On the controllability problem with pointwise observation for the parabolic equation with free convection term

IRINA ASTASHOVA
Lomonosov Moscow State University
Department of Mathematics and Mechanics
Leninskie Gory, 1, 119991, Moscow
Plekhanov Russian University of Economics
Stremyanny lane, 36, 117997, Moscow
RUSSIAN FEDERATION
ast.diffiety@gmail.com

ALEXEY FILINOVSKIY
Bauman Moscow State Technical University
2 nd Baumanskaya str., 5, 105005, Moscow
Lomonosov Moscow State University
Department of Mathematics and Mechanics
Leninskie Gory, 1, 119991, Moscow
RUSSIAN FEDERATION
flnv@yandex.ru

Abstract: A mathematical model of the temperature control in industrial greenhouse is based on a one-dimensional parabolic equation with a free convection term and a quadratic cost functional with the point observation. The existence and uniqueness of a control function from some convex set of functions are proved and the structure of the set of accessible temperature functions is studied. We also prove the dense controllability of the problem for some set of control functions.

Key–Words: Parabolic equation, free convection, pointwise observation, extremal problem, exact controllability, dense controllability.

1 Introduction

Many models in physics and technics can be described by parabolic equations. We pay special attention to extreme problems related to corresponding boundary value problems. Let us consider in the rectangle $Q_T = (0,1) \times (0,T)$ the mixed problem for the heat equation with a free convection term

$$u_t = u_{xx} + b(x)u_x, \quad (x,t) \in Q_T, \qquad (1)$$
$$u(0,t) = \varphi(t), \quad u_x(1,t) = \psi(t), \quad t > 0, \qquad (2)$$
$$u(x,0) = \xi(x), \quad 0 < x < 1, \qquad (3)$$

for a sufficiently smooth coefficient b satisfies $|b(x)| \le b_1$, $x \in [0,1]$, and the functions $\varphi \in W_2^1(0,T)$, $\psi \in W_2^1(0,T)$, $\xi \in L_2(0,1)$. We mean that the functions b, ξ, ψ are fixed and φ is a control function to be found. The mixed problem (1) – (3) arises in the engineering temperature control problem in an extended industrial greenhouse with lower heating and upper ventilation (see [3], [4], [5]). The problem is to find control function $\varphi_0(t)$ making the temperature $u(x,t)$ at $x = c \in (0,1)$, maximally close to the given one $z(t)$ during the whole time interval $(0,T)$. The quality of the control is estimated by a quadratic cost functional.

Let us note that extremum problems for parabolic equations with integral functionals were considered by different authors (see [18], [8], [10], [11]). One of the first studies is the paper [10] where the heat equation with the third type boundary condition contains the control function is considered. In [10] for the extremal problem with the final observation functional the existence of minimizer is proved in the class of measurable control functions not exceeding to some constant. The existence and uniqueness of minimizer are also proved (see [10]) in the case of a functional with an additional quadratic term. Some of the later results deal with non-homogeneous equation and right-hand side as a distributed in Q_T control function and the distributed or boundary observation ([18], [19]). The other problems of minimization with final observation and the problem of the control optimal time are considered in [8], [11], [9], [23]. The review of early results is contained in [9], survey of later works is contained in [23], see also [3], [13]. Note that our formulation of the extremal problem with pointwise observation is different from those formulated in the papers listed. Also the case of the equation with a free convection term was not previously considered. Closely relates to our formulation of problem is the problem with distributed control and pointwise observation with an additional quadratic control function term (see [18]). For such problem in [18] the existence and uniqueness of minimizer are proved.

We prove the existence and uniqueness of the control function $\varphi_0(t)$ from a convex set (the minimizer) giving the minimum to this functional, and study the structure of the set of accessible temperature functions. We also prove the "dense controllability" of the problem for some set of control functions. In comparison with our previous results ([3] —[6]) now we consider the parabolic equation with non self-adjoint elliptic operator and an arbitrary convex closed bounded set of control functions. To do this we use methods of qualitative theory of differential equations and, in particular, some methods described in [1] and [2].

2 Main Results

Propose a mathematical approach to solve the problem.

Definition 1 . *By $V_2^{1,0}(Q_T)$ denote the Banach space of functions $u \in W_2^{1,0}(Q_T)$ with the finite norm*

$$\|u\|_{V_2^{1,0}(Q_T)} = \sup_{0\le t\le T} \|u(x,t)\|_{L_2(0,1)} + \|u_x\|_{L_2(Q_T)} \quad (4)$$

and such that $t \mapsto u(\cdot,t)$ is a continuous mapping $[0,T] \to L_2(0,1)$.

