From fuzzy dependences to fuzzy formulas and vice versa, for Kleene-Dienes fuzzy implication operator

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Abstract: To prove that a fuzzy dependency follows from a set of fuzzy dependences can be a very demanding task. As far as we know, an algorithm or an application that generally and automatically solves the problem, does not exist. The main goal of this paper is to offer such an algorithm. In order to achieve our goal we consider fuzzy dependences as fuzzy formulas. In particular, we fix fuzzy logic operators: conjunction, disjunction and implication, and allow only these operators to appear within fuzzy formulas. Ultimately, we prove that a fuzzy dependency follows from a set of fuzzy dependences if and only if the corresponding fuzzy formula is a logical consequence of the corresponding set of fuzzy formulas. To prove an implication of the last type, one usually uses the resolution principle, i.e., the steps that can be fully automated. Our methodology assumes the use of soundness and completeness of fuzzy dependences inference rules as well as the extensive use of active fuzzy multivalued dependences fulfillment.

Key–Words: Fuzzy multivalued dependences, Fuzzy functional dependences, Fuzzy relation instances, Fuzzy formulas, Fuzzy logic

1 Introduction

In [18], the authors first introduced the formal definitions of fuzzy functional and fuzzy multivalued dependences based on conformance values and the degree of clarity of dependency itself. The inference rules for both types of dependences are enumerated and are proved to be consistent, sound and complete.

In this paper, we join fuzzy formulas to fuzzy dependences described in [18].

For $c_1 \notin C^+$, the completeness of inference rules allows anyone to construct a fuzzy relation instance r_1 such that all $c \in C^+$ are satisfied by r_1 but c_1 is not, where c_1 denotes a fuzzy dependency and C is a set of fuzzy dependences with the closure C^+. In this paper we prove that r_1 may be chosen to have only two elements.

The soundness of inference rules, however, yields that any $c_1 \in C^+$ is satisfied by any r_1 satisfying all $c \in C$. Consequently, we prove that arbitrary c_1 is satisfied by any r_1 satisfying all $c \in C$ if and only if c_1 is satisfied by any two-element r_1 satisfying all $c \in C$. Furthermore, we prove that c_1 is satisfied by any two-element r_1 satisfying all $c \in C$ if and only if $c_1^{'}$ is valid whenever all $c^{'} \in C^{'}$ are valid, where $c_1^{'}$ resp. $C^{'}$ denote the fuzzy formula resp. the set of fuzzy formulas joined to c_1 resp. C (see, [9] for Yager's fuzzy implication operator and [10] for Reichenbach's fuzzy implication operator).

Summarizing what is said above, we can say that the main result of our paper claims that: arbitrary c_1 is satisfied by any r_1 satisfying all $c \in C$ if and only if $c_1^{'}$ is valid whenever all $c^{'} \in C^{'}$ are valid.

The structure of the paper is as follows: Section 2 provides some necessary background and preliminary material. In Section 3 we recall the inference rules, introduce closures, limit strengths of dependences, dependency basis and assemble those facts we will need. In Section 4 we prove a number of auxiliary results related to two-element fuzzy relation instances satisfying actively some fuzzy multivalued dependency. In Section 5 we state and prove the main results. In Section 6 we give concluding remarks.

2 Preliminaries

We use the following operators

$$\tau(a \wedge b) = \min(\tau(a), \tau(b)), \tag{1}$$

$$\tau(a \vee b) = \max(\tau(a), \tau(b)), \tag{2}$$

$\tau(\neg a) = 1 - \tau(a)$, where $\tau(a)$, $\tau(b) \in [0,1]$ and $\tau(x)$ denotes the truth value of the formula x.

An interpretation $\mathscr{I}$ is said to satisfy resp. falsify the formula f if $\tau(f) \geq \frac{1}{2}$ resp. $\tau(f) \leq \frac{1}{2}$ under $\mathscr{I}$ (see, [11, p. 111]).

The notation that will be applied in the sequel follows similarity-based fuzzy relational database approach [18] (see also, [4]–[6]).

Recall that an entity is an object that exists. It does not have to do anything. It just has to exist. An entity can be a single thing, a person or a place. Each entity has its own identity and a number of properties that characterize itself [7, p. 22]. A set of entities that have the same properties constitute an entity type. More freely, an entity type comprises a group of entities that are similar. When referring to the notation of an entity type, each of its entities is often referred to as an entity instance. For example, The University of Oklahoma ia an entity instance of the university entity type. Attributes are the properties that describe entities. For example, if the entity is a faculty, attributes could include name, faculty ID, address and salary. More precisely, attributes (or attribute names) are associated with entity types and attribute values are associated with entity instances. Thus, a particular entity will have a value for each of its attributes.

Let $R(U)$ be a scheme on domains D_1, D_2,..., D_n, where U is the set of all attributes A_1, A_2,..., A_n on D_1, D_2,..., D_n. Here, we assume that the domain of A_i is D_i, $i = 1, 2,..., n$. Moreover, we assume that D_i is a finite set for $i = 1, 2,..., n$.

A fuzzy instance (or fuzzy relation instance) r on $R(U)$ is a subset of the cross product $2^{D_1} \times 2^{D_2} \times ... \times 2^{D_n}$.

A tuple t of r is then of the form

$$(d_1, d_2, ..., d_n),$$

where $d_i \subseteq D_i$, $i = 1, 2,..., n$. Here, we assume that $d_i \neq \emptyset$ for $i = 1, 2,..., n$. Furthermore, we consider d_i the value of A_i on t, each attribute A_i is the name of a role played by domain D_i, and the scheme $R(U)$ describes the structure of a relation. In accordance with what has already been said, we can underline that attributes are associated with schemes and attribute values are associated with fuzzy relation instances.

A fuzzy relation instance r on $R(U)$ can be visibly represented as a two-dimensional table with the table headings $(A_1, A_2, ..., A_n)$ together constituting the scheme $R(U)$, each horizontal row of the table being a tuple of r, and each column of the table containing the attribute values under the corresponding heading.

For example, if $U = \{Name, Age, Height\}$ is the set of all attributes on domains $D_1 = \{Jane, Ana, John\}$, $D_2 = \{n \in \mathbb{N} \mid 15 \leq n \leq 45\}$, $D_3 = \{n \in \mathbb{N} \mid 100 \leq n \leq 220\}$, then, a two-element fuzzy relation instance on scheme $R(U)$ (on domains D_1, D_2 and D_3) is given by Table 1 below.

Table 1:

	Name	Age	Height
t_1	$\{Ana\}$	$\{18, 25, 35\}$	$\{158, 160\}$
t_2	$\{John\}$	$\{43\}$	$\{100, 101, 203\}$

Recall that the relational database model was first introduced by Codd [8]. In such a model, for each $i \in \{1, 2, ..., n\}$, the elements of D_i are mutually unrelated, i.e., they are mutually distinct. Moreover, d_i is a single element od D_i (see, e.g., [7, pp. 3–6]).

