ is the membership value of the element $u_i \in U_i$ to the fuzzy set $t[A_i]$. The value $t[A_i]$ of the attribute A_i on the tuple t may be represented as

$$t[A_i] = \left\{\langle u_i, \left[a^t_{u_i}, a^t_{u_i}\right]\rangle : u_i \in U_i\right\}.$$

Hence, the fuzzy relation instance r on $R(A_1, A_2, ..., A_n)$ may be represented as a vague relation instance on $R(A_1, A_2, ..., A_n)$.

Similarly, if

$$t[A_i] = \left\{u^t_i\right\}$$

for all $i \in I$, and all tuples $t \in r$, where u^t_i is some element in U_i, and r is some relation instance on $R(A_1, A_2, ..., A_n)$, then, we may write

$$t[A_i] = \langle u^t_i, [1,1]\rangle \cup \left\{\langle u_i, [0,0]\rangle : u_i \in U_i \setminus \left\{u^t_i\right\}\right\}.$$

Therefore, the relation instance r on $R(A_1, A_2, ..., A_n)$ may also be represented as a vague relation instance on $R(A_1, A_2, ..., A_n)$.

The discussion given above, shows that the vague relational concept generalizes in a natural way both, the classical relational concept as well as the fuzzy relational concept.

For the basic relational concepts, we refer to [19] (see also, [13], [4], [34]).

In [17], we applied the similarity measures defined as follows (see also, [21], [8]-[18], [20], [31], [14]).

Let $x = [a, 1-b] \subseteq [0,1]$ and $y = [c, 1-d] \subseteq [0,1]$ be some vague values. It is not required these values to be associated to the elements of the same universe of discourse.

The similarity measure $SE(x,y)$ between the vague values x and y is given by

$$\begin{aligned} &SE(x,y) \\ =&\sqrt{1-\frac{|(a-c)-(b-d)|}{2}}\cdot \\ &\sqrt{1-|(a-c)+(b-d)|}. \end{aligned}$$

It is known that $SE(x,y) \in [0,1]$, $SE(x,y) = SE(y,x)$, $SE(x,y) = 1$ if and only if $x = y$, and $SE(x,y) = 0$ if and only if $x = [0,0]$, $y = [1,1]$ or $x = [0,1]$, $y = [q,q]$, $0 \leq q \leq 1$.

If

$$\begin{aligned} A &= \{\langle u, [t_A(u), 1-f_A(u)]\rangle \,:\, u \in U\}, \\ B &= \{\langle u, [t_B(u), 1-f_B(u)]\rangle \,:\, u \in U\} \end{aligned}$$

are two vague sets in some universe of discourse U, then, the similarity measure $SE(A,B)$ between the vague sets A and B is introduced accordingly.

It is easily deduced that $SE(A,B) \in [0,1]$, $SE(A,B) = SE(B,A)$, $SE(A,B) = 1$ if and only if $A = B$, and $SE(A,B) = 0$ if and only if $[t_A(u), 1-f_A(u)] = [0,0]$, $[t_B(u), 1-f_B(u)] = [1,1]$ for all $u \in U$ or $[t_A(u), 1-f_A(u)] = [0,1]$, $[t_B(u), 1-f_B(u)] = [q,q]$, for all $u \in U$, where $0 \leq q \leq 1$.

The equality $A = B$ means that $A \subseteq B$ and $B \subseteq A$, where A is contained in B, i.e., $A \subseteq B$ holds true, if and only if $t_A(u) \leq t_B(u)$ and $1 - f_A(u) \leq 1 - f_B(u)$ for all $u \in U$.

So, $A = B$ if and only if $t_A(u) = t_B(u)$, $f_A(u) = f_B(u)$ for all $u \in U$.

Finally, if $R(A_1, A_2, ..., A_n)$ is a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$, r is a vague relation instance on $R(A_1, A_2, ..., A_n)$, t_1 and t_2 are any two tuples in r, and $X \subseteq \{A_1, A_2, ..., A_n\}$ is a set of attributes, then, the similarity measure $SE_X(t_1, t_2)$ between the tuples t_1 and t_2 on the attribute set X is given by

$$SE_X(t_1, t_2) = \min_{A \in X} \{SE(t_1[A], t_2[A])\}.$$

It is also easily deduced that $SE_X(t_1, t_2) \in [0,1]$, $SE_X(t_1, t_2) = SE_X(t_2, t_1)$, $SE_X(t_1, t_2) = 1$ if and only if $t_1[A_k] = t_2[A_k]$ for all $A_k \in X$ if and only if $t_{t_1[A_k]}(u) = t_{t_2[A_k]}(u)$, $f_{t_1[A_k]}(u) = f_{t_2[A_k]}(u)$ for all $u \in U_k$, and all $A_k \in X$, and $SE_X(t_1, t_2) = 0$ if and only if there exists $A_k \in X$ such that $\left[t_{t_1[A_k]}(u), 1 - f_{t_1[A_k]}(u)\right] = [0,0]$, $\left[t_{t_2[A_k]}(u), 1 - f_{t_2[A_k]}(u)\right] = [1,1]$ for all $u \in U_k$ or $\left[t_{t_1[A_k]}(u), 1 - f_{t_1[A_k]}(u)\right] = [0,1]$, $\left[t_{t_2[A_k]}(u), 1 - f_{t_2[A_k]}(u)\right] = [q,q]$ for all $u \in U_k$, where $0 \leq q \leq 1$.

In [17], we have proved the assertions:

1) $SE_Y(t_1, t_2) \geq SE_X(t_1, t_2)$ for t_1, t_2 in r, if $Y \subseteq X \subseteq \{A_1, A_2, ..., A_n\}$,

2) $SE_X(t_1, t_2) \geq \theta$ if $SE(t_1[A], t_2[A]) \geq \theta$ for all $A \in X$,

3) $SE_X(t_1, t_2) \geq \theta$ and $SE_X(t_2, t_3) \geq \theta$ do not necessarily imply that $SE_X(t_1, t_3) \geq \theta$, where t_1, t_2 and t_3 are some mutually distinct tuples in r.

In this paper we introduce the similarity measures in the following way.

Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$.

Denote by $Vag(U_i)$ the set of all vague values associated to the elements $u_i \in U_i$, $i \in I$.

A similarity measure on $Vag(U_i)$ is a mapping $SE_i : Vag(U_i) \times Vag(U_i) \to [0,1]$, such that $SE_i(x,x) = 1$, $SE_i(x,y) = SE_i(y,x)$, and $SE_i(x,z) \geq \max_{y \in Vag(U_i)} (\min(SE_i(x,y), SE_i(y,z)))$ for all x, y, $z \in Vag(U_i)$.

Suppose that SE_i is a similarity measure on $Vag(U_i)$, $i \in I$.

Let

$$\begin{aligned} A_i &= \{\langle u, [t_{A_i}(u), 1-f_{A_i}(u)]\rangle \,:\, u \in U_i\} \\ &= \{a_u^i \,:\, u \in U_i\}, \\ B_i &= \{\langle u, [t_{B_i}(u), 1-f_{B_i}(u)]\rangle \,:\, u \in U_i\} \\ &= \{b_u^i \,:\, u \in U_i\} \end{aligned}$$

be two vague sets in U_i.

The similarity measure $SE(A_i, B_i)$ between the vague sets A_i and B_i is given by

$$SE(A_i, B_i)$$

$$= \min \left\{ \min_{a_u^i \in A_i} \left\{ \max_{b_u^i \in B_i} \left\{ SE_i\Big(\left[t_{A_i}(u), 1 - f_{A_i}(u)\right], \left[t_{B_i}(u), 1 - f_{B_i}(u)\right] \Big) \right\} \right\}, \min_{b_u^i \in B_i} \left\{ \max_{a_u^i \in A_i} \left\{ SE_i\Big(\left[t_{B_i}(u), 1 - f_{B_i}(u)\right], \left[t_{A_i}(u), 1 - f_{A_i}(u)\right] \Big) \right\} \right\} \right\}.$$

Now, if r is a vague relation instance on $R(A_1, A_2, ..., A_n)$, t_1 and t_2 are any two tuples in r, and X is a subset of $\{A_1, A_2, ..., A_n\}$, then, the similarity measure $SE_X(t_1, t_2)$ between the tuples t_1 and t_2 on the attribute set X has the same form as before, i.e.,

$$SE_X(t_1, t_2) = \min_{A \in X} \{SE(t_1[A], t_2[A])\}.$$

Note that the assertions 1) and 2) remain valid if we take them with respect to new similarity measures. Namely, it is obvious that the proofs of these assertions do not depend on the choice of function $SE : V(U_i) \times V(U_i) \to [0, 1]$ (see, [17]).

The assertion 3), however, does not hold anymore. In particular, the following assertion holds true.

Lemma 1. *Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$. Let r be a vague relation instance on $R(A_1, A_2, ..., A_n)$. If $SE_X(t_i, t_j) \geq \theta$ and $SE_X(t_j, t_k) \geq \theta$, where t_i, t_j and t_k are any three tuples in r, and X is a subset of $\{A_1, A_2, ..., A_n\}$, then $SE_X(t_i, t_k) \geq \theta$.*

2 Vague multivalued dependencies

Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$. Suppose that r is a relation instance on $R(A_1, A_2, ..., A_n)$. Furthermore, let X and Y be subsets of $\{A_1, A_2, ..., A_n\}$, and $Z = \{A_1, A_2, ..., A_n\} \setminus (X \cup Y)$.

Relation instance r is said to satisfy the multivalued dependency $X \to\to Y$, if for every pair of tuples t_1 and t_2 in r, $t_1[X] = t_2[X]$ implies that there exists a tuple t_3 in r, such that $t_3[X] = t_1[X]$, $t_3[Y] = t_1[Y]$, and $t_3[Z] = t_2[Z]$.

Note the following facts:

Multivalued dependencies are introduced by Fagin [12],

$t_1[X] = t_2[X]$ means that $t_1[A] = t_2[A]$ for all $A \in X$,

$t[A_i] \in U_i$ for all $i \in I$, and all $t \in r$,

there exists the identity relation $i_j : U_j \times U_j \to \{0, 1\}$, $j \in I$, such that $i_j(t_k[A_j], t_l[A_j]) = 1$ if and only if $t_k[A_j] = t_l[A_j]$, and $i_j(t_k[A_j], t_l[A_j]) = 0$ if and only if $t_k[A_j] \neq t_l[A_j]$, where $t_k, t_l \in r$.