The formula to the norm in the space $V_2^{1,0}(Q_T)$ is introduced in [16], p.26. This norm naturally corresponds to the energy balance equation for the mixed problem to the parabolic equation ([16], ch. 3, formula (2.22)).

Definition 2 . *By $\widetilde{W}_2^1(Q_T)$ denote the space of all functions $\eta \in W_2^1(Q_T)$ such that $\eta(x,T) = 0$, $\eta(0,t) = 0$.*

The values of the functions $\eta(x,T)$ and $\eta(0,t)$ we consider in the trace sense (see [16], ch. 1, th. 6.3, p. 71).

Definition 3 . *We call the function $u \in V_2^{1,0}(Q_T)$ an energy class weak solution to problem (1)–(3) if it satisfies the boundary condition $u(0,t) = \varphi(t)$ and the integral identity*

$$\int_{Q_T} (u_x\eta_x - bu_x\eta - u\eta_t)\, dx\, dt = \int_0^1 \xi(x)\eta(x,0)\, dx + \int_0^T \psi(t)\,\eta(1,t)\, dt \quad (5)$$

for any function $\eta(x,t) \in \widetilde{W}_2^1(Q_T)$.

Under the conditions $\varphi,\psi \in W_2^1(0,T)$, and $\xi \in L_2(0,1)$ the weak solution from the set $W_2^{1,0}(Q_T)$ automatically belongs to $V_2^{1,0}(Q_T)$ ([16], ch. 3, par. 3). By standard technique (see [15], [16]) we can obtain the following estimate for the solution to problem (1)–(3):

Theorem 4 . *There exists a unique weak solution to problem (1)–(3) belonging to $V_2^{1,0}(Q_T)$ with the following inequality:*

$$\|u_\varphi\|_{V_2^{1,0}(Q_T)} \le C_1(\|\xi\|_{L_2(0,1)} + \|\varphi\|_{W_2^1(0,T)} + \|\psi\|_{W_2^1(0,T)}), \quad (6)$$

where the constant C_1 is independent of φ, ψ and ξ.

Hereafter denote by u_φ the unique solution to the problem (1) – (3) with $\varphi,\psi \in W_2^1(0,T), \xi \in L_2(0,1)$, existing according to theorem 4.

Suppose $z \in L_2(0,T)$. Let $\Phi \subset W_2^1(0,T)$ be a bounded closed convex set.

For some $c \in (0,1)$ define the functional

$$J[z,\varphi] = \int_0^T (u_\varphi(c,t) - z(t))^2 dt. \quad (7)$$

The value of the function $u_\varphi(c,t) \in L_2(0,T)$ we also consider in the trace sense.

Consider the minimization problem for this functional and put

$$m[z,\Phi] = \inf_{\varphi\in\Phi} J[z,\varphi]. \quad (8)$$

Theorem 5 . *There exists a unique function $\varphi_0(t) \in \Phi$ such that $m[z,\Phi] = J[z,\varphi_0]$.*

Definition 6 . *We call the problem (1)–(3), (8) exactly controllable from $\Phi \subset W_2^1(0,T)$ to $Z \subset L_2(0,T)$, if for any $z \in Z$ there exists $\varphi_0 \in \Phi$ such that*

$$J[z,\varphi_0] = 0. \quad (9)$$

Definition 7 . *By the exact control we denote the function $\varphi_0 \in \Phi$ making the functional $J[z,\varphi]$ to vanish:*

$$J[z,\varphi_0] = \int_0^T (u_{\varphi_0}(c,t) - z(t))^2\, dt = 0.$$

The next theorem shows that the set Z of functions $z \in L_2(0,T)$ admitting exact controllability is sufficiently "small" subset of $L_2(0,T)$.

Theorem 8 . *The set Z of all functions $z \in L_2(0,T)$ admitting exact controllability for $\Phi = W_2^1(0,T)$ is a first Baire category subset in $L_2(0,T)$.*

Definition 9 . *We call the problem (1)–(3), (8) densely controllable from $\Phi \subset W_2^1(0,T)$ to $Z \subset L_2(0,T)$, if for any $z \in Z$ we have*

$$\inf_{\varphi\in\Phi} J[z,\varphi] = \inf_{\varphi\in\Phi} \int_0^T (u_\varphi(c,t) - z(t))^2 dt = 0.$$

The following result proves the dense controllability for $Z = L_2(0,T)$ and $\Phi = W_2^1(0,T)$.