The similarity based framework [4, 5] generalizes these two assumptions of the classical relational database model. Namely, it allows each domain to be associated with a similarity relation (instead of just an identity relation). Furthermore, it allows each attribute value to be a subset of the corresponding domain rather than just a single element of the domain.

A similarity relation s_i on D_i is a mapping $s_i : D_i \times D_i \to [0,1]$ such that for all $x, y, z \in D_i$,

$$s_i(x,x) = 1,$$
$$s_i(x,y) = s_i(y,x),$$
$$s_i(x,z) \geq \max_{q \in D_i}(\min(s_i(x,q), s_i(q,z))).$$

In the classical relational database model, two tuples t_1 and t_2 agree on an attribute if and only if the attribute values on t_1 and t_2 are identical. In the similarity based model, similarity relations on domains enable us to define how conformant (or similar) two tuples are on attributes.

Let r be a fuzzy relation instance on $R(U)$, where $R(U) = R(A_1, A_2, ..., A_n)$ is a scheme on domains D_1, D_2,..., D_n, and U is the set of all attributes A_1, A_2,..., A_n (U is the universal set of attributes A_1, A_2,..., A_n). Suppose that s_i is a similarity relation on D_i, $i = 1, 2,..., n$.

The conformance of attribute $A_i \in \{A_1, A_2, ..., A_n\}$ (defined on domain D_i) on any two tuples t_1 and t_2 present in the fuzzy relation instance r and denoted by $\varphi(A_i[t_1, t_2])$, is defined by

$$\varphi(A_i[t_1,t_2]) = \min\left\{ \min_{x\in d_1}\left\{ \max_{y\in d_2}\{s_i(x,y)\}\right\}, \min_{x\in d_2}\left\{ \max_{y\in d_1}\{s_i(x,y)\}\right\}\right\},$$

where d_1 resp. d_2 denote the value of attribute A_i on tuple t_1 resp. t_2.

Let $\alpha \in [0,1]$. If $\varphi(A_i[t_1,t_2]) \geq \alpha$, tuples t_1 and t_2 are said to be conformant on attribute A_i with α.

The conformance of attribute set $X \subseteq \{A_1,A_2,...,A_n\}$ on any two tuples t_1 and t_2 present in the fuzzy relation instance r and denoted by $\varphi(X[t_1,t_2])$ is defined by $\varphi(X[t_1,t_2]) = \min_{A_i\in X}\{\varphi(A_i[t_1,t_2])\}$.

The following statements hold true: (see, [18, pp. 166–167])

1. If $X \supseteq Y$, then $\varphi(Y[t_1,t_2]) \geq \varphi(X[t_1,t_2])$,
2. If $X = \{A_{i_1},A_{i_2},...,A_{i_k}\} \subseteq \{A_1,A_2,...,A_n\}$ and for some $\alpha \in [0,1]$, $\varphi(A_{i_j}[t_1,t_2]) \geq \alpha$ for all $j \in \{1,2,...,k\}$, then $\varphi(X[t_1,t_2]) \geq \alpha$,
3. $\varphi(X[t,t]) = 1$ for any tuple t in r.

Let r be a relation instance on $R(U)$, where $R(U)$ is a scheme on domains D_1, D_2,..., D_n, U is the universal set of attributes A_1, A_2,..., A_n on D_1, D_2,..., D_n. Let $X,Y \subseteq U$.

Recall that the relation instance r is said to satisfy the functional dependency $X \to Y$ if for every pair of tuples t_1 and t_2 in r, $t_1[X] = t_2[X]$ implies $t_1[Y] = t_2[Y]$. Here, $t_i[X]$ ($t_i[Y]$) denotes the values of the set of attributes X (Y) on t_i, $i = 1$, 2.

Furthermore, the relation instance r is said to satisfy the multivalued dependency $X \to\to Y$ if for any pair of tuples t_1 and t_2 in r, $t_1[X] = t_2[X]$ yields that there exists a tuple t_3 in r such that $t_3[X] = t_1[X]$, $t_3[Y] = t_1[Y]$ and $t_3[Z] = t_2[Z]$, where $Z = U \setminus (X \cup Y)$.

Note that the definition of functional dependency is not directly applicable to similarity-based databases. Namely, that definition is based on the concept of equality. Hence, it is not possible to check if two imprecise values are equal. However, it is possible to check if such values are similar (see, [18, p. 165]). If $t_1[X]$ is similar to $t_2[X]$, then we require that $t_1[Y]$ be similar to $t_2[Y]$. Moreover, we require that the similarity between Y values be greater or equal to the similarity between X values.

Consider the following examples: "Employees with similar experiences must have similar salaries" and "The intelligence level of a person more or less determines the degree of success". In the first case, the defined dependency is precise. However, it is not the case in the second example. Such examples force us to introduce the linguistic strength of the dependency. Thus, we can say that the dependency from the first example has linguistic strength $1 \in [0,1]$, while the dependency from the second example has linguistic strength $0.7 \in [0,1]$. In this way, we obtain a method for describing imprecise dependences as well as precise ones. The linguistic strength of the dependency (shorter, strength of the dependency) will be denoted by $\theta \in [0,1]$.

As earlier, D_i will be assumed to denote the domain of the attribute A_i, $i = 1$, 2,..., n. We shall simplify our notation by omitting to mention this fact in the sequel.

Now, we come to the fuzzy functional dependency definition.

Let r be a fuzzy relation instance on scheme $R(A_1,A_2,...,A_n)$, U be the universal set of attributes and $X,Y \subseteq U$.

Fuzzy relation instance r is said to satisfy the fuzzy functional dependency $X \xrightarrow{\theta}_F Y$ if for every pair of tuples t_1 and t_2 in r,

$$\varphi(Y[t_1,t_2]) \geq \min(\theta, \varphi(X[t_1,t_2])).$$

Here, $\theta \in [0,1]$ denotes the linguistic strength of the dependency. When $\theta = 1$, we omit to write it in the dependency notation.

As in the functional dependences case, multivalued dependences face the problem of the inability to handle imprecise attribute values. Reasoning in the same way as in the case of functional dependences, we come to the fuzzy multivalued dependency definition (see, [18, p. 172]).

Let r be a fuzzy relation instance on scheme $R(A_1,A_1,...,A_n)$, U be the universal set of attributes A_1, A_2, ..., A_n and $X,Y \subseteq U$.

Fuzzy relation instance r is said to satisfy the fuzzy multivalued dependency $X \to\xrightarrow{\theta}_F Y$ if for every pair of tuples t_1 and t_2 in r, there exists a tuple t_3 in r such that:

$$\varphi(X[t_3,t_1]) \geq \min(\theta, \varphi(X[t_1,t_2])),$$
$$\varphi(Y[t_3,t_1]) \geq \min(\theta, \varphi(X[t_1,t_2])),$$
$$\varphi(Z[t_3,t_2]) \geq \min(\theta, \varphi(X[t_1,t_2])),$$

where $Z = U \setminus (X \cup Y)$. Here, $\theta \in [0,1]$ denotes the linguistic strength of the dependency. If $\theta = 1$, we omit to write it in the dependency notation.