If we put $\emptyset \neq t[A_i] \subseteq U_i$ for all $i \in I$, and all $t \in r$, then the relation instance r becomes a fuzzy relation instance on $R(A_1, A_2, ..., A_n)$. In this setting we are able to determine how similar (or how conformant) $t_1[X]$ and $t_2[X]$ are. More precisely, we calculate the conformance $\varphi(X[t_1, t_2])$ of the attribute set X on tuples t_1 and t_2 as

$$\varphi(X[t_1, t_2]) = \min_{A_k \in X} \{\varphi(A_k[t_1, t_2])\},$$

where the conformance $\varphi(A_k[t_1, t_2])$ of the attribute A_k on tuples t_1 and t_2 is given by

$$\varphi(A_k[t_1, t_2]) = \min \left\{ \min_{x \in t_1[A_k]} \left\{ \max_{y \in t_2[A_k]} \{s_k(x, y)\} \right\}, \min_{x \in t_2[A_k]} \left\{ \max_{y \in t_1[A_k]} \{s_k(x, y)\} \right\} \right\}.$$

Here, $s_k : U_k \times U_k \to [0, 1]$ is a similarity relation on U_k, $k \in I$, i.e., $s_k(x, x) = 1$, $s_k(x, y) = s_k(y, x)$, and $s_k(x, z) \geq \max_{y \in U_k}(\min(s_k(x, y), s_k(y, z)))$ for all x, y, $z \in U_k$.

For the similarity-based fuzzy relational database approach, see, [5]-[7].

Now, it would be natural to state that some fuzzy relation instance r on $R(A_1, A_2, ..., A_n)$ satisfies the fuzzy multivalued dependency $X \to\to_F Y$, if for every pair of tuples t_1 and t_2 in r, there exists a tuple t_3 in r, such that

$$\begin{aligned} \varphi(X[t_3, t_1]) &\geq \varphi(X[t_1, t_2]), \\ \varphi(Y[t_3, t_1]) &\geq \varphi(X[t_1, t_2]), \\ \varphi(Z[t_3, t_2]) &\geq \varphi(X[t_1, t_2]). \end{aligned} \tag{1}$$

However, it is not so hard to select both, a fuzzy relation instance r on $R(A_1, A_2, ..., A_n)$, and a fuzzy multivalued dependency $X \to\to_F Y$, X, $Y \subseteq \{A_1, A_2, ..., A_n\}$, such that r satisfies $X \to\to_F Y$ in reality, and $X \to\to_F Y$ makes perfectly sense by itself, but there are tuples t_1 and t_2 in r, such that (1) fails for every t_3 in r. In other words, the scenario where some of the elements $\varphi(X[t_3, t_1])$, $\varphi(Y[t_3, t_1])$ and $\varphi(Z[t_3, t_2])$ are slightly smaller than $\varphi(X[t_1, t_2])$ for all $t_3 \in r$, may occur. Therefore, the condition (1) is not adequate for determining if some fuzzy relation instance satisfies some fuzzy multivalued dependency. In particular, if (1) holds true, the instance r satisfies $X \to\to_F Y$. Otherwise, r may or may not satisfy $X \to\to_F Y$.

Note that several authors, including Tripathy-Saxena [33] and Nakata [26], have been taken attempts in order to express the fuzzy multivalued dependencies in various fuzzy relational database models.

Sozat and Yazici [30], adapted (1) in the following way.

Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$. Suppose that r is a fuzzy relation instance on $R(A_1, A_2, ..., A_n)$. Furthermore, let X and Y be subsets of $\{A_1, A_2, ..., A_n\}$, $Z = \{A_1, A_2, ..., A_n\} \setminus (X \cup Y)$, and $\theta \in [0, 1]$. Fuzzy relation instance r is said to satisfy the fuzzy multivalued dependency $X \to\overset{\theta}{\to}_F Y$, if for every pair of tuples t_1 and t_2 in r, there exists a tuple t_3 in r, such that

$$\begin{aligned} &\varphi (X [t_3, t_1]) \geq \min (\theta, \varphi (X [t_1, t_2])), \\ &\varphi (Y [t_3, t_1]) \geq \min (\theta, \varphi (X [t_1, t_2])), \\ &\varphi (Z [t_3, t_2]) \geq \min (\theta, \varphi (X [t_1, t_2])). \end{aligned} \tag{2}$$

Thus, if it happens that for some $t_1^{'}$ and $t_2^{'}$ in r, some of the elements $\varphi\left(X\left[t_3, t_1^{'}\right]\right)$, $\varphi\left(Y\left[t_3, t_1^{'}\right]\right)$ and $\varphi\left(Z\left[t_3, t_2^{'}\right]\right)$ are smaller than $\varphi\left(X\left[t_1^{'}, t_2^{'}\right]\right)$, and the elements in

$$\left\{\varphi\left(X\left[t_3, t_1^{'}\right]\right), \varphi\left(Y\left[t_3, t_1^{'}\right]\right), \varphi\left(Z\left[t_3, t_2^{'}\right]\right)\right\}$$

that are smaller than $\varphi\left(X\left[t_1^{'}, t_2^{'}\right]\right)$ are larger than θ for all $t_3 \in r$, then, the condition (2) will be fulfilled, so the instance r will satisfy $X \to\overset{\theta}{\to}_F Y$ (assuming that (2) is fulfilled for $(t_1, t_2) \in r \times r$, $(t_1, t_2) \neq \left(t_1^{'}, t_2^{'}\right)$).

The value $\theta \in [0, 1]$ that appears in the notation $X \to\overset{\theta}{\to}_F Y$ is called the linguistic strength of the fuzzy multivalued dependency. If $\theta = 1$, the fuzzy multivalued dependency $X \to\overset{\theta}{\to}_F Y$ becomes $X \to\to_F Y$.

Now, reasoning as in the fuzzy case, we first state that some vague relation instance r on $R(A_1, A_2, ..., A_n)$ satisfies the vague multivalued dependency $X \to\to_V Y$, if for every pair of tuples t_1 and t_2 in r, there exists a tuple t_3 in r, such that

$$\begin{aligned} &SE_X (t_3, t_1) \geq SE_X (t_1, t_2), \\ &SE_Y (t_3, t_1) \geq SE_X (t_1, t_2), \\ &SE_Z (t_3, t_2) \geq SE_X (t_1, t_2). \end{aligned} \tag{3}$$

Then, we adapt (3) to the following form.

Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$. Suppose that r is a vague relation instance on $R(A_1, A_2, ..., A_n)$. Furthermore, let X and Y be subsets of $\{A_1, A_2, ..., A_n\}$, $Z = \{A_1, A_2, ..., A_n\} \setminus (X \cup Y)$, and $\theta \in [0, 1]$. Vague relation instance r is said to satisfy the vague multivalued dependency $X \to\overset{\theta}{\to}_V Y$, if for every pair of tuples t_1 and t_2 in r, there exists a tuple t_3 in r, such that

$$\begin{aligned} &SE_X (t_3, t_1) \geq \min (\theta, SE_X (t_1, t_2)), \\ &SE_Y (t_3, t_1) \geq \min (\theta, SE_X (t_1, t_2)), \\ &SE_Z (t_3, t_2) \geq \min (\theta, SE_X (t_1, t_2)). \end{aligned}$$

If $\theta = 1$, the vague multivalued dependency $X \to\overset{\theta}{\to}_V Y$ becomes $X \to\to_V Y$.

For yet another definition of vague multivalued dependency, called α-vague multivalued dependency, see [25].

Note that by [17], r satisfies the vague functional dependency $X \overset{\theta}{\to}_V Y$, if for every pair of tuples t_1 and t_2 in r,

$$SE_Y (t_1, t_2) \geq \min (\theta, SE_X (t_1, t_2)).$$

$X \overset{\theta}{\to}_V Y$ becomes $X \to_V Y$ if $\theta = 1$.

3 Soundness of inference rules for vague multivalued dependencies

The following rules are the inference rules for vague multivalued dependencies (VMVDs).

VM1 Inclusive rule for VMVDs: If $X \to\overset{\theta_1}{\to}_V Y$ holds, and $\theta_1 \geq \theta_2$, then $X \to\overset{\theta_2}{\to}_V Y$ holds.

VM2 Complementation rule for VMVDs: If $X \to\overset{\theta}{\to}_V Y$ holds, then $X \to\overset{\theta}{\to}_V Q$ holds, where $Q = \{A_1, A_2, ..., A_n\} \setminus (X \cup Y)$.

VM3 Augmentation rule for VMVDs: If $X \to\overset{\theta}{\to}_V Y$ holds, and $W \supseteq Z$, then $W \cup X \to\overset{\theta}{\to}_V Y \cup Z$ also holds.

VM4 Transitivity rule for VMVDs: If $X \to\overset{\theta_1}{\to}_V Y$ and $Y \to\overset{\theta_2}{\to}_V Z$ hold true, then $X \overset{\min(\theta_1,\theta_2)}{\to\to}_V Z \setminus Y$ holds true.

VM5 Replication rule: If $X \overset{\theta}{\to}_V Y$ holds, then $X \to\overset{\theta}{\to}_V Y$ holds.

VM6 Coalescence rule for VFDs and VMVDs: If $X \to\xrightarrow{\theta_1}_V Y$ holds, $Z \subseteq Y$, and for some W disjoint from Y, we have that $W \xrightarrow{\theta_2}_V Z$ holds true, then $X \overset{\min(\theta_1,\theta_2)}{\rightarrow}_V Z$ also holds true.

In [17], we listed the inference rules for vague functional dependencies. There, we proved that the rules are sound, and that the set of these rules, i.e., the set $\{VF1, VF2, VF3, VF4\}$ is complete set. Additional inference rules (labeled as $VF5$, $VF6$ and $VF7$) are also proved to be sound.

Since the proofs of the corresponding theorems in [17] do not depend on the choice of similarity measures between vague values and vagues sets, these theorems remain valid in the present setting, i.e., for the choice: SE_i, $i \in I$, SE, and SE_X.

Theorem 2. *The inference rules: VM1, VM2, VM3, VM4, VM5 and VM6 are sound.*

4 Soundness of additional inference rules for vague multivalued dependencies

The following inference rules are additional inference rules for vague multivalued dependencies.

VM7 Union rule for VMVDs: If $X \to\xrightarrow{\theta_1}_V Y$ and $X \to\xrightarrow{\theta_2}_V Z$ hold true, then $X \overset{\min(\theta_1,\theta_2)}{\to\to}_V Y \cup Z$ holds true.

VM8 Pseudo-transitivity rule for VMVDs: If $X \to\xrightarrow{\theta_1}_V Y$ and $W \cup Y \to\xrightarrow{\theta_2}_V Z$ hold true, then $W \cup X \overset{\min(\theta_1,\theta_2)}{\to\to}_V Z \setminus (W \cup Y)$ holds also true.

VM9 Decomposition rule for VMVDs: If $X \to\xrightarrow{\theta_1}_V Y$ and $X \to\xrightarrow{\theta_2}_V Z$ hold true, then $X \overset{\min(\theta_1,\theta_2)}{\to\to}_V Y \cap Z$, $X \overset{\min(\theta_1,\theta_2)}{\to\to}_V Y \setminus Z$, and $X \overset{\min(\theta_1,\theta_2)}{\to\to}_V Z \setminus Y$ hold also true.

VM10 Mixed pseudo-transitivity rule: If $X \to\xrightarrow{\theta_1}_V Y$ and $X \cup Y \xrightarrow{\theta_2}_V Z$ hold true, then $X \overset{\min(\theta_1,\theta_2)}{\rightarrow}_V Z \setminus Y$ holds true.