Theorem 10 . *For any $z \in L_2(0,T)$ the following equality holds*

$$\inf_{\varphi \in W_2^1(0,T)} J[z,\varphi] = 0. \quad (10)$$

3 Proofs

Proof of theorem 5. The proofs of results on the existence and uniqueness are based on the following statement concerning the best approximation in Hilbert spaces.

Theorem 11 *([4]). Let A be a convex closed set in a Hilbert space H. Then for any $x \in H$ there exists a unique element $y \in A$ such that*

$$\|x - y\| = \inf_{z \in A} \|x - z\|.$$

Let

$$B = \{y = u_\varphi(c,\cdot) : \varphi \in \Phi\} \subset L_2(0,T).$$

Use the convexity of Φ we see that B is a convex set. By theorem 4 we obtain that B is a bounded set in $L_2(0,T)$. Now we prove that B is a closed subset of $L_2(0,T)$. Let $\{y_j\}_{j=1}^\infty \subset B$ be a fundamental sequence in $L_2(0,T)$ having the limit $y \in L_2(0,T)$, $\|y - y_j\|_{L_2(0,T)} \to 0$, $j \to \infty$. The corresponding sequence $\{\varphi_j\} \subset \Phi$ by the boundedness of Φ is a weakly precompact set in $W_2^1(0,T)$. By the convexity of Φ and Mazur theorem [21] Φ is a weakly closed subset of $W_2^1(0,T)$. Therefore, there exists a subsequence (we denote it by $\{\varphi_j\}_{j=1}^\infty$, too) such that $w - \lim_{j\to\infty} \varphi_j = \varphi \in \Phi$. Hence, by Mazur theorem there exist the numbers $\alpha_{jl} \geq 0$, $1 \leq j \leq l$, $l = 1, 2, \dots$, $\sum_{j=1}^l \alpha_{jl} = 1$, such that for some $\varphi \in \Phi$ we have

$$\|\tilde\varphi_l - \varphi\|_{W_2^1(0,T)} \to 0, \quad l \to \infty, \quad (11)$$

$$\tilde\varphi_l = \sum_{j=1}^l \alpha_{jl}\varphi_j.$$

Therefore, for the corresponding sequence of solutions

$$u_{\tilde\varphi_l} = \sum_{j=1}^l \alpha_{jl} u_{\varphi_j} \quad (12)$$

we obtain

$$\|u_{\tilde\varphi_l} - u_{\tilde\varphi_p}\|_{V_2^{1,0}(Q_T)} \quad (13)$$
$$\leq C_1 \|\tilde\varphi_l - \tilde\varphi_p\|_{W_2^1(0,T)} \to 0, \quad l,p \to \infty.$$

This means that $u_{\tilde\varphi_l}(0,t) = \tilde\varphi_l(t)$, and the integral identity

$$\int_{Q_T} ((u_{\tilde\varphi_l})_x \eta_x - b(x)(u_{\tilde\varphi_l})_x \eta - u_{\tilde\varphi_l}\eta_t)\, dx\, dt$$
$$= \int_0^1 \xi(x)\eta(x,0)\, dx + \int_0^T \psi(t)\,\eta(1,t)\, dt \quad (14)$$

holds for any function $\eta(x,t) \in \widetilde{W}_2^1(Q_T)$. Taking into account relations (11), (13), and (14), we see that there exists the limit function $u \in V_2^{1,0}(Q_T)$, which is a weak solution to problem (1)–(3) with the boundary function φ, and

$$\|u - u_{\tilde\varphi_l}\|_{V_2^{1,0}(Q_T)} \leq C_1 \|\varphi - \tilde\varphi_l\|_{W_2^1(0,T)}.$$

So, by the embedding estimate (see [16], Ch. 1, sec. 6, form. 6.15) we obtain

$$\begin{aligned} &\|u(c,\cdot) - u_{\tilde\varphi_l}(c,\cdot)\|_{L_2(0,T)} \\ \leq\ & C_2 \|u - u_{\tilde\varphi_l}\|_{V_2^{1,0}(Q_T)} \\ \leq\ & C_1 C_2 \|\varphi - \tilde\varphi_l\|_{W_2^1(0,T)}, \end{aligned}$$

whence $y = u(c,\cdot) \in B$ and B is a closed subset in $L_2(0,T)$.