Fuzzy relation instance r is said to satisfy the fuzzy multivalued dependency $X \to\xrightarrow{\theta}_F Y$, θ-actively if r satisfies $X \to\xrightarrow{\theta}_F Y$ and $\varphi(A[t_1,t_2]) \geq \theta$ for all $A \in X$ and all t_1, $t_2 \in r$.

Obviously, r satisfies $X \to\xrightarrow{\theta}_F Y$, $\theta-$ actively if and only if r satisfies $X \to\xrightarrow{\theta}_F Y$ and $\varphi(X[t_1,t_2]) \geq \theta$ for all t_1, $t_2 \in r$.

3 Inference rules, closures and dependency sets

The authors in [18] derived the inference rules IR1-IR20 for fuzzy functional dependences (shorter $FFDs$) and fuzzy multivalued dependences (shorter $FMVDs$).

There, U denotes some universal set of attributes and X, Y, Z, W denote subsets of U. Moreover, $U - XY$, for example, means $U \setminus (X \cup Y)$.

The inference rules IR1–IR17 are sound (see, [18, pp. 168–176]). In the case of the IR1 rule, for example, this means that any fuzzy relation instance r on $R(U)$, which satisfies $X \xrightarrow{\theta_1}_F Y$, satisfies also $X \xrightarrow{\theta_2}_F Y$, where $\theta_1 \geq \theta_2$.

Let $\mathscr{F}$ and $\mathscr{G}$ be sets of $FFDs$ and $FMVDs$ on some universal set of attributes U, respectively. The closure of $\mathscr{F} \cup \mathscr{G}$, denoted by $(\mathscr{F},\mathscr{G})^+$, is the set of all $FFDs$ and $FMVDs$ that can be derived from $\mathscr{F} \cup \mathscr{G}$ by repeated applications of the rules IR1–IR10. Since these rules are sound, we know that any fuzzy dependency in $(\mathscr{F},\mathscr{G})^+$ is valid in each fuzzy relation instance on $R(U)$ that obeys all dependences in $\mathscr{F} \cup \mathscr{G}$.

The limit strength of a FFD $X \xrightarrow{\theta}_F Y$, with respect to $\mathscr{F}$ and $\mathscr{G}$, is a number $\theta_l \in [0,1]$ such that $X \xrightarrow{\theta_l}_F Y$ is in $(\mathscr{F},\mathscr{G})^+$ and $\theta_l \geq \theta'$ for any $X \xrightarrow{\theta'}_F Y$ in $(\mathscr{F} \cup \mathscr{G})^+$.

The limit strength of a $FMVD$ $X \to\xrightarrow{\theta}_F Y$, with respect to $\mathscr{F}$ and $\mathscr{G}$, is a number $\theta_l \in [0,1]$ such that $X \to\xrightarrow{\theta_l}_F Y$ is in $(\mathscr{F},\mathscr{G})^+$ and $\theta_l \geq \theta'$ for any $X \to\xrightarrow{\theta'}_F Y$ in $(\mathscr{F},\mathscr{G})^+$.

The closure $X^+(\theta)$ of $X \subseteq U$, with respect to $\mathscr{F}$ and $\mathscr{G}$, is the set of all attributes $A \in U$ such that $X \xrightarrow{\theta}_F A$ is in $(\mathscr{F},\mathscr{G})^+$. Consequently, $X \xrightarrow{\theta}_F Y$ belongs to $(\mathscr{F},\mathscr{G})^+$ if and only if $Y \subseteq X^+(\theta)$.

By [18, p. 177, Lemma 4.4.], for given $X \subseteq U$ and $\theta \in [0,1]$, we can partition U into sets of attributes Y_1, Y_2, ... , Y_k, such that $X \to\xrightarrow{\theta}_F Z$ if and only if Z is the union of some of the Y_i's. We call the set $dep(X,\theta) = \{Y_1,Y_2,...,Y_k\}$, constructed for X with respect to θ, the dependency basis for X.

By [18, p. 177, Th. 4.2.], the inference rules IR1–IR10 are complete. This means the following.

Let $\mathscr{F}$ and $\mathscr{G}$ be some sets of $FFDs$ and $FMVDs$ on some universal set of attributes U. Suppose that some FFD $X \xrightarrow{\theta}_F Y$ on U ($FMVD$ $X \to\xrightarrow{\theta}_F Y$ on U) does not belong to $(\mathscr{F},\mathscr{G})^+$. Then, there exists a fuzzy relation instance r^* on $R(U)$ which satisfies all dependences in $(\mathscr{F},\mathscr{G})^+$ and does not satisfy the dependency $X \xrightarrow{\theta}_F Y$ ($X \to\xrightarrow{\theta}_F Y$).

Recall that the authors in [18, pp. 177–178] constructed r^* as follows (see, Table 2).

Let W_1, W_2, ... , W_m, where $m \geq 1$, be the sets in the dependency basis $dep(X,\theta)$ of X with respect to θ, that cover $U \setminus X^+(\theta)$. Thus, $X^+(\theta)$, W_1, W_2, ... , W_m form a partition of U. Here, $X^+(\theta)$ is the closure of X with respect to $\mathscr{F}$ and $\mathscr{G}$. Note that $X^+(\theta)$ is a proper subset of U since otherwise each FFD or $FMVD$ with left side X belongs to $(\mathscr{F},\mathscr{G})^+$.

Table 2:

$X^+(\theta)$	W_1	...	W_{m-1}	W_m
$a, ... , a$	$a, ... , a$	...	$a, ... , a$	$a, ... , a$
$a, ... , a$	$a, ... , a$	...	$a, ... , a$	$b, ... , b$
$\vdots$	$\vdots$	...	$\vdots$	$\vdots$
$a, ... , a$	$b, ... , b$	...	$b, ... , b$	$a, ... , a$
$a, ... , a$	$b, ... , b$	...	$b, ... , b$	$b, ... , b$

The set $\{a,b\}$ is chosen to be the domain of each of the attributes in U. A similarity relation s ($s = s_1 = s_2 = ... = s_n$ now if $U = \{A_1,A_2,...,A_n\}$) is defined by $s(a,b) = \theta'$, where θ' is a number less than θ and greater than or equal to the strength of any dependency in $(\mathscr{F},\mathscr{G})^+$ whose limit strength is less than θ. If there are no such dependences, then $\theta' = 0$.

Note that $\varphi(Q[t_1,t_2]) \geq \theta'$ for any attribute set $Q \subseteq U$ and any two tuples t_1 and t_2 in r^*.