Theorem 3. *The inference rules: VM7, VM8, VM9 and VM10 are sound.*

5 Main result

For various definitions of similarity measures, see, [21], [8], [9], [18] and [20].

For various definitions of vague functional and vague multivalued dependencies, see, [21], [24], [35] and [25].

The most important classes of fuzzy implications are: S-implications, R-implications and QL-implications.

For precise definitions and description of S-, R-, QL-implications, as well as for the definitions of various additional fuzzy implications, see, [29] and [3].

In this paper we use the following operators:

$$\begin{aligned} T_M(x,y) &= \min\{x,y\}, \\ S_M(x,y) &= \max\{x,y\}, \\ I_L(x,y) &= \min\{1-x+y,1\}, \end{aligned}$$

where T_M is the minimum t-norm, S_M is the maximum t-co-norm, and I_L is the Lukasiewicz fuzzy implication.

The Lukasiewics fuzzy implication is an S-, an R-, and a QL-fuzzy implication at the same time (see, [29], [3]).

Some of the works that deal with S-, R-, and QL-implications are the following ones: [1], [2], [22], [32], [28], [23], [27].

Let $r = \{t_1, t_2\}$ be a two-element vague relation instance on $R(A_1, A_2, ..., A_n)$, and $\beta \in [0,1]$ be a number.

We say that $i_{r,\beta}$ is a valuation joined to r and β if $i_{r,\beta} : \{A_1, A_2, ..., A_n\} \to [0,1]$, and

$$\begin{aligned} i_{r,\beta}(A_k) &> \frac{1}{2} \quad \text{if} \quad SE(t_1[A_k], t_2[A_k]) \geq \beta, \\ i_{r,\beta}(A_k) &\leq \frac{1}{2} \quad \text{if} \quad SE(t_1[A_k], t_2[A_k]) < \beta, \end{aligned}$$

$k \in \{1, 2, ..., n\}$.

Note that the fact that $i_{r,\beta}(A_k) \in [0,1]$ for $k \in \{1, 2, ..., n\}$ yields that the attributes A_k, $k \in \{1, 2, ..., n\}$ are actually fuzzy formulas with respect to $i_{r,\beta}$.

Through the rest of the paper we shall assume that each time some $r = \{t_1, t_2\}$ and some $\beta \in [0,1]$ are given, the fuzzy formula

$$(\wedge_{A \in X} A) \Rightarrow (\wedge_{B \in Y} B)$$

resp.

$$(\wedge_{A \in X} A) \Rightarrow ((\wedge_{B \in Y} B) \vee (\wedge_{C \in Z} C))$$

with respect to $i_{r,\beta}$ is joined to $X \overset{\theta}{\rightarrow}_V Y$ resp. $X \overset{\theta}{\rightarrow\rightarrow}_V Y$, where $X \overset{\theta}{\rightarrow}_V Y$ resp. $X \overset{\theta}{\rightarrow\rightarrow}_V Y$ is a vague functional resp. vague multivalued dependency on $\{A_1, A_2, ..., A_n\}$, and $Z = \{A_1, A_2, ..., A_n\} \setminus (X \cup Y)$.

Theorem 4. *Let $R(A_1, A_2, ..., A_n)$ be a relation scheme on domains U_1, U_2,..., U_n, where A_i is an attribute on the universe of discourse U_i, $i \in I$. Let C be some set of vague functional and vague multivalued dependencies on $\{A_1, A_2, ..., A_n\}$. Suppose that $X \overset{\theta}{\rightarrow}_V Y$ resp. $X \overset{\theta}{\rightarrow\rightarrow}_V Y$ is some vague functional resp. vague multivalued dependency on $\{A_1, A_2, ..., A_n\}$. The following two conditions are equivalent:*

(a) Any two-element vague relation instance on $R(A_1, A_2, ..., A_n)$ which satisfies all dependencies in C, satisfies the dependency $X \overset{\theta}{\rightarrow}_V Y$ resp. $X \overset{\theta}{\rightarrow\rightarrow}_V Y$.

(b) Let r be any two-element vague relation instance on $R(A_1, A_2, ..., A_n)$, and $\beta \in [0,1]$. Suppose that $i_{r,\beta}(\mathcal{K}) > \frac{1}{2}$ for all $\mathcal{K} \in C'$, where C' is the set of fuzzy formulas with respect to $i_{r,\beta}$, joined to the elements of C. Then,

$$i_{r,\beta}\left((\wedge_{A\in X} A) \Rightarrow (\wedge_{B\in Y} B)\right) > \frac{1}{2}$$

resp.

$$i_{r,\beta}\left((\wedge_{A\in X} A) \Rightarrow ((\wedge_{B\in Y} B) \vee (\wedge_{C\in Z} C))\right) > \frac{1}{2},$$

where $Z = \{A_1, A_2, ..., A_n\} \setminus (X \cup Y)$.

Proof. Suppose that $U_1 = U_2 = ... = U_n = \{u\} = U$.

Put

$$\theta' = \min\{\theta, \theta_C\},$$

where

$$\theta_C = \min_{K \overset{1\theta}{\rightarrow}_V L \in C, K \overset{1\theta}{\rightarrow\rightarrow}_V L \in C} \{{}_1\theta\}.$$

We may assume that $\theta' < 1$.

Namely, if $\theta' = 1$, then $\theta = 1$ and $\theta_C = 1$

This means that $\theta = 1$, and ${}_1\theta = 1$ for all $K \overset{1\theta}{\rightarrow}_V L \in C$ and all $K \overset{1\theta}{\rightarrow\rightarrow}_V L \in C$

This case, however, is not interesting.

Choose some $\theta'' < \theta'$.

Put

$$\begin{aligned} V_1 &= \{\langle u, [t_{V_1}(u), 1 - f_{V_1}(u)]\rangle \ : \ u \in U\} \\ &= \{\langle u, [t_{V_1}(u), 1 - f_{V_1}(u)]\rangle\} = \{\langle u, a\rangle\}, \\ V_2 &= \{\langle u, [t_{V_2}(u), 1 - f_{V_2}(u)]\rangle \ : \ u \in U\} \\ &= \{\langle u, [t_{V_2}(u), 1 - f_{V_2}(u)]\rangle\} = \{\langle u, b\rangle\} \end{aligned}$$

to be two vague sets in U, such that

$$SE_U(a, b) = \theta''$$

where $SE_U : Vag(U) \times Vag(U) \to [0,1]$, is a similarity measure on $Vag(U) = \{a, b\}$.

It follows that

$$\begin{aligned} &SE(V_1, V_2) \\ &= \min\Big\{ \min_{\langle u,a\rangle \in V_1} \Big\{ \max_{\langle u,b\rangle \in V_2} \Big\{ SE_U\Big(a, b\Big)\Big\}\Big\}, \\ &\min_{\langle u,b\rangle \in V_2} \Big\{ \max_{\langle u,a\rangle \in V_1} \Big\{ SE_U\Big(b, a\Big)\Big\}\Big\}\Big\} \\ &= \min\Big\{\theta'', \theta''\Big\} = \theta''. \end{aligned}$$

Similarly, $SE(V_1, V_1) = SE(V_2, V_2) = 1$.

Since $Vag(U) = \{a, b\}$, it follows that

$$\begin{aligned} &SE(A, B) \\ &= \min\Big\{ \min_{\langle u,x\rangle \in A} \Big\{ \max_{\langle u,y\rangle \in B} \Big\{ SE_U\Big(x, y\Big)\Big\}\Big\}, \\ &\min_{\langle u,y\rangle \in B} \Big\{ \max_{\langle u,x\rangle \in A} \Big\{ SE_U\Big(y, x\Big)\Big\}\Big\}\Big\} \\ &\geq \min\Big\{\theta'', \theta''\Big\} = \theta'' \end{aligned}$$

for any two vague sets $A = \{\langle u, x\rangle\}$ and $B = \{\langle u, y\rangle\}$ in U.

$(a) \Rightarrow (b)$ Suppose that (a) holds true.

Furthermore, suppose that (b) does not hold true.

Now, there is some two-element vague relation instance r on $R(A_1, A_2, ..., A_n)$, and $\beta \in [0,1]$, such that

$$i_{r,\beta}\left((\wedge_{A\in K} A) \Rightarrow (\wedge_{B\in L} B)\right) > \frac{1}{2},$$

$$i_{r,\beta}\left((\wedge_{A\in K} A) \Rightarrow ((\wedge_{B\in L} B) \vee (\wedge_{C\in M} C))\right) > \frac{1}{2}$$

for all

$$(\wedge_{A\in K}A)\Rightarrow(\wedge_{B\in L}B)\in C',$$

$$(\wedge_{A\in K}A)\Rightarrow((\wedge_{B\in L}B)\vee(\wedge_{C\in M}C))\in C',$$

$M=\{A_1,A_2,...,A_n\}\setminus(K\cup L)$, and

$$i_{r,\beta}((\wedge_{A\in X}A)\Rightarrow(\wedge_{B\in Y}B))\leq\frac{1}{2}$$

resp.

$$i_{r,\beta}((\wedge_{A\in X}A)\Rightarrow((\wedge_{B\in Y}B)\vee(\wedge_{C\in Z}C)))\leq\frac{1}{2}.$$

Put

$$W=\left\{A\in\{A_1,A_2,...,A_n\}\ :\ i_{r,\beta}(A)>\frac{1}{2}\right\}.$$

Suppose that $W=\emptyset$.
Then, $i_{r,\beta}(A)\leq\frac{1}{2}$ for all $A\in\{A_1,A_2,...,A_n\}$.
Consequently,

$$\begin{aligned}&i_{r,\beta}(\wedge_{A\in M}A)\\=&\min\{i_{r,\beta}(A)\ :\ A\in M\}\leq\frac{1}{2}\end{aligned}$$

for all $M\subseteq\{A_1,A_2,...,A_n\}$.
Since

$$i_{r,\beta}((\wedge_{A\in X}A)\Rightarrow(\wedge_{B\in Y}B))\leq\frac{1}{2}$$

resp.

$$i_{r,\beta}((\wedge_{A\in X}A)\Rightarrow((\wedge_{B\in Y}B)\vee(\wedge_{C\in Z}C)))\leq\frac{1}{2}$$

it follows that

$$\begin{aligned}&\min\{1-i_{r,\beta}(\wedge_{A\in X}A)+i_{r,\beta}(\wedge_{B\in Y}B),1\}\\\leq&\frac{1}{2}\end{aligned}$$

resp.

$$\begin{aligned}&\min\Big\{1-i_{r,\beta}(\wedge_{A\in X}A)+\\&\max\{i_{r,\beta}(\wedge_{B\in Y}B),i_{r,\beta}(\wedge_{C\in Z}C)\},1\Big\}\\=&\min\Big\{1-i_{r,\beta}(\wedge_{A\in X}A)+\\&i_{r,\beta}((\wedge_{B\in Y}B)\vee(\wedge_{C\in Z}C)),1\Big\}\\\leq&\frac{1}{2}.\end{aligned}$$