Therefore, by theorem 11, there exists a unique function $y = u(c,\cdot)$, where $u \in V_2^{1,0}(Q_T)$ is a solution to problem (1)–(3) with some $\varphi_0 \in \Phi$ such that

$$\inf_{\varphi\in\Phi} J[z,\varphi] = J[z,\varphi_0].$$

Now we prove that such $\varphi_0 \in \Phi$ is unique. If not, consider a pair of such functions φ_1, φ_2 and the corresponding pair of solutions u_{φ_1}, u_{φ_2}. The function $\tilde u = u_{\varphi_1} - u_{\varphi_2}$ is a solution to the problem

$$\begin{aligned} \tilde u_t &= \tilde u_{xx} + b(x)\tilde u_x, && (15)\\ & \quad 0 < t < T, \quad 0 < x < 1, \\ \tilde u(0,t) &= \tilde\varphi(t), \quad 0 < t < T, && (16)\\ \tilde\varphi(t) &= \varphi_1(t) - \varphi_2(t), \\ \tilde u(c,t) &= 0, \quad 0 < t < T, && (17)\\ \tilde u_x(1,t) &= 0, \quad 0 < t < T, && (18)\\ \tilde u(x,0) &= 0, \quad 0 < x < 1. && (19) \end{aligned}$$

Taking into account integral identity (5) with the function $\eta(x,t)$ equal to 0 on $[0,c]\times[0,T]$, we obtain that the function $\tilde u$ on the rectangle $Q_T^{(c)} = (c,1) \times (0,T)$ is equal to the solution of the problem

$$\begin{aligned} \hat u_t &= \hat u_{xx} + b(x)\hat u_x, && (20)\\ & \quad 0 < t < T, \quad c < x < 1, \\ \hat u(c,t) &= 0, \quad 0 < t < T, && (21)\\ \hat u_x(1,t) &= 0, \quad 0 < t < T, && (22)\\ \hat u(x,0) &= 0, \quad c < x < 1. && (23) \end{aligned}$$

But the solution to problem (20)–(23) vanishes on $[c, 1] \times [0, T]$, whence we have

$$\tilde{u}(x,t) = 0, \quad c < x < 1, \quad 0 < t < T. \tag{24}$$

Now we prove that

$$\tilde{u}(x,t) = 0, \quad 0 < x < 1, \quad 0 < t < T. \tag{25}$$

Note that by theorem 2 ([14], Sec. 11), the weak solution $\tilde{u}$ is a classical solution to equation (15) in Q_T. Now we use theorem 5 ([17], Sec. 3). It establishes the following.

Consider a function $u(x,t) \in C^{2,1}(\Omega)$, $\Omega \subset R^2$ such that $u_t = u_{xx} + q(x)u$ in Ω. Suppose G_0 is a connected component of the set $\Omega \cap \{t = t_0\}$, and $\widetilde{G}$ is a connected open subset of G_0. If $u|_{\widetilde{G}} = 0$, then $u|_{G_0} = 0$.

Let us consider the function

$$v(x,t) = ue^{-\frac{1}{2}\int_0^x b(z)dz} \tag{26}$$

which is a classical solution to equation

$$v_t = v_{xx} - \left(\frac{b'(x)}{2} + \frac{b^2(x)}{4}\right)v. \tag{27}$$

By the equalities (24) and (26) we obtain $v(x,t) = 0$, $c < x < 1$, $0 < t < T$. Applying this theorem to the solution v of the equation (27) with $q = -\left(\frac{b'(x)}{2} + \frac{b^2(x)}{4}\right)$ for any $t_0 \in (0,T)$ with $G_0 = (0,1) \times \{t_0\}$ and $\widetilde{G} = (c,1) \times \{t_0\}$, we obtain that $v(x,t) = 0$, $0 < x < 1$, $0 < t < T$. So, by (26) we have (24) which gives the equality (25). Therefore, $\tilde{u}(x,t) = 0$ for any $x \in (0,1)$ and $t \in (0,T)$. This means that $\tilde{\varphi}(t) = \tilde{u}(0,t) = 0$. The proof of Theorem 5 is complete. □

Corollary 12 . *From the theorem 5 we can obtain the existence and uniqueness theorems for some practically important classes of control functions (see [3], [3]).*

Proof of theorem 8. At first we prove the following analog to the classical maximum principle.