Each attribute in $X^+(\theta)$ has the value a in all rows of r^*. Every row in r^* corresponds to some finite sequence of the length m, whose elements belong to $\{a,b\}$. For example, the row that corresponds to the sequence $(a,a,...,a,b)$ has all a's in the $X^+(\theta)$ columns, all a's in the W_i columns, $i \in \{1,2,...,m-1\}$, all b's in the W_m columns. Since the number of such sequences is 2^m, the fuzzy relation instance r^* has 2^m rows.

4 Auxiliary results

In the earlier literature, different authors proposed many individual definitions of fuzzy implications (see, e.g., [17, p. 3]).

Besides these individual definitions, two classes of fuzzy implications, generated from the fuzzy logic operators: negation, conjunction and disjunction have emerged. They are strong implications (S) and residuated implications (R).

Besides (S) and (R) implications, there is another class of fuzzy implications generated from the fuzzy logic operators: negation, conjunction and disjunction coming from quantum logic. Such implications are called quantum logic implications (QL).

(S), (R) and (QL) implications are the most important classes of fuzzy implications which were widely studied from the beginning until now.

Thus, in [1] and [2], the authors work on the characterization of (S) implications generated from continuous negations. [12], [19] and [16] deals with the properties of a group of (QL) implications. The authors in [14] and [15] work on (R) implications and left-continuous t-norms, etc.

Besides (S), (R) and (QL) implications, there are other classes of fuzzy implications which are not generated from the fuzzy logic operators: negation, conjunction and disjunction. Examples of such classes are: two parameterized classes of fuzzy implications generated from aditive generating functions [20], fuzzy implications generated from uninorms [3], etc.

For the intersections between classes of fuzzy implication operators we refer to [3] and [13]. For detailed general study on the fuzzy implication operators we refer to [3] and [17].

Note that here and through the rest of the paper we shall assume that the fuzzy implication operator is given by

$$\tau(a \Rightarrow b) = \max(1 - \tau(a), \tau(b)), \qquad (3)$$

where, as before, $\tau(a)$, $\tau(b) \in [0,1]$ and $\tau(x)$ denotes the truth value of the formula x.

Recall that this fuzzy implication operator is widely-known as Kleene-Dienes or (KD) operator. It represents a classical example of (S) and (QL) implications.

Let $r = \{t_1, t_2\}$ be any two-element fuzzy relation instance on scheme $R(A_1, A_2, ..., A_n)$, U be the universal set of attributes A_1, A_2, ..., A_n, and $\beta \in [0,1]$.

A valuation joined to r and β is a mapping $i_{r,\beta} : \{A_1, A_2, ..., A_n\} \to [0,1]$ such that

$$i_{r,\beta}(A_k) > \frac{1}{2} \quad \text{if} \quad \varphi(A_k[t_1,t_2]) \geq \beta,$$
$$i_{r,\beta}(A_k) \leq \frac{1}{2} \quad \text{if} \quad \varphi(A_k[t_1,t_2]) < \beta,$$

$k \in \{1, 2, ..., n\}$. Note that here, as well as in the fuzzy functional and fuzzy multivalued dependences definitions, we assume that each attribute domain is equipped with a similarity relation. We apply these relations to calculate conformances $\varphi(A_k[t_1,t_2])$, $k \in \{1, 2, ..., n\}$. Comparing these values to β, we define the values $i_{r,\beta}(A_k)$, $k \in \{1, 2, ..., n\}$. More precisely, for $\varphi(A_k[t_1,t_2]) \geq \beta$, we put $i_{r,\beta}(A_k)$ to be some value in $\left(\frac{1}{2}, 1\right]$. Otherwise, we put $i_{r,\beta}(A_k)$ to be some value in $\left[0, \frac{1}{2}\right]$. In this way, we consider the attributes $A_1, A_2, ..., A_n$ as fuzzy formulas with respect to $i_{r,\beta}$. Now, for A, $B \in \{A_1, A_2, ..., A_n\}$, it makes sense to consider $A \wedge B$, $A \vee B$ and $A \Rightarrow B$ as fuzzy formulas with respect to $i_{r,\beta}$ if we put $i_{r,\beta}(A \wedge B)$, $i_{r,\beta}(A \vee B)$ and $i_{r,\beta}(A \Rightarrow B)$ to be in line with (1), (2) and (3), respectively, i.e., if we put that

$$i_{r,\beta}(A \wedge B) = \min\left(i_{r,\beta}(A), i_{r,\beta}(B)\right),$$
$$i_{r,\beta}(A \vee B) = \max\left(i_{r,\beta}(A), i_{r,\beta}(B)\right),$$
$$i_{r,\beta}(A \Rightarrow B) = \max\left(1 - i_{r,\beta}(A), i_{r,\beta}(B)\right).$$

Consequently, expressions like $\wedge_{A \in X} A$, $\vee_{B \in Y} B$, $(\wedge_{B \in Y} B) \vee (\wedge_{C \in Z} C)$, $(\wedge_{A \in X} A) \Rightarrow (\wedge_{B \in Y} B)$, $(\wedge_{A \in X} A) \Rightarrow ((\wedge_{B \in Y} B) \vee (\wedge_{C \in Z} C))$, etc., where X, Y, $Z \subseteq U$, become fuzzy formulas as well.

Let $X \xrightarrow{\theta}_F Y$ $\left(X \rightarrow\xrightarrow{\theta}_F Y\right)$ be some FFD ($FMVD$) on U.

We join the fuzzy formula

$$(\wedge_{A \in X} A) \Rightarrow (\wedge_{B \in Y} B) \quad \text{resp.}$$
$$(\wedge_{A \in X} A) \Rightarrow ((\wedge_{B \in Y} B) \vee (\wedge_{C \in Z} C))$$

to the fuzzy functional dependency $X \xrightarrow{\theta}_F Y$ resp. the fuzzy multivalued dependency $\left(X \rightarrow\xrightarrow{\theta}_F Y\right)$, where $Z = U \setminus (X \cup Y)$.