If

$$\min\{1-i_{r,\beta}(\wedge_{A\in X}A)+i_{r,\beta}(\wedge_{B\in Y}B),1\}=1$$

resp.

$$\begin{aligned}&\min\Big\{1-i_{r,\beta}(\wedge_{A\in X}A)+\\&\max\{i_{r,\beta}(\wedge_{B\in Y}B),i_{r,\beta}(\wedge_{C\in Z}C)\},1\Big\}=1,\end{aligned}$$

then, $1\leq\frac{1}{2}$. This is a contradiction.
Hence,

$$\frac{1}{2}+i_{r,\beta}(\wedge_{B\in Y}B)\leq i_{r,\beta}(\wedge_{A\in X}A)$$

resp.

$$\begin{aligned}&\frac{1}{2}+\max\{i_{r,\beta}(\wedge_{B\in Y}B),i_{r,\beta}(\wedge_{C\in Z}C)\}\\\leq&i_{r,\beta}(\wedge_{A\in X}A).\end{aligned}$$

Since $i_{r,\beta}(\wedge_{B\in Y}B)\geq 0$ resp.

$$\max\{i_{r,\beta}(\wedge_{B\in Y}B),i_{r,\beta}(\wedge_{C\in Z}C)\}\geq 0,$$

it follows that $i_{r,\beta}(\wedge_{A\in X}A)\geq\frac{1}{2}$.
This contradicts the fact that $i_{r,\beta}(\wedge_{A\in X}A)\leq\frac{1}{2}$.
Therefore, $W\neq\emptyset$.
Suppose that $W=\{A_1,A_2,...,A_n\}$.
It follows that $i_{r,\beta}(A)>\frac{1}{2}$ for all $A\in\{A_1,A_2,...,A_n\}$.
Hence, $i_{r,\beta}(\wedge_{A\in M}A)>\frac{1}{2}$ for all $M\subseteq\{A_1,A_2,...,A_n\}$.
Since

$$i_{r,\beta}((\wedge_{A\in X}A)\Rightarrow(\wedge_{B\in Y}B))\leq\frac{1}{2}$$

resp.

$$i_{r,\beta}((\wedge_{A\in X}A)\Rightarrow((\wedge_{B\in Y}B)\vee(\wedge_{C\in Z}C)))\leq\frac{1}{2},$$

we have that

$$\frac{1}{2}+i_{r,\beta}(\wedge_{B\in Y}B)\leq i_{r,\beta}(\wedge_{A\in X}A)$$

resp.

$$\begin{aligned}&\frac{1}{2}+\max\{i_{r,\beta}(\wedge_{B\in Y}B),i_{r,\beta}(\wedge_{C\in Z}C)\}\\\leq&i_{r,\beta}(\wedge_{A\in X}A).\end{aligned}$$

Now, $i_{r,\beta}(\wedge_{B\in Y}B) > \frac{1}{2}$ resp.

$$\max\{i_{r,\beta}(\wedge_{B\in Y}B), i_{r,\beta}(\wedge_{C\in Z}C)\} > \frac{1}{2},$$

yields that $i_{r,\beta}(\wedge_{A\in X}A) > 1$.

This is a contradiction.

We conclude, $W \neq \{A_1, A_2, ..., A_n\}$.

Let $r' = \{t', t''\}$ be the vague relation instance on $R(A_1, A_2, ..., A_n)$ given by Table 1.

Table 1:

	attributes of W	other attributes
t'	$V_1, V_1, ..., V_1$	$V_1, V_1, ..., V_1$
t''	$V_1, V_1, ..., V_1$	$V_2, V_2, ..., V_2$

We shall prove that r' satisfies $K \overset{1\theta}{\rightarrow}_V L$ resp. $K \overset{1\theta}{\rightarrow\rightarrow}_V L$ if $K \overset{1\theta}{\rightarrow}_V L$ resp. $K \overset{1\theta}{\rightarrow\rightarrow}_V L$ belongs to C, and violates $X \overset{\theta}{\rightarrow}_V Y$ resp. $X \overset{\theta}{\rightarrow\rightarrow}_V Y$.

Let $K \overset{1\theta}{\rightarrow}_V L$ belongs to C.

Suppose that $i_{r,\beta}(\wedge_{A\in K}A) \leq \frac{1}{2}$.

It follows that $i_{r,\beta}(A_0) \leq \frac{1}{2}$ for some $A_0 \in K$.

Therefore, $A_0 \notin W$.

Since $A_0 \in K$, we obtain that $SE_K(t', t'') = \theta''$.

By definition of r', we have that $SE_M(t', t'') \geq \theta''$ for all $M \subseteq \{A_1, A_2, ..., A_n\}$.

In particular, $SE_L(t', t'') \geq \theta''$.

Consequently,

$$\begin{aligned} SE_L(t', t'') &\geq \theta'' = \min\{{}_1\theta, \theta''\} \\ &= \min\{{}_1\theta, SE_K(t', t'')\}. \end{aligned}$$

We conclude, r' satisfies $K \overset{1\theta}{\rightarrow}_V L$.

Suppose that $i_{r,\beta}(\wedge_{A\in K}A) > \frac{1}{2}$.

Since

$$i_{r,\beta}((\wedge_{A\in K}A) \Rightarrow (\wedge_{B\in L}B)) > \frac{1}{2},$$

it follows that

$$\min\{1 - i_{r,\beta}(\wedge_{A\in K}A) + i_{r,\beta}(\wedge_{B\in L}B), 1\} > \frac{1}{2}.$$

Suppose that $i_{r,\beta}(\wedge_{B\in L}B) \leq \frac{1}{2}$.

We have,

$$\begin{aligned} &1 - i_{r,\beta}(\wedge_{A\in K}A) + i_{r,\beta}(\wedge_{B\in L}B) \\ &< 1 - \frac{1}{2} + \frac{1}{2} = 1. \end{aligned}$$

Thus,

$$\begin{aligned} &\min\{1 - i_{r,\beta}(\wedge_{A\in K}A) + i_{r,\beta}(\wedge_{B\in L}B), 1\} \\ &= 1 - i_{r,\beta}(\wedge_{A\in K}A) + i_{r,\beta}(\wedge_{B\in L}B). \end{aligned}$$

We obtain,

$$\frac{1}{2} + i_{r,\beta}(\wedge_{B\in L}B) > i_{r,\beta}(\wedge_{A\in K}A).$$

This inequality must always hold true.

Therefore, $i_{r,\beta}(\wedge_{A\in K}A) < \frac{1}{2}$.

This is a contradiction.

Hence, $i_{r,\beta}(\wedge_{B\in L}B) > \frac{1}{2}$.

This immediately implies that $SE_L(t', t'') = 1$.

Consequently,

$$SE_L(t', t'') = 1 \geq \min\{{}_1\theta, SE_K(t', t'')\}.$$

We conclude, r' satisfies $K \overset{1\theta}{\rightarrow}_V L$.

Let $K \overset{1\theta}{\rightarrow\rightarrow}_V L$ belongs to C.

Suppose that $i_{r,\beta}(\wedge_{A\in K}A) \leq \frac{1}{2}$.

We obtain, $SE_K(t', t'') = \theta''$.

Now, there exists $t_3 \in r'$, $t_3 = t'$, such that

$$\begin{aligned} SE_K(t_3, t') &= 1 \geq \min\{{}_1\theta, SE_K(t', t'')\}, \\ SE_L(t_3, t') &= 1 \geq \min\{{}_1\theta, SE_K(t', t'')\}, \\ SE_M(t_3, t'') &\geq \theta'' = \min\{{}_1\theta, \theta''\} \\ &= \min\{{}_1\theta, SE_K(t', t'')\}. \end{aligned}$$

This means that r' satisfies $K \overset{1\theta}{\rightarrow\rightarrow}_V L$.

Suppose that $i_{r,\beta}(\wedge_{A\in K}A) > \frac{1}{2}$.

Since

$$i_{r,\beta}((\wedge_{A\in K}A) \Rightarrow ((\wedge_{B\in L}B) \vee (\wedge_{C\in M}C))) > \frac{1}{2},$$

we have that

$$\min\left\{1 - i_{r,\beta}\left(\wedge_{A\in K}A\right) + \max\left\{i_{r,\beta}\left(\wedge_{B\in L}B\right), i_{r,\beta}\left(\wedge_{C\in M}C\right)\right\}, 1\right\} > \frac{1}{2}.$$

Suppose that

$$\max\left\{i_{r,\beta}\left(\wedge_{B\in L}B\right), i_{r,\beta}\left(\wedge_{C\in M}C\right)\right\} \leq \frac{1}{2}.$$

We obtain,

$$1 - i_{r,\beta}\left(\wedge_{A\in K}A\right) + \max\left\{i_{r,\beta}\left(\wedge_{B\in L}B\right), i_{r,\beta}\left(\wedge_{C\in M}C\right)\right\} < 1 - \frac{1}{2} + \frac{1}{2} = 1.$$

Hence,

$$\min\left\{1 - i_{r,\beta}\left(\wedge_{A\in K}A\right) + \max\left\{i_{r,\beta}\left(\wedge_{B\in L}B\right), i_{r,\beta}\left(\wedge_{C\in M}C\right)\right\}, 1\right\} = 1 - i_{r,\beta}\left(\wedge_{A\in K}A\right) + \max\left\{i_{r,\beta}\left(\wedge_{B\in L}B\right), i_{r,\beta}\left(\wedge_{C\in M}C\right)\right\}.$$

Thus,

$$\frac{1}{2} + \max\left\{i_{r,\beta}\left(\wedge_{B\in L}B\right), i_{r,\beta}\left(\wedge_{C\in M}C\right)\right\} > i_{r,\beta}\left(\wedge_{A\in K}A\right).$$

This inequality must always hold true. Consequently, $i_{r,\beta}\left(\wedge_{A\in K}A\right) < \frac{1}{2}$.

This is a contradiction.

We conclude,

$$\max\left\{i_{r,\beta}\left(\wedge_{B\in L}B\right), i_{r,\beta}\left(\wedge_{C\in M}C\right)\right\} > \frac{1}{2}.$$

This implies that $i_{r,\beta}\left(\wedge_{B\in L}B\right) > \frac{1}{2}$ or $i_{r,\beta}\left(\wedge_{C\in M}C\right) > \frac{1}{2}$.

Suppose that $i_{r,\beta}\left(\wedge_{B\in L}B\right) > \frac{1}{2}$.

Now, $SE_L\left(t', t''\right) = 1$.

Thus, there is $t_3 \in r'$, $t_3 = t''$, such that

$$\begin{aligned} SE_K\left(t_3, t'\right) &= 1 \geq \min\left\{{}_1\theta, SE_K\left(t', t''\right)\right\}, \\ SE_L\left(t_3, t'\right) &= 1 \geq \min\left\{{}_1\theta, SE_K\left(t', t''\right)\right\}, \\ SE_M\left(t_3, t''\right) &= 1 \geq \min\left\{{}_1\theta, SE_K\left(t', t''\right)\right\}. \end{aligned}$$

This means that r' satisfies $K \overset{1\theta}{\rightarrow\!\!\rightarrow}_V L$.