Theorem 13 . *Let $u \in V_2^{1,0}(Q_T)$ be a solution to the problem*

$$\begin{aligned} u_t &= u_{xx} + b(x)u_x, \quad (x,t) \in Q_T, && (28)\\ u(0,t) &= \varphi(t), \quad u_x(1,t) = 0, && (29)\\ & \quad 0 < x < 1, \quad 0 < t < T, &&\\ u(x,0) &= 0, \quad 0 < x < 1. && (30) \end{aligned}$$

Then for almost all $(x,t) \in Q_T$ the following inequalities hold:

$$\begin{aligned} &\min\{0, \operatorname{ess}\inf_{t\in[0,T]} \varphi(t)\} \\ \le\ & u(x,t) \le \max\{0, \operatorname{ess}\sup_{t\in[0,T]} \varphi(t)\}. \end{aligned} \tag{31}$$

Proof of theorem 13. Let $k = \operatorname{ess}\sup_{t\in[0,T]} \varphi(t)$. We define the function $u^{(k)} = \max\{u - k, 0\}$. By the same way as in the proof ([15], Ch. 3, Sec. 7, th. 7.1) we can obtain for $0 < t_1 < T$ the following relations

$$\begin{aligned} &\int_0^1 (u^{(k)}(x,t_1))^2 dx + 2\int_0^1\int_0^{t_1} (u_x^{(k)})^2 dxdt \\ =\ & 2\int_0^1\int_0^{t_1} b(x)u^{(k)}u_x dxdt \\ \le\ & b_1^2 \int_0^1\int_0^{t_1} (u^{(k)})^2 dxdt \\ +\ & \int_0^1\int_0^{t_1} (u_x^{(k)})^2 dxdt. \end{aligned} \tag{32}$$

Let $y(t) = \sup_{0\le\tau\le t} \|u^{(k)}(x,\tau)\|_{L_2(0,1)}$. It follows from (32) that

$$\begin{aligned} y^2(t_1) &+ 2\|u_x^{(k)}\|^2_{L_2(Q_{t_1})} \\ &\le b_1^2 t_1 y^2(t_1) + \|u_x^{(k)}\|^2_{L_2(Q_{t_1})}. \end{aligned} \tag{33}$$

Taking $t_1 < (2b_1^2)^{-1}$ we obtain by (33) that $\|u^{(k)}\|_{V_2^{1,0}(Q_{t_1})} \le 0$. Therefore, for almost all $(x,t) \in Q_{t_1}$ we have $u(x,t) \le \max\{0, \operatorname{ess}\sup_{t\in[0,t_1]} \varphi(t)\}$. Repeat this process to $(0,1) \times (t_1, 2t_1)$, $(0,1) \times (2t_1, 3t_1)$, ..., we obtain the right inequality from (31). Similar considerations with the function $-u$ proves the left inequality from (31). □

Consider the solutions $u_{\varphi_j}(x,t) \in V_2^{1,0}(Q_T)$, $j = 1, 2$. Denote $\tilde{u} = u_{\varphi_1} - u_{\varphi_2}$. The function $\tilde{u}$ is a solution of equation (1) with the boundary conditions

$$\begin{aligned} \tilde{u}(0,t) &= \tilde{\varphi}(t) = \varphi_1(t) - \varphi_2(t), && (34)\\ \tilde{u}_x(1,t) &= 0, && (35) \end{aligned}$$

and the initial condition

$$\tilde{u}(x,0) = 0. \tag{36}$$

From (31) we obtain

$$\|\tilde{u}\|_{L_\infty(Q_T)} \le \|\varphi_1 - \varphi_2\|_{L_\infty(0,T)}, \tag{37}$$

and, consequently, by the continuity of solution to equation (28),

$$\sup_{t\in[0,T]} |\tilde{u}(c,t)| \le \|\varphi_1 - \varphi_2\|_{L_\infty(0,T)}. \tag{38}$$

By integrating inequality (38), we obtain

$$\|\tilde{u}(c,t)\|_{L_2(0,T)} \leq \sqrt{T}\|\varphi_1 - \varphi_2\|_{L_\infty(0,T)}. \quad (39)$$

Suppose the functions φ_1 and φ_2 are the exact control functions for given z_1 and z_2. This means that

$$J[z,\varphi_j] = \int_0^T (u_{\varphi_j}(c,t) - z_j(t))^2 dt = 0, \quad j = 1,2.$$

In this situation inequality (39) invokes the inequality

$$\|z_1 - z_2\|_{L_2(0,T)} \leq \sqrt{T}\|\varphi_1 - \varphi_2\|_{L_\infty(0,T)} \quad (40)$$

for arbitrary functions $z_1(t)$ and $z_2(t)$ admitting exact controllability.