Theorem 1 *Let $r = \{t_1, t_2\}$ be any two-element, fuzzy relation instance on scheme $R(A_1, A_2, ..., A_n)$, U be the universal set of attributes A_1, A_2, ..., A_n and X, Y be subsets of U. Let $Z = U \setminus (X \cup Y)$. Then, r satisfies the fuzzy multivalued dependency $X \rightarrow\xrightarrow{\theta}_F Y$, $\theta-$actively if and only if*

$$\varphi(X[t_1,t_2]) \geq \theta, \ \ \varphi(Y[t_1,t_2]) \geq \theta \quad or$$
$$\varphi(X[t_1,t_2]) \geq \theta, \ \ \varphi(Z[t_1,t_2]) \geq \theta.$$

Proof: $(\Rightarrow)$ Assume that r satisfies $X \rightarrow\xrightarrow{\theta}_F Y$, $\theta-$actively. Then, as already mentioned, we have that

$\varphi(X[t_1,t_2]) \geq \theta$. Since r satisfies $X \to\overset{\theta}{\to}_F Y$, there exists a tuple $t_3 \in r$ such that

$$\varphi(X[t_3,t_1]) \geq \min(\theta, \varphi(X[t_1,t_2])) = \theta,$$
$$\varphi(Y[t_3,t_1]) \geq \min(\theta, \varphi(X[t_1,t_2])) = \theta,$$
$$\varphi(Z[t_3,t_2]) \geq \min(\theta, \varphi(X[t_1,t_2])) = \theta.$$

If $t_3 = t_1$, then $\varphi(Z[t_1,t_2]) \geq \theta$. If $t_3 = t_2$, we have that $\varphi(Y[t_1,t_2]) \geq \theta$.
$(\Leftarrow)$ Suppose that either conditions $\varphi(X[t_1,t_2]) \geq \theta$, $\varphi(Y[t_1,t_2]) \geq \theta$ or $\varphi(X[t_1,t_2]) \geq \theta$, $\varphi(Z[t_1,t_2]) \geq \theta$ hold true.

Let $\varphi(X[t_1,t_2]) \geq \theta$ and $\varphi(Y[t_1,t_2]) \geq \theta$ hold true.
Then, there exists $t_3 \in r$, $t_3 = t_2$ such that

$$\varphi(X[t_3,t_1]) \geq \theta = \min(\theta, \varphi(X[t_1,t_2])),$$
$$\varphi(Y[t_3,t_1]) \geq \theta = \min(\theta, \varphi(X[t_1,t_2])),$$
$$\varphi(Z[t_3,t_2]) = 1 \geq \theta = \min(\theta, \varphi(X[t_1,t_2])).$$

Similarly, if $\varphi(X[t_1,t_2]) \geq \theta$ and $\varphi(Z[t_1,t_2]) \geq \theta$ hold true, then there exists $t_3 \in r$, $t_3 = t_1$ such that

$$\varphi(X[t_3,t_1]) \geq \min(\theta, \varphi(X[t_1,t_2])),$$
$$\varphi(Y[t_3,t_1]) \geq \min(\theta, \varphi(X[t_1,t_2])),$$
$$\varphi(Z[t_3,t_2]) \geq \min(\theta, \varphi(X[t_1,t_2])).$$

Therefore, r satisfies $X \to\overset{\theta}{\to}_F Y$. Since $\varphi(X[t_1,t_2]) \geq \theta$, we conclude that r satisfies $X \to\overset{\theta}{\to}_F Y$, $\theta-$actively. This completes the proof. ◻

Theorem 2 *Let $r = \{t_1,t_2\}$ be any two-element, fuzzy relation instance on scheme $R(A_1,A_2,...,A_n)$, U be the universal set of attributes A_1, A_2, ..., A_n and X, Y be subsets of U. Let $Z = U \setminus (X \cup Y)$. Then, r satisfies the fuzzy multivalued dependency $X \to\overset{\theta}{\to}_F Y$, $\theta-$actively if and only if $\varphi(X[t_1,t_2]) \geq \theta$ and $i_{r,\theta}(\mathscr{H}) > 0.5$, where $\mathscr{H}$ denotes the fuzzy formula $(\wedge_{A\in X}A) \Rightarrow ((\wedge_{B\in Y}B) \vee (\wedge_{C\in Z}C))$ associated to $X \to\overset{\theta}{\to}_F Y$.*

Proof: $(\Rightarrow)$ Assume that r satisfies the dependency $X \to\overset{\theta}{\to}_F Y$, $\theta-$actively. By Theorem 1, $\varphi(X[t_1,t_2]) \geq \theta$, $\varphi(Y[t_1,t_2]) \geq \theta$ or $\varphi(X[t_1,t_2]) \geq \theta$, $\varphi(Z[t_1,t_2]) \geq \theta$.

Let $\varphi(X[t_1,t_2]) \geq \theta$ and $\varphi(Y[t_1,t_2]) \geq \theta$ hold true.
Since $\min_{B\in Y}\{\varphi(B[t_1,t_2])\} = \varphi(Y[t_1,t_2]) \geq \theta$ hold true, we conclude that $\varphi(B[t_1,t_2]) \geq \theta$ for all $B \in Y$. Hence, $i_{r,\theta}(B) > 0.5$ for $B \in Y$. Therefore, $i_{r,\theta}(\wedge_{B\in Y}B) = \min\{i_{r,\theta}(B) \,|\, B \in Y\} > 0.5$. We have,

$$\begin{aligned} & i_{r,\theta}(\mathscr{H}) \\ &= \max(1 - i_{r,\theta}(\wedge_{A\in X}A), i_{r,\theta}((\wedge_{B\in Y}B) \vee (\wedge_{C\in Z}C))) \\ &= \max\big(1 - \min\{i_{r,\theta}(A) \,|\, A \in X\}, \\ &\quad \max(i_{r,\theta}(\wedge_{B\in Y}B), i_{r,\theta}(\wedge_{C\in Z}C))\big) \\ &= \max\big(1 - \min\{i_{r,\theta}(A) \,|\, A \in X\}, \\ &\quad i_{r,\theta}(\wedge_{B\in Y}B), i_{r,\theta}(\wedge_{C\in Z}C)\big). \end{aligned}$$

Since, $i_{r,\theta}(\wedge_{B\in Y}B) > 0.5$, it follows immediately that $i_{r,\theta}(\mathscr{H}) > 0.5$

Similarly, if we assume that $\varphi(X[t_1,t_2]) \geq \theta$ and $\varphi(Z[t_1,t_2]) \geq \theta$ hold true, we obtain that $i_{r,\theta}(\wedge_{C\in Z}C) > 0.5$ and hence $i_{r,\theta}(\mathscr{H}) > 0.5$.
The assertion follows.
$(\Leftarrow)$ Let $\varphi(X[t_1,t_2]) \geq \theta$ and $i_{r,\theta}(\mathscr{H}) > 0.5$. Suppose that $i_{r,\theta}(\mathscr{H}) = 1 - \min\{i_{r,\theta}(A) \,|\, A \in X\}$. Then, there exists $A_0 \in X$ such that $1 - i_{r,\theta}(A_0) = 1 - \min\{i_{r,\theta}(A) \,|\, A \in X\} = i_{r,\theta}(\mathscr{H}) > 0.5$. We obtain, $i_{r,\theta}(A_0) < 0.5$. Therefore, $\varphi(A_0[t_1,t_2]) < \theta$ and hence $\varphi(X[t_1,t_2]) = \min_{A\in X}\{\varphi(A[t_1,t_2])\} < \theta$, which contradicts our assumption $\varphi(X[t_1,t_2]) \geq \theta$. As a consequence, we have that either $i_{r,\theta}(\mathscr{H}) = i_{r,\theta}(\wedge_{B\in Y}B)$ or $i_{r,\theta}(\mathscr{H}) = i_{r,\theta}(\wedge_{C\in Z}C)$.