Note that $SE_K\left(t_3, t'\right) = 1$ since $i_{r,\beta}\left(\wedge_{A\in K}A\right) > \frac{1}{2}$.

Suppose that $i_{r,\beta}\left(\wedge_{C\in M}C\right) > \frac{1}{2}$.

In this case, $SE_M\left(t', t''\right) = 1$.

Thus, there is $t_3 \in r'$, $t_3 = t'$, such that

$$\begin{aligned} SE_K\left(t_3, t'\right) &= 1 \geq \min\left\{{}_1\theta, SE_K\left(t', t''\right)\right\}, \\ SE_L\left(t_3, t'\right) &= 1 \geq \min\left\{{}_1\theta, SE_K\left(t', t''\right)\right\}, \\ SE_M\left(t_3, t''\right) &= 1 \geq \min\left\{{}_1\theta, SE_K\left(t', t''\right)\right\}. \end{aligned}$$

Therefore, r' satisfies $K \overset{1\theta}{\rightarrow\!\!\rightarrow}_V L$.

It remains to prove that r' violates $X \overset{\theta}{\rightarrow}_V Y$ resp. $X \overset{\theta}{\rightarrow\!\!\rightarrow}_V Y$.

Suppose that

$$i_{r,\beta}\left(\left(\wedge_{A\in X}A\right) \Rightarrow \left(\wedge_{B\in Y}B\right)\right) \leq \frac{1}{2}.$$

We obtain,

$$\frac{1}{2} + i_{r,\beta}\left(\wedge_{B\in Y}B\right) \leq i_{r,\beta}\left(\wedge_{A\in X}A\right).$$

If $i_{r,\beta}\left(\wedge_{B\in Y}B\right) > \frac{1}{2}$, then $i_{r,\beta}\left(\wedge_{A\in X}A\right) > 1$.

Hence, $i_{r,\beta}\left(\wedge_{B\in Y}B\right) \leq \frac{1}{2}$.

Since

$$\frac{1}{2} + i_{r,\beta}\left(\wedge_{B\in Y}B\right) \leq i_{r,\beta}\left(\wedge_{A\in X}A\right)$$

holds always true, it follows that $i_{r,\beta}\left(\wedge_{A\in X}A\right) = 1$.

Hence, $SE_Y\left(t', t''\right) = \theta''$, $SE_X\left(t', t''\right) = 1$.

Now,

$$\begin{aligned} SE_Y\left(t', t''\right) &= \theta'' < \theta' \leq \theta \\ &= \min\left\{\theta, 1\right\} \\ &= \min\left\{\theta, SE_X\left(t', t''\right)\right\}. \end{aligned}$$

This means that r' violates $X \overset{\theta}{\rightarrow}_V Y$.

Suppose that

$$i_{r,\beta}\left((\wedge_{A\in X}A) \Rightarrow ((\wedge_{B\in Y}B) \vee (\wedge_{C\in Z}C))\right) \leq \frac{1}{2}.$$

It follows that,

$$\frac{1}{2} + \max\left\{i_{r,\beta}\left(\wedge_{B\in Y}B\right), i_{r,\beta}\left(\wedge_{C\in Z}C\right)\right\} \leq i_{r,\beta}\left(\wedge_{A\in X}A\right).$$

If

$$\max\left\{i_{r,\beta}\left(\wedge_{B\in Y}B\right), i_{r,\beta}\left(\wedge_{C\in Z}C\right)\right\} > \frac{1}{2},$$

then $i_{r,\beta}\left(\wedge_{A\in X}A\right) > 1$.

Therefore,

$$\max\left\{i_{r,\beta}\left(\wedge_{B\in Y}B\right), i_{r,\beta}\left(\wedge_{C\in Z}C\right)\right\} \leq \frac{1}{2}.$$

Since

$$\frac{1}{2} + \max\left\{i_{r,\beta}\left(\wedge_{B\in Y}B\right), i_{r,\beta}\left(\wedge_{C\in Z}C\right)\right\} \leq i_{r,\beta}\left(\wedge_{A\in X}A\right)$$

holds always true, it follows that $i_{r,\beta}\left(\wedge_{A\in X}A\right) = 1$.

Furthermore,

$$\max\left\{i_{r,\beta}\left(\wedge_{B\in Y}B\right), i_{r,\beta}\left(\wedge_{C\in Z}C\right)\right\} \leq \frac{1}{2}$$

yields that $i_{r,\beta}\left(\wedge_{B\in Y}B\right) \leq \frac{1}{2}$, $i_{r,\beta}\left(\wedge_{C\in Z}C\right) \leq \frac{1}{2}$.

We obtain, $SE_X\left(t', t''\right) = 1$, $SE_Y\left(t', t''\right) = \theta''$, $SE_Z\left(t', t''\right) = \theta''$.

If $t_3 \in r'$, $t_3 = t'$, then

$$\begin{aligned}
SE_X\left(t_3, t'\right) &= 1 \geq \min\left\{\theta, SE_X\left(t', t''\right)\right\}, \\
SE_Y\left(t_3, t'\right) &= 1 \geq \min\left\{\theta, SE_X\left(t', t''\right)\right\}, \\
SE_Z\left(t_3, t''\right) &= \theta'' < \theta' \leq \theta \\
&= \min\left\{\theta, 1\right\} \\
&= \min\left\{\theta, SE_X\left(t', t''\right)\right\}.
\end{aligned}$$

If $t_3 \in r'$, $t_3 = t''$, then

$$\begin{aligned}
SE_X\left(t_3, t'\right) &= 1 \geq \min\left\{\theta, SE_X\left(t', t''\right)\right\}, \\
SE_Y\left(t_3, t'\right) &= \theta'' < \theta' \leq \theta \\
&= \min\left\{\theta, 1\right\} \\
&= \min\left\{\theta, SE_X\left(t', t''\right)\right\}, \\
SE_Z\left(t_3, t''\right) &= 1 \geq \min\left\{\theta, SE_X\left(t', t''\right)\right\}.
\end{aligned}$$

This means that r' violates $X \overset{\theta}{\rightarrow\rightarrow}_V Y$.

Hence, r' satisfies $K \overset{1\theta}{\rightarrow}_V L$ resp. $K \overset{1\theta}{\rightarrow\rightarrow}_V L$ if $K \overset{1\theta}{\rightarrow}_V L$ resp. $K \overset{1\theta}{\rightarrow\rightarrow}_V L$ belongs to C, and violates $X \overset{\theta}{\rightarrow}_V Y$ resp. $X \overset{\theta}{\rightarrow\rightarrow}_V Y$.

This contradicts the fact that (a) holds true.

Hence, (b) holds true.

$(b) \Rightarrow (a)$ Suppose that (b) holds true.

Furthermore, suppose that (a) does not hold true.

Now, there is some two-element vague relation instance $r' = \left\{t', t''\right\}$ on $R\left(A_1, A_2, ..., A_n\right)$ which satisfies all dependencies in C, and violates $X \overset{\theta}{\rightarrow}_V Y$ resp. $X \overset{\theta}{\rightarrow\rightarrow}_V Y$.

Define

$$W = \left\{A \in \left\{A_1, A_2, ..., A_n\right\} : SE\left(t'[A], t''[A]\right) = 1\right\}.$$

Suppose that $W = \emptyset$.

It follows that $SE\left(t'[A], t''[A]\right) = \theta''$ for all $A \in \left\{A_1, A_2, ..., A_n\right\}$.

Consequently, $SE_M\left(t', t''\right) = \theta''$ for all $M \subseteq \left\{A_1, A_2, ..., A_n\right\}$.

Suppose that r' violates $X \overset{\theta}{\rightarrow}_V Y$.

We obtain,

$$SE_Y\left(t', t''\right) < \min\left\{\theta, SE_X\left(t', t''\right)\right\},$$

i.e.,

$$\theta'' < \min\left\{\theta, \theta''\right\} = \theta''.$$

This is a contradiction.

Suppose that r' violates $X \overset{\theta}{\rightarrow\rightarrow}_V Y$.

It follows that the inequalities

$$SE_X\left(t^{'},t^{'}\right)\geq\min\left\{\theta,SE_X\left(t^{'},t^{''}\right)\right\},$$
$$SE_Y\left(t^{'},t^{'}\right)\geq\min\left\{\theta,SE_X\left(t^{'},t^{''}\right)\right\},$$
$$SE_Z\left(t^{'},t^{''}\right)\geq\min\left\{\theta,SE_X\left(t^{'},t^{''}\right)\right\}$$

don't hold at the same time.

The first and the second inequality hold obviously true. Hence,

$$SE_Z\left(t^{'},t^{''}\right)<\min\left\{\theta,SE_X\left(t^{'},t^{''}\right)\right\},$$

i.e.,

$$\theta^{''}<\min\left\{\theta,\theta^{''}\right\}=\theta^{''}.$$

This is a contradiction.

We conclude, $W\neq\emptyset$.

Suppose that $W=\{A_1,A_2,...,A_n\}$.

It follows that $SE\left(t^{'}[A],t^{''}[A]\right)=1$ for all $A\in\{A_1,A_2,...,A_n\}$.

Consequently, $SE_M\left(t^{'},t^{''}\right)=1$ for all $M\subseteq\{A_1,A_2,...,A_n\}$.

Suppose that $r^{'}$ violates $X\overset{\theta}{\rightarrow}_V Y$.

Then,

$$SE_Y\left(t^{'},t^{''}\right)<\min\left\{\theta,SE_X\left(t^{'},t^{''}\right)\right\},$$

i.e.,

$$1<\min\{\theta,1\}=\theta.$$

This is a contradiction.

Suppose that $r^{'}$ violates $X\overset{\theta}{\rightarrow\rightarrow}_V Y$.

Now, the inequalities

$$SE_X\left(t^{'},t^{'}\right)\geq\min\left\{\theta,SE_X\left(t^{'},t^{''}\right)\right\},$$
$$SE_Y\left(t^{'},t^{'}\right)\geq\min\left\{\theta,SE_X\left(t^{'},t^{''}\right)\right\},$$
$$SE_Z\left(t^{'},t^{''}\right)\geq\min\left\{\theta,SE_X\left(t^{'},t^{''}\right)\right\}$$

don't hold at the same time.

Since the first and the second inequality hold true, we obtain that

$$SE_Z\left(t^{'},t^{''}\right)<\min\left\{\theta,SE_X\left(t^{'},t^{''}\right)\right\},$$

i.e.,

$$1<\min\{\theta,1\}=\theta.$$

This is a contradiction.

We conclude, $W\neq\{A_1,A_2,...,A_n\}$.

Since $r^{'}$ is a two-element vague relation instance on $R(A_1,A_2,...,A_n)$, and $1\in[0,1]$ is a number, we may define $i_{r^{'},1}$.