Let $Z \subset L_2(0,T)$ be a set of exactly controllable functions. We have $Z = \cup_{M=1}^{\infty} Z_M$, where $Z_M \subset L_2(0,T)$ is the set of functions exactly controllable with $\varphi \in \Phi_M = \{\varphi \in W_2^1(0,T), \|\varphi\|_{W_2^1(0,T)} \leq M\}$. For any $M = 1,2,\ldots$ consider an arbitrary sequence of control functions $\{\varphi_j^M\} \subset \Phi_M$ and the corresponding sequence $\{z_j(t)\} = \{u_{\varphi_j^M}(c,t)\} \subset Z_M$. The set Φ_M is a bounded set in $W_2^1(0,T)$. By the embedding theorem for $W_2^1(0,T)$, we have

$$\|\varphi_{j_l}^M - \varphi_{j_p}^M\|_{L_\infty(0,T)} \to 0, \quad l,p \to \infty, \quad (41)$$

for some subsequence $\varphi_{j_l}^M$. Therefore, by (40), (41) we get for the sequence $\{z_{j_l}\} \subset Z_M$ the relation

$$\|z_{j_l} - z_{j_p}\|_{L_2(0,T)} \leq \sqrt{T}\|\varphi_{j_l}^M - \varphi_{j_p}^M\|_{L_\infty(0,T)} \to 0, \quad l,p \to \infty. \quad (42)$$

It follows from (42) that Z_M is a pre-compact set in $L_2(0,T)$. So, Z_M is nowhere dense in $L_2(0,T)$. Thus, since $Z = \cup_{M=1}^{\infty} Z_M$, we conclude that Z is a first Baire category set in $L_2(0,T)$. Theorem 8 is proved. □

Proof of theorem 10. Let us represent the solution of the problem (1) – (3) in the form

$$u_\varphi = v + w \quad (43)$$

where v and w are solutions of the following boundary value problems

$$\begin{aligned} v_t - v_{xx} - b(x)v_x &= 0, \quad 0<x<1, \quad 0<t<T, && (44)\\ v(0,t) &= \varphi(t), \quad 0<t<T, && (45)\\ v_x(1,t) &= 0, \quad 0<t<T, && (46)\\ v(x,0) &= 0, \quad 0<x<1. && (47) \end{aligned}$$

and

$$\begin{aligned} w_t - w_{xx} - b(x)w_x &= 0, \quad 0<x<1, \quad 0<t<T, && (48)\\ w(0,t) &= 0, \quad 0<t<T, && (49)\\ w_x(1,t) &= \psi(t), \quad 0<t<T, && (50)\\ w(x,0) &= \xi(x), \quad 0<x<1. && (51) \end{aligned}$$

Therefore, denote $v = v_\varphi$ we have

$$J[z,\varphi] = \int_0^T (v_\varphi(c,t) - z_1(t))^2 dt, \ c \in (0,1), \quad (52)$$

where $z_1(t) = z(t) - w(c,t) \in L_2(0,T)$. It follows from the inequality

$$\inf_{\varphi \in W_2^1(0,T)} J[z,\varphi] \leq \inf_{\varphi \in W_2^1(0,T), \varphi(0)=0} J[z,\varphi] = \inf_{\varphi \in W_2^1(0,T), \varphi(0)=0} \int_0^T (v_\varphi(c,t) - z_1(t))^2 dt \quad (53)$$

that to establish (10) it is sufficient to prove that

$$\inf_{\varphi \in W_2^1(0,T), \varphi(0)=0} \int_0^T (v_\varphi(c,t) - z_1(t))^2 dt = 0. \quad (54)$$

Let us construct the weak solution $v_\varphi \in W_2^{1,0}(Q_T)$ of problem (44) – (47) for $\varphi \in W_2^1(0,T)$, $\varphi(0) = 0$. At first we consider the function $P \in V_2^{1,0}(Q_T)$ which is a weak solution of the mixed problem

$$\begin{aligned} P_t - P_{xx} - b(x)P_x &= 0, \quad 0<x<1, \quad 0<t<T, && (55)\\ P(0,t) &= 1, \quad 0<t<T, && (56)\\ P_x(1,t) &= 0, \quad 0<t<T, && (57)\\ P(x,0) &= 0, \quad 0<x<1. && (58) \end{aligned}$$