Assume that $\min\{i_{r,\theta}(B) \,|\, B \in Y\} = i_{r,\theta}(\wedge_{B\in Y}B) = i_{r,\theta}(\mathscr{H}) > 0.5$. We obtain $i_{r,\theta}(B) > 0.5$ for all $B \in Y$. Hence, $\varphi(B[t_1,t_2]) \geq \theta$ for $B \in Y$. This implies that $\varphi(Y[t_1,t_2]) = \min_{B\in Y}\{\varphi(B[t_1,t_2])\} \geq \theta$. Now, by Theorem 1, $\varphi(X[t_1,t_2]) \geq \theta$ and $\varphi(Y[t_1,t_2]) \geq \theta$ yield the result.

Similarly, if we assume that $i_{r,\theta}(\mathscr{H}) = i_{r,\theta}(\wedge_{C\in Z}C)$, we obtain that $\varphi(Z[t_1,t_2]) \geq \theta$. Since $\varphi(X[t_1,t_2]) \geq \theta$, the theorem follows. This completes the proof. ◻

Theorem 3 *Let $r = \{t_1,t_2\}$ and $q = \{u_1,u_2\}$ be any two, two-element, fuzzy relation instances on scheme $R(A_1,A_2,...,A_n)$. Let U be the universal set of attributes A_1, A_2, ..., A_n and X, Y subsets of U. Put $Z = U \setminus (X \cup Y)$. Assume that r satisfies the fuzzy multivalued dependency $X \to\overset{\theta}{\to}_F Y$, $\theta-$actively. Suppose that $\varphi(A[u_1,u_2]) \geq \theta$ for every attribute $A \in U$ such that $\varphi(A[t_1,t_2]) \geq \theta$. Then, q satisfies $X \to\overset{\theta}{\to}_F Y$, $\theta-$actively.*

Proof: Since r satisfies $X \to\overset{\theta}{\to}_F Y$, $\theta-$actively, it follows from Theorem 1 that either conditions

$\varphi(X[t_1,t_2]) \geq \theta$, $\varphi(Y[t_1,t_2]) \geq \theta$ or $\varphi(X[t_1,t_2]) \geq \theta$, $\varphi(Z[t_1,t_2]) \geq \theta$ hold true.

Suppose that $\varphi(X[t_1,t_2]) \geq \theta$ and $\varphi(Y[t_1,t_2]) \geq \theta$ hold true. We have, $\min_{A\in X}\{\varphi(A[t_1,t_2])\} = \varphi(X[t_1,t_2]) \geq \theta$ and $\min_{B\in Y}\{\varphi(B[t_1,t_2])\} = \varphi(Y[t_1,t_2]) \geq \theta$. Hence, $\varphi(A[t_1,t_2]) \geq \theta$ for all $A \in X$ and $\varphi(B[t_1,t_2]) \geq \theta$ for all $B \in Y$. Therefore, by our assumption, $\varphi(A[u_1,u_2]) \geq \theta$ for all $A \in X$ and $\varphi(B[u_1,u_2]) \geq \theta$ for all $B \in Y$. Now, $\varphi(X[u_1,u_2]) = \min_{A\in X}\{\varphi(A[u_1,u_2])\} \geq \theta$ and $\varphi(Y[u_1,u_2]) = \min_{B\in Y}\{\varphi(B[u_1,u_2])\} \geq \theta$. Hence, by Theorem 1, q satisfies $X \rightarrow\xrightarrow{\theta}_F Y$, $\theta-$actively.

If $\varphi(X[t_1,t_2]) \geq \theta$ and $\varphi(Z[t_1,t_2]) \geq \theta$ hold true, the theorem follows analogously. This completes the proof. ◻

Lemma 4 *Let r be any fuzzy relation instance on scheme $R(A_1,A_2,...,A_n)$, U be the universal set of attributes A_1, A_2, ..., A_n and X, Y be subsets of U. Then, r satisfies the fuzzy functional dependency $X \xrightarrow{\theta}_F Y$ if and only if r satisfies $X \xrightarrow{\theta}_F B$ for all $B \in Y$.*

Proof: $(\Rightarrow)$ Suppose that r satisfies the dependency $X \xrightarrow{\theta}_F Y$. Then, $\varphi(B[t_1,t_2]) \geq \varphi(Y[t_1,t_2]) \geq \min(\theta, \varphi(X[t_1,t_2]))$ for any $B \in Y$ and any t_1 and t_2 in r. Hence, r satisfies $X \xrightarrow{\theta}_F B$ for all $B \in Y$.

$(\Leftarrow)$ Suppose that r does not satisfy the dependency $X \xrightarrow{\theta}_F Y$. Then, there exist tuples t_1 and t_2 in r such that $\varphi(Y[t_1,t_2]) < \min(\theta, \varphi(X[t_1,t_2]))$. Since $\varphi(Y[t_1,t_2]) = \min_{B\in Y}\{\varphi(B[t_1,t_2])\}$, we conclude that there exists $B \in Y$ such that $\varphi(B[t_1,t_2]) = \varphi(Y[t_1,t_2])$. Now, $\varphi(B[t_1,t_2]) < \min(\theta, \varphi(X[t_1,t_2]))$. Hence, r does not satisfy the dependency $X \xrightarrow{\theta}_F B$. This completes the proof. ◻

5 Main result

Firstly, we prove the following result regarding the fuzzy relation instance r^* introduced in Section 3.

Theorem 5 *There exists some two-element, fuzzy relation instance $s \subseteq r^*$ such that s satisfies all dependences from the set $(\mathscr{F},\mathscr{G})^+$ but does not satisfy the dependency $X \xrightarrow{\theta}_F Y$ $\left(X \rightarrow\xrightarrow{\theta}_F Y\right)$.*

Proof: We distinguish between two cases:

1) r^* does not satisfy the fuzzy functional dependency $X \xrightarrow{\theta}_F Y$,

2) r^* does not satisfy the fuzzy multivalued dependency $X \rightarrow\xrightarrow{\theta}_F Y$.

1) Suppose that r^* does not satisfy the dependency $X \xrightarrow{\theta}_F Y$. Then, $\varphi(Y[u_1,u_2]) < \min(\theta, \varphi(X[u_1,u_2]))$ for some u_1, $u_2 \in r^*$. Now, $\{u_1,u_2\} \subseteq r^*$ is some two-element, fuzzy relation instance which does not satisfy the dependency $X \xrightarrow{\theta}_F Y$. Since r^* satisfies all fuzzy functional dependences from the set $(\mathscr{F},\mathscr{G})^+$, we immediately obtain that each, two- element, fuzzy relation instance $s^* \subseteq r^*$ satisfies all fuzzy functional dependences from $(\mathscr{F},\mathscr{G})^+$.