We have,

$$i_{r^{'},1}(A_k)>\frac{1}{2}\ \text{ if }\ SE\left(t^{'}[A_k],t^{''}[A_k]\right)\geq 1,$$
$$i_{r^{'},1}(A_k)\leq\frac{1}{2}\ \text{ if }\ SE\left(t^{'}[A_k],t^{''}[A_k]\right)<1,$$

$k\in\{1,2,...,n\}$, i.e.,

$$i_{r^{'},1}(A_k)>\frac{1}{2}\ \text{ if }\ SE\left(t^{'}[A_k],t^{''}[A_k]\right)=1,$$
$$i_{r^{'},1}(A_k)\leq\frac{1}{2}\ \text{ if }\ SE\left(t^{'}[A_k],t^{''}[A_k]\right)=\theta^{''},$$

$k\in\{1,2,...,n\}$, i.e.,

$$i_{r^{'},1}(A_k)>\frac{1}{2}\ \text{ if }\ A_k\in W,$$
$$i_{r^{'},1}(A_k)\leq\frac{1}{2}\ \text{ if }\ A_k\notin W,$$

$k\in\{1,2,...,n\}$.

Thus,

$$i_{r^{'},1}(A)>\frac{1}{2}\ \text{ if }\ A\in W,$$
$$i_{r^{'},1}(A)\leq\frac{1}{2}\ \text{ if }\ A\notin W.$$

We shall prove that

$$i_{r^{'},1}\left((\wedge_{A\in K}A)\Rightarrow(\wedge_{B\in L}B)\right)>\frac{1}{2},$$

$$i_{r^{'},1}\left((\wedge_{A\in K}A)\Rightarrow((\wedge_{B\in L}B)\vee(\wedge_{C\in M}C))\right)>\frac{1}{2}$$

for all $(\wedge_{A\in K}A)\Rightarrow(\wedge_{B\in L}B)\in C^{'}$, $(\wedge_{A\in K}A)\Rightarrow((\wedge_{B\in L}B)\vee(\wedge_{C\in M}C))\in C^{'}$, where $M=\{A_1,A_2,...,A_n\}\setminus(K\cup L)$, and

$$i_{r',1}\left(\left(\wedge_{A\in X}A\right)\Rightarrow\left(\wedge_{B\in Y}B\right)\right)\leq\frac{1}{2}$$

resp.

$$i_{r',1}\left(\left(\wedge_{A\in X}A\right)\Rightarrow\left(\left(\wedge_{B\in Y}B\right)\vee\left(\wedge_{C\in Z}C\right)\right)\right)>\frac{1}{2}.$$

Suppose that $(\wedge_{A\in K}A)\Rightarrow(\wedge_{B\in L}B)\in C'$ corresponds to $K\overset{1\theta}{\to}_V L\in C$.

Suppose that

$$i_{r',1}\left(\left(\wedge_{A\in K}A\right)\Rightarrow\left(\wedge_{B\in L}B\right)\right)\leq\frac{1}{2}.$$

It follows that

$$\frac{1}{2}+i_{r',1}\left(\wedge_{B\in L}B\right)\leq i_{r',1}\left(\wedge_{A\in K}A\right).$$

Now, $i_{r',1}(\wedge_{B\in L}B)\leq\frac{1}{2}$, $i_{r',1}(\wedge_{A\in K}A)=1$.

Consequently, $SE_L\left(t',t''\right)=\theta''$, $SE_K\left(t',t''\right)=1$.

Therefore,

$$\begin{aligned}SE_L\left(t',t''\right)&=\theta''<\theta'\leq_1\theta\\&=\min\left\{{}_1\theta,1\right\}\\&=\min\left\{{}_1\theta,SE_K\left(t',t''\right)\right\}.\end{aligned}$$

This contradicts the fact that r' satisfies $K\overset{1\theta}{\to}_V L$.

Hence,

$$i_{r',1}\left(\left(\wedge_{A\in K}A\right)\Rightarrow\left(\wedge_{B\in L}B\right)\right)>\frac{1}{2}.$$

Suppose that $(\wedge_{A\in K}A)\Rightarrow((\wedge_{B\in L}B)\vee(\wedge_{C\in M}C))\in C'$ corresponds to $K\overset{1\theta}{\to}\to_V L\in C$.

Suppose that

$$i_{r',1}\left(\left(\wedge_{A\in K}A\right)\Rightarrow\left(\left(\wedge_{B\in L}B\right)\vee\left(\wedge_{B\in L}B\right)\right)\right)\leq\frac{1}{2}.$$

It follows that

$$\begin{aligned}&\frac{1}{2}+\max\left\{i_{r',1}\left(\wedge_{B\in L}B\right),i_{r',1}\left(\wedge_{C\in M}C\right)\right\}\\&\leq i_{r',1}\left(\wedge_{A\in K}A\right).\end{aligned}$$

Now,

$$\max\left\{i_{r',1}\left(\wedge_{B\in L}B\right),i_{r',1}\left(\wedge_{C\in M}C\right)\right\}\leq\frac{1}{2},$$

$i_{r',1}(\wedge_{A\in K}A)=1$, i.e., $i_{r',1}(\wedge_{B\in L}B)\leq\frac{1}{2}$, $i_{r',1}(\wedge_{C\in M}C)\leq\frac{1}{2}$, $i_{r',1}(\wedge_{A\in K}A)=1$.

Consequently, $SE_L\left(t',t''\right)=\theta''$, $SE_M\left(t',t''\right)=\theta''$, $SE_K\left(t',t''\right)=1$.

We have,

$$\begin{aligned}SE_M\left(t',t''\right)&=\theta''<\theta'\leq_1\theta\\&=\min\left\{{}_1\theta,1\right\}\\&=\min\left\{{}_1\theta,SE_K\left(t',t''\right)\right\}.\end{aligned}$$

Hence, the third inequality of the inequalities

$$\begin{aligned}SE_K\left(t',t'\right)&\geq\min\left\{{}_1\theta,SE_K\left(t',t''\right)\right\},\\SE_L\left(t',t'\right)&\geq\min\left\{{}_1\theta,SE_K\left(t',t''\right)\right\},\\SE_M\left(t',t''\right)&\geq\min\left\{{}_1\theta,SE_K\left(t',t''\right)\right\}\end{aligned}$$

is not satisfied.

Furthermore,

$$\begin{aligned}SE_L\left(t',t''\right)&=\theta''<\theta'\leq_1\theta\\&=\min\left\{{}_1\theta,1\right\}\\&=\min\left\{{}_1\theta,SE_K\left(t',t''\right)\right\}.\end{aligned}$$

Therefore, the second inequality of the inequalities

$$\begin{aligned}SE_K\left(t'',t'\right)&\geq\min\left\{{}_1\theta,SE_K\left(t',t''\right)\right\},\\SE_L\left(t'',t'\right)&\geq\min\left\{{}_1\theta,SE_K\left(t',t''\right)\right\},\\SE_M\left(t'',t''\right)&\geq\min\left\{{}_1\theta,SE_K\left(t',t''\right)\right\}\end{aligned}$$

is not satisfied.

This means that $r^{'}$ violates $K \overset{1\theta}{\rightarrow}_{V} L$.

This contradicts the fact that $r^{'}$ satisfies $K \overset{1\theta}{\rightarrow}_{V} L$.

Hence,

$$i_{r',1}\left((\wedge_{A\in K}A) \Rightarrow ((\wedge_{B\in L}B) \vee (\wedge_{C\in M}C))\right) > \frac{1}{2}.$$

It remains to prove that

$$i_{r',1}\left((\wedge_{A\in X}A) \Rightarrow (\wedge_{B\in Y}B)\right) \leq \frac{1}{2}$$

resp.

$$i_{r',1}\left((\wedge_{A\in X}A) \Rightarrow ((\wedge_{B\in Y}B) \vee (\wedge_{B\in Z}C))\right) \leq \frac{1}{2}.$$

Suppose that

$$i_{r',1}\left((\wedge_{A\in X}A) \Rightarrow (\wedge_{B\in Y}B)\right) > \frac{1}{2}.$$

Let $i_{r',1}(\wedge_{A\in X}A) \leq \frac{1}{2}$.

It follows that $SE_X\left(t^{'}, t^{''}\right) = \theta^{''}$.

Since $SE_M\left(t^{'}, t^{''}\right) \geq \theta^{''}$ for all $M \subseteq \{A_1, A_2, ..., A_n\}$, we obtain that

$$\begin{aligned} SE_Y\left(t^{'}, t^{''}\right) \geq & \theta^{''} = \min\left\{\theta, \theta^{''}\right\} \\ & = \min\left\{\theta, SE_X\left(t^{'}, t^{''}\right)\right\}. \end{aligned}$$

This contradicts the fact that $r^{'}$ violates $X \overset{\theta}{\rightarrow}_{V} Y$.

Let $i_{r',1}(\wedge_{A\in X}A) > \frac{1}{2}$.

Since

$$i_{r',1}\left((\wedge_{A\in X}A) \Rightarrow (\wedge_{B\in Y}B)\right) > \frac{1}{2},$$

it follows that

$$\begin{aligned} &\min\left\{1 - i_{r',1}(\wedge_{A\in X}A) + i_{r',1}(\wedge_{B\in Y}B), 1\right\} \\ &> \frac{1}{2}. \end{aligned}$$

Suppose that $i_{r',1}(\wedge_{B\in Y}B) \leq \frac{1}{2}$.

We obtain,

$$\begin{aligned} &1 - i_{r',1}(\wedge_{A\in X}A) + i_{r',1}(\wedge_{B\in Y}B) \\ <&1 - \frac{1}{2} + \frac{1}{2} = 1. \end{aligned}$$

Hence,

$$\begin{aligned} &\min\left\{1 - i_{r',1}(\wedge_{A\in X}A) + i_{r',1}(\wedge_{B\in Y}B), 1\right\} \\ =&1 - i_{r',1}(\wedge_{A\in X}A) + i_{r',1}(\wedge_{B\in Y}B). \end{aligned}$$

Thus,

$$\frac{1}{2} + i_{r',1}(\wedge_{B\in Y}B) > i_{r',1}(\wedge_{A\in X}A).$$

This inequality must hold always true.

Consequently, $i_{r',1}(\wedge_{A\in X}A) < \frac{1}{2}$.

This is a contradiction.

Therefore, $i_{r',1}(\wedge_{B\in Y}B) > \frac{1}{2}$, i.e., $SE_Y\left(t^{'}, t^{''}\right) = 1$.

We obtain,

$$SE_Y\left(t^{'}, t^{''}\right) = 1 \geq \min\left\{\theta, SE_X\left(t^{'}, t^{''}\right)\right\}.$$

This contradicts the fact that $r^{'}$ violates $X \overset{\theta}{\rightarrow}_{V} Y$.

We conclude,

$$i_{r',1}\left((\wedge_{A\in X}A) \Rightarrow (\wedge_{B\in Y}B)\right) \leq \frac{1}{2}.$$

Suppose that

$$i_{r',1}\left((\wedge_{A\in X}A) \Rightarrow ((\wedge_{B\in Y}B) \vee (\wedge_{C\in Z}C))\right) > \frac{1}{2}.$$

Let $i_{r',1}(\wedge_{A\in X}A) \leq \frac{1}{2}$.

It follows that $SE_X\left(t^{'}, t^{''}\right) = \theta^{''}$.