It means that P satisfies the integral identity

$$\int_{Q_T} (P_x(\eta_x - b(x)\eta) - P\eta_t)\, dx\, dt = 0 \quad (59)$$

for any function $\eta \in \widetilde{W}_2^1(Q_T)$ and $P(0,t) = 1$ in the trace sense. At first we establish the existence to the function

$$\tilde{P} = P - 1 \quad (60)$$

which is a solution of the problem

$$\begin{aligned} \tilde{P}_t - \tilde{P}_{xx} - b(x)\tilde{P}_x &= 0, \quad 0<x<1, \quad 0<t<T, && (61)\\ \tilde{P}(0,t) &= 0, \quad 0<t<T, && (62)\\ \tilde{P}_x(1,t) &= 0, \quad 0<t<T, && (63)\\ \tilde{P}(x,0) &= -1, \quad 0<x<1. && (64) \end{aligned}$$

Then we can prove the existence and uniqueness of such solution by Galerkin method by the same way as the solution u in theorem 11. Now we prove the following representation formula:

$$v_\varphi(x,t) = \int_0^t \varphi'(\tau)P(x,t-\tau)d\tau. \qquad (65)$$

The weak solution $v = v_\varphi$ satisfies the integral identity

$$\int_{Q_T} (v_x(\eta_x - b(x)\eta) - v\eta_t)\, dx\, dt = 0 \qquad (66)$$

for any function $\eta \in \widetilde{W}_2^1(Q_T)$ and $v(0,t) = \varphi(t)$ in the trace sense. It follows from (59), (65) that for any function $\eta \in \widetilde{W}_2^1(Q_T)$ we have the equalities

$$\begin{aligned} &\int_{Q_T} (v_x(\eta_x - b(x)\eta) - v\eta_t)\, dx\, dt \qquad (67)\\ =\ & \int_0^1\int_0^T \Big(\Big(\int_0^t \varphi'(\tau)P(x,t-\tau)d\tau\Big)_x(\eta_x - b(x)\eta)\\ -\ & \int_0^t \varphi'(\tau)P(x,t-\tau)d\tau\eta_t\Big)\, dx\, dt\\ =\ & \int_0^1\int_0^T \varphi'(\tau)\int_\tau^T (P_x(x,t-\tau)(\eta_x - b(x)\eta)\\ -\ & P(x,t-\tau)\eta_t)dx\, d\tau\, dt\\ =\ & \int_0^T \varphi'(\tau)\Big(\int_0^1\int_0^{T-\tau}(P_x(x,s)(\eta_x(x,s+\tau)\\ -\ & b\eta(x,s+\tau))\\ -\ & P(x,s)\eta_t(x,s+\tau))dx\, ds\Big)\, d\tau = 0. \end{aligned}$$

The last equation is valid because $\eta(x,s+\tau)|_{s=T-\tau} = \eta(x,T) = 0$. Therefore, the integral identity holds for any function $\eta \in \widetilde{W}_2^1(Q_T)$. Moreover, taking the sequence of smooth functions $\varphi_j(t)$ vanishes in the neighborhood of zero and such that $\|\varphi - \varphi_j\|_{W_2^1(0,T)} \to 0$, $j \to \infty$, we obtain from (56), (59) that $v_j(0,t) = \varphi_j(t) \to \varphi(t)$, $j \to \infty$ in $W_2^1(0,T)$. It means that $v(0,t) = \varphi(t)$ in the trace sense and the equality (65) is proved.

We can define the trace $P(c,\cdot) \in L_2(0,T)$, $c \in (0,1)$.

We use the following property of linear manifolds in Hilbert space ([20], ch. 2, par. 4, lemma 2):

Theorem 14 . *The linear manifold G is dense in Hilbert space H if and only if there are no non-zero element which is orthogonal to any element of G.*

Now we apply these theorem to $H = L_2(0,T)$ and the linear manifold $G = \{v_\varphi(c,t), \varphi(t) \in D(0,T) = \overset{0}{C^\infty}(0,T)\}$. To prove (10) it is sufficient to prove that if for any $\varphi \in D(0,T)$ we have

$$\begin{aligned} &\int_0^T z_1(t)v_\varphi(c,t)dt \qquad (68)\\ =\ & \int_0^T z_1(t)\Big(\int_0^t P(c,t-\tau)\varphi'(\tau)d\tau\Big)dt = 0, \end{aligned}$$

then $z_1(t) = 0$. We can transform (68) as

$$\begin{aligned} &\int_0^T z_1(t)\int_0^t P(c,t-\tau)\varphi'(\tau)d\tau dt \qquad (69)\\ =\ & \int_0^T \varphi'(\tau)\int_\tau^T z_1(t)P(c,t-\tau)dtd\tau = 0. \end{aligned}$$