Let $\{t_1,t_2\} = s \subseteq r^*$ be two element, fuzzy relation instance such that s does not satisfy the dependency $X \xrightarrow{\theta}_F Y$ and s actively satisfies the maximal number of the fuzzy multivalued dependences in $(\mathscr{F},\mathscr{G})^+$. In the sequel, we prove that s satisfies the claim of the theorem.

Let $V \rightarrow\xrightarrow{\theta_1}_F W$ be any fuzzy multivalued dependency from the set $(\mathscr{F},\mathscr{G})^+$.

Suppose that $\varphi(V[t_1,t_2]) < \theta_1$.

Put $Z = U \setminus (V \cup W)$. Now, $\varphi(V[t_1,t_2]) < \theta_1 \leq 1$ and then $\theta' = \varphi(V[t_1,t_2]) < \theta_1$. We obtain

$$\varphi(V[t_1,t_1]) = 1 \geq \min(\theta_1, \varphi(V[t_1,t_2])),$$
$$\varphi(W[t_1,t_1]) = 1 \geq \min(\theta_1, \varphi(V[t_1,t_2])),$$
$$\varphi(Z[t_1,t_2]) \geq \theta' = \min(\theta_1, \varphi(V[t_1,t_2])).$$

Hence, s satisfies the dependency $V \rightarrow\xrightarrow{\theta_1}_F W$.

Let $\varphi(V[t_1,t_2]) \geq \theta_1$.

Suppose that s does not actively satisfy the dependency $V \rightarrow\xrightarrow{\theta_1}_F W$. Then, by Theorem 1, $\varphi(W[t_1,t_2]) < \theta_1$ and $\varphi(Z[t_1,t_2]) < \theta_1$. Hence, $\varphi(W[t_1,t_2]) = \theta' < \theta_1$ and $\varphi(Z[t_1,t_2]) = \theta' < \theta_1$. Now, $\varphi(V[t_1,t_2]) \geq \theta_1 > \theta'$.

Since s does not satisfy the dependency $X \xrightarrow{\theta}_F Y$, it follows from Lemma 4 that s does not satisfy $X \xrightarrow{\theta}_F B$ for some $B \in Y$. Therefore, $\varphi(B[t_1,t_2]) < \min(\theta, \varphi(X[t_1,t_2]))$.

By construction of the relation instance r^* we know that $\varphi(X[t_1,t_2]) = 1$. Hence, $\varphi(B[t_1,t_2]) < \theta$. Now, $\varphi(B[t_1,t_2]) < \theta \leq 1$ and then $\theta' = \varphi(B[t_1,t_2]) < \theta$. Since, $\varphi(V[t_1,t_2]) > \theta'$ and $\varphi(W[t_1,t_2]) = \theta'$, $\varphi(Z[t_1,t_2]) = \theta'$, we conclude that either $B \in W$ or $B \in Z$. Without loss of generality, we may assume that $B \in Z$.

By construction of the instance r^*, there exists some tuple $t_3 \in r^*$ which coincides with the tuple t_1 on V, W and coincides with the tuple t_2 on Z.

Put $q = \{t_1,t_3\}$.

Since $B \in Z$, $\varphi(B[t_1,t_2]) = \theta'$ and t_2, t_3 coincide on Z, we obtain $\varphi(B[t_1,t_3]) = \theta'$. Hence, $\varphi(B[t_1,t_3])$

$= \theta' < \theta = \min(\theta, 1) = \min(\theta, \varphi(X[t_1, t_3]))$, i.e., q does not satisfy the dependency $X \xrightarrow{\theta}_F B$. Now, by Lemma 4, q does not satisfy the dependency $X \xrightarrow{\theta}_F Y$.

Since t_1 and t_3 coincide on V and W, we have that $\varphi(V[t_1, t_3]) = 1 \geq \theta_1$, $\varphi(W[t_1, t_3]) = 1 \geq \theta_1$. Hence, by Theorem 1, the instance q actively satisfies the dependency $V \rightarrow\xrightarrow{\theta_1}_F W$.

Let $P \rightarrow\xrightarrow{\theta_2}_F Q$ be any fuzzy multivalued dependency from the set $(\mathscr{F}, \mathscr{G})^+$ which is actively satisfied by the instance s.

Suppose that $\varphi(A[t_1, t_2]) \geq \theta_2$ for some attribute $A \in U$. If $A \in V \cup W$, then $\varphi(A[t_1, t_3]) = 1 \geq \theta_2$. If $A \in Z$, then $\varphi(A[t_1, t_3]) = \varphi(A[t_1, t_2]) \geq \theta_2$. Therefore, by Theorem 3, q actively satisfies the dependency $P \rightarrow\xrightarrow{\theta_2}_F Q$. Now, q actively satisfies more fuzzy multivalued dependences from the set $(\mathscr{F}, \mathscr{G})^+$ than s does, i.e., a contradiction. Therefore, s actively satisfies the dependency $V \rightarrow\xrightarrow{\theta_1}_F W$. As mentioned before, this implies that s satisfies the dependency $V \rightarrow\xrightarrow{\theta_1}_F W$. In any case, i.e., either $\varphi(V[t_1, t_2]) < \theta_1$ or $\varphi(V[t_1, t_2]) \geq \theta_1$, the instance s satisfies the dependency $V \rightarrow\xrightarrow{\theta_1}_F W$. Since, $V \rightarrow\xrightarrow{\theta_1}_F W$ was arbitrary, we conclude that the instance s satisfies all fuzzy multivalued dependences from the set $(\mathscr{F}, \mathscr{G})^+$. This completes the proof in this case.

2) Suppose that r^* does not satisfy the dependency $X \rightarrow\xrightarrow{\theta}_F Y$. Put $Z = U \setminus (X \cup Y)$. Then, there exist some tuples u_1 and u_2 in r^* such that the conditions

$$\varphi(X[u_3, u_1]) \geq \min(\theta, \varphi(X[u_1, u_2])),$$
$$\varphi(Y[u_3, u_1]) \geq \min(\theta, \varphi(X[u_1, u_2])),$$
$$\varphi(Z[u_3, u_2]) \geq \min(\theta, \varphi(X[u_1, u_2]))$$

don't hold at the same time for any $u_3 \in r^*$. Now, $\{u_1, u_2\} \subseteq r^*$ is a two-element, fuzzy relation instance which does not satisfy the dependency $X \rightarrow\xrightarrow{\theta}_F Y$. Let $\{t_1, t_2\} = s \subseteq r^*$ be two-element, fuzzy relation instance such that the conditions

$$\begin{aligned} \varphi\left(X\left[t_3', t_1\right]\right) &\geq \min(\theta, \varphi(X[t_1, t_2])), \\ \varphi\left(Y\left[t_3', t_1\right]\right) &\geq \min(\theta, \varphi(X[t_1, t_2])), \\ \varphi\left(Z\left[t_3', t_2\right]\right) &\geq \min(\theta, \varphi(X[t_1, t_2])) \end{aligned} \tag{4}$$

don't hold at the same time for any $t_3' \in r^*$ and s actively satisfies the maximal number of the fuzzy multivalued dependences from the set $(\mathscr{F}, \mathscr{G})^+$. The instance s does not satisfy the dependency $X \rightarrow\xrightarrow{\theta}_F Y$. In the sequel, we prove that s satisfies the claim of the theorem.