Since $SE_M\left(t^{'}, t^{''}\right) \geq \theta^{''}$ for all $M \subseteq \{A_1, A_2, ..., A_n\}$, we obtain that

$$\begin{aligned} SE_Z\left(t^{'}, t^{''}\right) \geq & \theta^{''} = \min\left\{\theta, \theta^{''}\right\} \\ & = \min\left\{\theta, SE_X\left(t^{'}, t^{''}\right)\right\}. \end{aligned}$$

Hence,

$$SE_X\left(t^{'},t^{'}\right) \geq \min\left\{\theta, SE_X\left(t^{'},t^{''}\right)\right\},$$
$$SE_Y\left(t^{'},t^{'}\right) \geq \min\left\{\theta, SE_X\left(t^{'},t^{''}\right)\right\},$$
$$SE_Z\left(t^{'},t^{''}\right) \geq \min\left\{\theta, SE_X\left(t^{'},t^{''}\right)\right\}.$$

This contradicts the fact that $r^{'}$ violates $X \overset{\theta}{\to}\to_V Y$.

Let $i_{r^{'},1}\left(\wedge_{A\in X}A\right) > \frac{1}{2}$.

Since

$$i_{r^{'},1}\left(\left(\wedge_{A\in X}A\right) \Rightarrow \left(\left(\wedge_{B\in Y}B\right) \vee \left(\wedge_{C\in Z}C\right)\right)\right) > \frac{1}{2},$$

it follows that

$$\min\left\{1 - i_{r^{'},1}\left(\wedge_{A\in X}A\right) + \max\left\{i_{r^{'},1}\left(\wedge_{B\in Y}B\right), i_{r^{'},1}\left(\wedge_{C\in Z}C\right)\right\}, 1\right\} > \frac{1}{2}.$$

Suppose that

$$\max\left\{i_{r^{'},1}\left(\wedge_{B\in Y}B\right), i_{r^{'},1}\left(\wedge_{C\in Z}C\right)\right\} \leq \frac{1}{2}.$$

We obtain,

$$1 - i_{r^{'},1}\left(\wedge_{A\in X}A\right) + \max\left\{i_{r^{'},1}\left(\wedge_{B\in Y}B\right), i_{r^{'},1}\left(\wedge_{C\in Z}C\right)\right\} < 1 - \frac{1}{2} + \frac{1}{2} = 1.$$

Hence,

$$\min\left\{1 - i_{r^{'},1}\left(\wedge_{A\in X}A\right) + \max\left\{i_{r^{'},1}\left(\wedge_{B\in Y}B\right), i_{r^{'},1}\left(\wedge_{C\in Z}C\right)\right\}, 1\right\} = 1 - i_{r^{'},1}\left(\wedge_{A\in X}A\right) + \max\left\{i_{r^{'},1}\left(\wedge_{B\in Y}B\right), i_{r^{'},1}\left(\wedge_{C\in Z}C\right)\right\}.$$

Thus,

$$\frac{1}{2} + \max\left\{i_{r^{'},1}\left(\wedge_{B\in Y}B\right), i_{r^{'},1}\left(\wedge_{C\in Z}C\right)\right\} > i_{r^{'},1}\left(\wedge_{A\in X}A\right).$$

Since this inequality holds always true, we obtain that $i_{r^{'},1}\left(\wedge_{A\in X}A\right) < \frac{1}{2}$.

This is a contradiction.

Therefore,

$$\max\left\{i_{r^{'},1}\left(\wedge_{B\in Y}B\right), i_{r^{'},1}\left(\wedge_{C\in Z}C\right)\right\} > \frac{1}{2}.$$

It follows that $i_{r^{'},1}\left(\wedge_{B\in Y}B\right) > \frac{1}{2}$ or $i_{r^{'},1}\left(\wedge_{C\in Z}C\right) > \frac{1}{2}$.

Thus, $SE_Y\left(t^{'},t^{''}\right) = 1$ or $SE_Z\left(t^{'},t^{''}\right) = 1$.

Since $i_{r^{'},1}\left(\wedge_{A\in X}A\right) > \frac{1}{2}$, we have that $SE_X\left(t^{'},t^{''}\right) = 1$.

Consequently, $SE_X\left(t^{'},t^{''}\right) = 1$, $SE_Y\left(t^{'},t^{''}\right) = 1$ or $SE_X\left(t^{'},t^{''}\right) = 1$, $SE_Z\left(t^{'},t^{''}\right) = 1$.

Suppose that $SE_X\left(t^{'},t^{''}\right) = 1$, $SE_Y\left(t^{'},t^{''}\right) = 1$.

We obtain,

$$SE_X\left(t^{''},t^{'}\right) \geq \min\left\{\theta, SE_X\left(t^{'},t^{''}\right)\right\},$$
$$SE_Y\left(t^{''},t^{'}\right) \geq \min\left\{\theta, SE_X\left(t^{'},t^{''}\right)\right\},$$
$$SE_Z\left(t^{''},t^{''}\right) \geq \min\left\{\theta, SE_X\left(t^{'},t^{''}\right)\right\}.$$

This contradicts the fact that $r^{'}$ violates $X \overset{\theta}{\to}\to_V Y$.

Suppose that $SE_X\left(t^{'},t^{''}\right) = 1$, $SE_Z\left(t^{'},t^{''}\right) = 1$.

We obtain,

$$SE_X\left(t^{'},t^{'}\right) \geq \min\left\{\theta, SE_X\left(t^{'},t^{''}\right)\right\},$$
$$SE_Y\left(t^{'},t^{'}\right) \geq \min\left\{\theta, SE_X\left(t^{'},t^{''}\right)\right\},$$
$$SE_Z\left(t^{'},t^{''}\right) \geq \min\left\{\theta, SE_X\left(t^{'},t^{''}\right)\right\}.$$

This contradicts the fact that $r^{'}$ violates $X \overset{\theta}{\to}\to_V Y$.

We conclude,

$$i_{r^{'},1}\left(\left(\wedge_{A\in X}A\right) \Rightarrow \left(\left(\wedge_{B\in Y}B\right) \vee \left(\wedge_{C\in Z}C\right)\right)\right) \leq \frac{1}{2}.$$

Thus,

$$i_{r',1}\left((\wedge_{A\in K}A)\Rightarrow(\wedge_{B\in L}B)\right)>\frac{1}{2},$$

$$i_{r',1}\left((\wedge_{A\in K}A)\Rightarrow((\wedge_{B\in L}B)\vee(\wedge_{C\in M}C))\right)>\frac{1}{2}$$

for all $(\wedge_{A\in K}A)\Rightarrow(\wedge_{B\in L}B)\in C'$, $(\wedge_{A\in K}A)\Rightarrow((\wedge_{B\in L}B)\vee(\wedge_{C\in M}C))\in C'$, where $M=\{A_1,A_2,...,A_n\}\setminus(K\cup L)$, and

$$i_{r',1}\left((\wedge_{A\in X}A)\Rightarrow(\wedge_{B\in Y}B)\right)\le\frac{1}{2}$$

resp.

$$i_{r',1}\left((\wedge_{A\in X}A)\Rightarrow((\wedge_{B\in Y}B)\vee(\wedge_{C\in Z}C))\right)\le\frac{1}{2}$$

This contradicts the fact that (b) holds true.
Hence, (a) holds true.
This completes the proof. □

6 Applications

Example 1. Let $R(A,B,...,K)$ be a relation scheme on domains U_1, U_2,... U_{11}, where A is an attribute on the universe of discourse U_1, B is an attribute on the universe of discourse U_2,..., K is an attribute on the universe of discourse U_{11}. Suppose that the following vague functional and vague multivalued dependencies on $\{A,B,...,K\}$ hold true:

$$\{A,B,C,D\}\overset{\theta_1}{\rightarrow\rightarrow}_V\{B,D,E,F,I,J\},$$
$$\{A,B,C,D\}\overset{\theta_2}{\rightarrow\rightarrow}_V\{C,D,F,G,H,I\},$$
$$\{B,C\}\overset{\theta_3}{\rightarrow}_V\{E,K\},$$
$$\{B,D,E,J\}\overset{\theta_4}{\rightarrow}_V\{G,E\}.$$

Then, the vague multivalued dependency

$$\{A,B,C,D\}\overset{\theta}{\rightarrow\rightarrow}_V\{E,G,K\}$$

on $\{A,B,...,K\}$ holds also true, where $\theta=\min\{\theta_1,\theta_2,\theta_3,\theta_4\}$.

Proof. I We may apply the inference rules VF1-VF7, VM1-VM10.

We obtain:

1) $\{A,B,C,D\}\overset{\theta_1}{\rightarrow\rightarrow}_V\{B,D,E,F,I,J\}$ (input)

2) $\{A,B,C,D\}\overset{\theta_2}{\rightarrow\rightarrow}_V\{C,D,F,G,H,I\}$ (input)

3) $\{A,B,C,D\}\overset{\min\{\theta_1,\theta_2\}}{\rightarrow\rightarrow}_V\{D,F,I\}$ (from 1), 2) and VM9)

4) $\{A,B,C,D\}\overset{\min\{\theta_1,\theta_2\}}{\rightarrow\rightarrow}_V\{E,G,H,J,K\}$ (from 3) and VM2)

5) $\{B,C\}\overset{\theta_3}{\rightarrow}_V\{E,K\}$ (input)

6) $\{A,B,C,D\}\overset{\min\{\theta_1,\theta_2,\theta_3\}}{\rightarrow}_V\{E,K\}$ (from 4), 5), VM6, and the fact that $\{B,C\}$ and $\{E,G,H,J,K\}$ are disjoint, $\{E,K\}\subset\{E,G,H,J,K\}$)

7) $\{A,B,C,D\}\overset{\min\{\theta_1,\theta_2\}}{\rightarrow\rightarrow}_V\{B,E,J\}$ (from 1), 2) and VM9)

8) $\{B,D,E,J\}\overset{\theta_4}{\rightarrow}_V\{G,E\}$ (input)

9) $\{B,D,E,J\}\overset{\theta_4}{\rightarrow\rightarrow}_V\{G,E\}$ (from 8) and VM5)

10) $\{A,B,C,D\}\overset{\min\{\theta_1,\theta_2,\theta_4\}}{\rightarrow\rightarrow}_V G$ (from 7), 9), VM8, where $W=\{D\}$)

11) $\{A,B,C,D\}\overset{\min\{\theta_1,\theta_2,\theta_3\}}{\rightarrow\rightarrow}_V\{E,K\}$ (from 6) and VM5)

12) $\{A,B,C,D\}\overset{\theta}{\rightarrow\rightarrow}_V\{E,G,K\}$ (from 10), 11) and VM7)

□

Proof. II We may apply Theorem 4.