By (69)

$$\int_\tau^T z_1(t)P(c,t-\tau)dt = \text{const}, \quad \tau \in [0,T], \quad (70)$$

but

$$\int_T^T z_1(t)P(c,t-T)dt = 0, \qquad (71)$$

so

$$\int_\tau^T z_1(t)P(c,t-\tau)dt = 0, \quad \tau \in [0,T]. \qquad (72)$$

After the transformation of variables we have

$$\begin{aligned} &\int_\tau^T z_1(t)P(c,t-\tau)dt \qquad (73)\\ =\ & \int_t^T z_1(\tau)P(c,\tau-t)d\tau\\ =\ & \int_0^{T-t} z_1(T-s)P(c,T-s-t)ds\\ =\ & \int_0^q z_1(T-s)P(c,q-s)ds\\ =\ & \int_0^q z_2(s)P(c,q-s)ds = 0, \end{aligned}$$

for almost all $q \in (0,T)$, here $z_2(t) = z_1(T-t) \in L_2(0,T) \subset L_1(0,T)$.

Now we apply Titchmarsh convolution theorem ([22], theorem 7):

Theorem 15 . *Let $\xi \in L_1(0,T)$, $\zeta \in L_1(0,T)$ are functions such that*

$$\int_0^t \xi(\tau)\zeta(t-\tau)d\tau = 0 \qquad (74)$$

almost everywhere in the interval $0 < t < T$, then $\xi(t) = 0$ almost everywhere in $(0,\alpha)$ and $\zeta(t) = 0$ almost everywhere in $(0,\beta)$, where $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta \geq T$.

We use Theorem 15 to the functions $P(c,\cdot)$ and $z_2(\cdot)$. By equality (73) we obtain that there exist $\alpha \geq 0$, $\beta \geq 0$, $\alpha+\beta \geq T$ such that $z_2(s) = 0$ almost everywhere in $(0,\alpha)$ and

$$P(c,s) = 0 \qquad (75)$$

almost everywhere in $(0,\beta)$.

Now we prove that $\beta = 0$. In the contrary let $\beta > 0$.

Applying maximum principle (31) from theorem 13 to problem (55) – (58) we obtain that $0 \leq P(x,t) \leq 1$ almost everywhere in Q_T. It follows from equality (64) and regularity theorem ([12], Ch. 7, par. 7.1, theorem 6) that P is a smooth function in $([0,1]\times[\varepsilon,T])$ for any $\varepsilon \in (0,T)$, and it is a classical solution of equation (55) in Q_T. Then

$$0 \leq P(x,t) \leq 1, \quad 0 \leq x \leq 1, \quad \varepsilon < t \leq T. \qquad (76)$$

Let us suppose that

$$P(c,t) = 0, \quad 0 < c < 1, \quad 0 < t < \beta \leq T, \qquad (77)$$

and consider the function $P(x,t)$ in the domain $Q_{\beta/2,T} = (0,1)\times(\beta/2,T)$. Note that by (76), (77)

$$P(c,\beta) = 0 = \inf_{(x,t)\in Q_{\beta/2,T}} P(x,t) \qquad (78)$$

and $(c,\beta) \in Q_T$. By the strong maximum principle ([12], Ch. 7, par. 7.1, theorem 11) we obtain that $P = 0$ in $(0,1)\times(\beta/2,\beta)$. It is impossible due to boundary condition (56). This contradiction means that $\beta = 0$. So, by the inequality $\alpha+\beta \geq T$ we have $\alpha \geq T$ and $z_2(t) = 0$ almost everywhere in $(0,T)$. Now, $z_1(t) = 0$ almost everywhere in $(0,T)$.

Therefore, by the Lemma 14 we obtain equality (10). Theorem 10 is proved. ☐

4 Conclusion

For the minimization problem for one-dimensional parabolic equation with free convection term and pointwise observation the existence and uniqueness of a control function from a prescribed set are proved, and the structure of the set of accessible temperature functions is studied. We also prove the dense controllability of the problem for some set of control functions.

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