Note that the condition (a) of Theorem 4 actually states that the dependency

$$\{A,B,C,D\}\overset{\theta}{\rightarrow\rightarrow}_V\{E,G,K\}$$

follows from the set

$$C=\Big\{\{A,B,C,D\}\overset{\theta_1}{\rightarrow\rightarrow}_V\{B,D,E,F,I,J\},\ \{A,B,C,D\}\overset{\theta_2}{\rightarrow\rightarrow}_V\{C,D,F,G,H,I\},\ \{B,C\}\overset{\theta_3}{\rightarrow}_V\{E,K\},\ \{B,D,E,J\}\overset{\theta_4}{\rightarrow}_V\{G,E\}\Big\}$$

of vague dependencies.

Since the conditions (a) and (b) of Theorem 4 are equivalent, it is enough to prove that the condition (b) is satisfied.

As it is usual, we apply the resolution principle.

Suppose that

$$\begin{aligned} & i_{r,\beta}\left(\mathcal{K}_1\right) \\ =& i_{r,\beta}\Big(\left(A\wedge B\wedge C\wedge D\right)\Rightarrow \\ & \left(\left(B\wedge D\wedge E\wedge F\wedge I\wedge J\right)\vee\left(G\wedge H\wedge K\right)\right)\Big) \\ >& \frac{1}{2}, \\ & i_{r,\beta}\left(\mathcal{K}_2\right) \\ =& i_{r,\beta}\Big(\left(A\wedge B\wedge C\wedge D\right)\Rightarrow \\ & \left(\left(C\wedge D\wedge F\wedge G\wedge H\wedge I\right)\vee\left(E\wedge J\wedge K\right)\right)\Big) \\ >& \frac{1}{2}, \end{aligned}$$

$$\begin{aligned} & i_{r,\beta}\left(\mathcal{K}_3\right) \\ =& i_{r,\beta}\Big(\left(B\wedge C\right)\Rightarrow \\ & \left(\left(E\wedge K\right)\vee\left(A\wedge D\wedge F\wedge G\wedge H\wedge I\wedge J\right)\right)\Big) \\ >& \frac{1}{2}, \\ & i_{r,\beta}\left(\mathcal{K}_4\right) \\ =& i_{r,\beta}\Big(\left(B\wedge D\wedge E\wedge J\right)\Rightarrow \\ & \left(\left(G\wedge E\right)\vee\left(A\wedge C\wedge F\wedge H\wedge I\wedge K\right)\right)\Big) \\ >& \frac{1}{2}, \end{aligned}$$

where r is a two-element vague relation instance on $R\left(A,B,...,K\right)$, and $\beta\in[0,1]$ is a number.

Our goal is to prove that

$$\begin{aligned} & i_{r,\beta}\left(c^{'}\right) \\ =& i_{r,\beta}\Big(\left(A\wedge B\wedge C\wedge D\right)\Rightarrow \\ & \left(\left(E\wedge G\wedge K\right)\vee\left(F\wedge H\wedge I\wedge J\right)\right)\Big) \\ >& \frac{1}{2}. \end{aligned}$$

First, we find the conjunctive normal forms of the formulas $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$, $\mathcal{K}_4$ and $\neg c^{'}$ (this is in line with the resolution principle).

We obtain,

$$\begin{aligned} \mathcal{K}_1\equiv & \left(\neg A\vee\neg B\vee\neg C\vee\neg D\vee E\vee G\right)\wedge \\ & \left(\neg A\vee\neg B\vee\neg C\vee\neg D\vee F\vee G\right)\wedge \\ & \left(\neg A\vee\neg B\vee\neg C\vee\neg D\vee G\vee I\right)\wedge \\ & \left(\neg A\vee\neg B\vee\neg C\vee\neg D\vee G\vee J\right)\wedge \\ & \left(\neg A\vee\neg B\vee\neg C\vee\neg D\vee E\vee H\right)\wedge \\ & \left(\neg A\vee\neg B\vee\neg C\vee\neg D\vee F\vee H\right)\wedge \\ & \left(\neg A\vee\neg B\vee\neg C\vee\neg D\vee H\vee I\right)\wedge \\ & \left(\neg A\vee\neg B\vee\neg C\vee\neg D\vee H\vee J\right)\wedge \\ & \left(\neg A\vee\neg B\vee\neg C\vee\neg D\vee E\vee K\right)\wedge \\ & \left(\neg A\vee\neg B\vee\neg C\vee\neg D\vee F\vee K\right)\wedge \\ & \left(\neg A\vee\neg B\vee\neg C\vee\neg D\vee I\vee K\right)\wedge \\ & \left(\neg A\vee\neg B\vee\neg C\vee\neg D\vee J\vee K\right), \end{aligned}$$

$$\begin{aligned} \mathcal{K}_2\equiv & \left(\neg A\vee\neg B\vee\neg C\vee\neg D\vee E\vee F\right)\wedge \\ & \left(\neg A\vee\neg B\vee\neg C\vee\neg D\vee E\vee G\right)\wedge \\ & \left(\neg A\vee\neg B\vee\neg C\vee\neg D\vee E\vee H\right)\wedge \\ & \left(\neg A\vee\neg B\vee\neg C\vee\neg D\vee E\vee I\right)\wedge \\ & \left(\neg A\vee\neg B\vee\neg C\vee\neg D\vee F\vee J\right)\wedge \\ & \left(\neg A\vee\neg B\vee\neg C\vee\neg D\vee G\vee J\right)\wedge \\ & \left(\neg A\vee\neg B\vee\neg C\vee\neg D\vee H\vee J\right)\wedge \\ & \left(\neg A\vee\neg B\vee\neg C\vee\neg D\vee I\vee J\right)\wedge \\ & \left(\neg A\vee\neg B\vee\neg C\vee\neg D\vee F\vee K\right)\wedge \\ & \left(\neg A\vee\neg B\vee\neg C\vee\neg D\vee G\vee K\right)\wedge \\ & \left(\neg A\vee\neg B\vee\neg C\vee\neg D\vee H\vee K\right)\wedge \\ & \left(\neg A\vee\neg B\vee\neg C\vee\neg D\vee I\vee K\right), \end{aligned}$$

$$\begin{aligned} & \mathcal{K}_3\equiv \\ & \left(A\vee\neg B\vee\neg C\vee E\right)\wedge\left(\neg B\vee\neg C\vee D\vee E\right)\wedge \\ & \left(\neg B\vee\neg C\vee E\vee F\right)\wedge\left(\neg B\vee\neg C\vee E\vee G\right)\wedge \\ & \left(\neg B\vee\neg C\vee E\vee H\right)\wedge\left(\neg B\vee\neg C\vee E\vee I\right)\wedge \\ & \left(\neg B\vee\neg C\vee E\vee J\right)\wedge\left(A\vee\neg B\vee\neg C\vee K\right)\wedge \\ & \left(\neg B\vee\neg C\vee D\vee K\right)\wedge\left(\neg B\vee\neg C\vee F\vee K\right)\wedge \\ & \left(\neg B\vee\neg C\vee G\vee K\right)\wedge\left(\neg B\vee\neg C\vee H\vee K\right)\wedge \\ & \left(\neg B\vee\neg C\vee I\vee K\right)\wedge\left(\neg B\vee\neg C\vee J\vee K\right), \end{aligned}$$

$$\begin{aligned} \mathcal{K}_4\equiv & \left(A\vee\neg B\vee\neg D\vee\neg E\vee G\vee\neg J\right)\wedge \\ & \left(\neg B\vee C\vee\neg D\vee\neg E\vee G\vee\neg J\right)\wedge \\ & \left(\neg B\vee\neg D\vee\neg E\vee F\vee G\vee\neg J\right)\wedge \end{aligned}$$

$$(\neg B \vee \neg D \vee \neg F \vee G \vee H \vee \neg J) \wedge$$
$$(\neg B \vee \neg D \vee \neg J \vee G \vee I \vee \neg J) \wedge$$
$$(\neg B \vee \neg D \vee \neg E \vee G \vee \neg J \vee K),$$

$$\neg c^{'} \equiv A \wedge B \wedge C \wedge D \wedge (\neg E \vee \neg G \vee \neg K) \wedge (\neg F \vee \neg H \vee \neg I \vee \neg J).$$

Let M be the of all conjunctive terms that appear within conjunctive normal forms of the formulas $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$, $\mathcal{K}_4$ and $\neg c^{'}$.

Applying the resolution principle to the elements of the set M, we obtain

1) $\neg A \vee \neg B \vee \neg C \vee \neg D \vee G \vee K$ (input)

2) A (input)

3) B (input)

4) C (input)

5) D (input)

6) $\neg B \vee \neg C \vee \neg D \vee G \vee K$ (resolvent from 1) and 2))

7) $\neg C \vee \neg D \vee G \vee K$ (resolvent from 6) and 3))

8) $\neg D \vee G \vee K$ (resolvent from 7) and 4))

9) $G \vee K$ (resolvent from 8) and 5))

10) $\neg E \vee \neg G \vee \neg K$ (input)

11) $\neg E$ (resolvent from 10) and 9))

12) $\neg A \vee \neg B \vee \neg C \vee \neg D \vee E \vee K$ (input)

13) $\neg B \vee \neg C \vee \neg D \vee E \vee K$ (resolvent from 12) and 2))

14) $\neg C \vee \neg D \vee E \vee K$ (resolvent from 13) and 3))

15) $\neg D \vee E \vee K$ (resolvent from 14) and 4))

16) $E \vee K$ (resolvent from 15) and 5))

17) K (resolvent from 16) and 11))

18) $\neg A \vee \neg B \vee \neg C \vee \neg D \vee E \vee G$ (input)

19) $\neg B \vee \neg C \vee \neg D \vee E \vee G$ (resolvent from 18) and 2))

20) $\neg C \vee \neg D \vee E \vee G$ (resolvent from 19) and 3))

21) $\neg D \vee E \vee G$ (resolvent from 20) and 4))

22) $E \vee G$ (resolvent from 21) and 5))

23) $\neg K$ (resolvent from 22) and 10))

Resolving 23) and 17), we obtain that the inequalities: $i_{r,\beta}(\mathcal{K}_1) > \frac{1}{2}$, $i_{r,\beta}(\mathcal{K}_2) > \frac{1}{2}$, $i_{r,\beta}(\mathcal{K}_3) > \frac{1}{2}$, $i_{r,\beta}(\mathcal{K}_4) > \frac{1}{2}$ and $i_{r,\beta}\left(c^{'}\right) \leq \frac{1}{2}$ cannot be satisfied at the same time.

Since, $i_{r,\beta}(\mathcal{K}_1) > \frac{1}{2}$, $i_{r,\beta}(\mathcal{K}_2) > \frac{1}{2}$, $i_{r,\beta}(\mathcal{K}_3) > \frac{1}{2}$ and $i_{r,\beta}(\mathcal{K}_4) > \frac{1}{2}$, it follows that $i_{r,\beta}\left(c^{'}\right) > \frac{1}{2}$.

Thus, the condition (b) of Theorem 4 is satisfied.

Consequently, the condition (a) of Theorem 4 is satisfied.

Therefore,

$$\{A, B, C, D\} \overset{\theta}{\twoheadrightarrow}_V \{E, G, K\}$$

follows. □

For analogous results in the case of fuzzy functional and fuzzy multivalued dependencies, we refer to [10], [11], [15], [16].

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