As in the previous case, we know that s satisfies all fuzzy functional dependences from the set $(\mathscr{F}, \mathscr{G})^+$.

If s satisfies all fuzzy multivalued dependences from the set $(\mathscr{F}, \mathscr{G})^+$, then s satisfies the claim of the theorem.

Suppose that s does not satisfy the fuzzy multivalued dependency $V \rightarrow\xrightarrow{\theta_1}_F W$ from the set $(\mathscr{F}, \mathscr{G})^+$. Let $Q = U \setminus (V \cup W)$. Now, the conditions

$$\varphi(V[t_1, t_1]) \geq \min(\theta_1, \varphi(V[t_1, t_2])),$$
$$\varphi(W[t_1, t_1]) \geq \min(\theta_1, \varphi(V[t_1, t_2])),$$
$$\varphi(Q[t_1, t_2]) \geq \min(\theta_1, \varphi(V[t_1, t_2]))$$

don't hold at the same time. Since the first and the second condition are obviously satisfied, we obtain $\varphi(Q[t_1, t_2]) < \min(\theta_1, \varphi(V[t_1, t_2]))$. Hence, $\varphi(Q[t_1, t_2]) < \min(\theta_1, \varphi(V[t_1, t_2])) \leq 1$ and then $\varphi(Q[t_1, t_2]) = \theta'$, $\varphi(V[t_1, t_2]) > \theta'$. Similarly, the conditions

$$\varphi(V[t_2, t_1]) \geq \min(\theta_1, \varphi(V[t_1, t_2])),$$
$$\varphi(W[t_2, t_1]) \geq \min(\theta_1, \varphi(V[t_1, t_2])),$$
$$\varphi(Q[t_2, t_2]) \geq \min(\theta_1, \varphi(V[t_1, t_2]))$$

don't hold at the same time. Since the first and the third condition are satisfied, we have $\varphi(W[t_1, t_2]) < \min(\theta_1, \varphi(V[t_1, t_2]))$. Therefore, $\varphi(W[t_1, t_2]) < \min(\theta_1, \varphi(V[t_1, t_2])) \leq 1$ and then $\varphi(W[t_1, t_2]) = \theta'$.

By construction of the relation instance r^*, there exist tuples t_3, $t_4 \in r^*$ such that t_3 resp. t_4 coincides with t_1 on V, Q resp. V, W and coincides with t_2 on W resp. Q.

Put $q_1 = \{t_1, t_3\}$ and $q_2 = \{t_1, t_4\}$. Since t_1 and t_3 coincide on V and Q, we have that $\varphi(V[t_1, t_3]) = 1 \geq \theta_1$, $\varphi(Q[t_1, t_3]) = 1 \geq \theta_1$. Hence, by Theorem 1, the instance q_1 actively satisfies the dependency $V \rightarrow\xrightarrow{\theta_1}_F W$. Similarly, t_1 and t_4 coincide on V and W. Then, $\varphi(V[t_1, t_4]) = 1 \geq \theta_1$, $\varphi(W[t_1, t_4]) = 1 \geq \theta_1$. Therefore, by Theorem 1, the instance q_2 actively satisfies the dependency $V \rightarrow\xrightarrow{\theta_1}_F W$.

Let $P \rightarrow\xrightarrow{\theta_2}_F T$ be any fuzzy multivalued dependency from the set $(\mathscr{F}, \mathscr{G})^+$ which is actively satisfied by s. Suppose that $\varphi(A[t_1, t_2]) \geq \theta_2$ for some attribute $A \in U$. If $A \in V \cup Q$, then $\varphi(A[t_1, t_3]) = 1 \geq \theta_2$. If $A \in W$, then $\varphi(A[t_1, t_3]) = \varphi(A[t_1, t_2]) \geq \theta_2$.

Therefore, by Theorem 3, q_1 actively satisfies the dependency $P \to\xrightarrow{\theta_2}_F T$. Similarly, if $A \in V \cup W$, then $\varphi(A[t_1,t_4]) = 1 \geq \theta_2$. If $A \in Q$, then $\varphi(A[t_1,t_4]) = \varphi(A[t_1,t_2]) \geq \theta_2$. Hence, q_2 actively satisfies the dependency $P \to\xrightarrow{\theta_2}_F T$. Now, the instances q_1 and q_2 actively satisfy more fuzzy multivalued dependences from the set $(\mathscr{F},\mathscr{G})^+$ than s does. Hence, if q_1 or q_2 does not satisfy the dependency $X \to\xrightarrow{\theta}_F Y$, we obtain a contradiction.

Suppose that q_1 and q_2 satisfy the dependency $X \to\xrightarrow{\theta}_F Y$. By construction of the relation instance r^* we have that $\varphi(X[t_1,t_3]) = 1 \geq \theta$ and $\varphi(X[t_1,t_4]) = 1 \geq \theta$. Hence, q_1 and q_2 actively satisfy the dependency $X \to\xrightarrow{\theta}_F Y$. Now, by Theorem 1, either conditions $\varphi(X[t_1,t_3]) \geq \theta$, $\varphi(Y[t_1,t_3]) \geq \theta$ or $\varphi(X[t_1,t_3]) \geq \theta$, $\varphi(Z[t_1,t_3]) \geq \theta$ hold true, and either conditions $\varphi(X[t_1,t_4]) \geq \theta$, $\varphi(Y[t_1,t_4]) \geq \theta$ or $\varphi(X[t_1,t_4]) \geq \theta$, $\varphi(Z[t_1,t_4]) \geq \theta$ hold true.

Since $\varphi(V[t_1,t_2]) > \theta'$, $\varphi(W[t_1,t_2]) = \theta'$ and $\varphi(Q[t_1,t_2]) = \theta'$, we introduce the sets

$$W^* = \left\{A \in W \,|\, \varphi(A[t_1,t_2]) = \theta'\right\},$$
$$Q^* = \left\{A \in Q \,|\, \varphi(A[t_1,t_2]) = \theta'\right\}.$$

The conditions (4) don't hold at the same time for any $t_3' \in r^*$. In particular, if $t_3' = t_1$, the first and the second condition in (4) are obviously satisfied. Hence, $\varphi(Z[t_1,t_2]) < \min(\theta, \varphi(X[t_1,t_2])) = \min(\theta,1) = \theta$. Therefore, $\varphi(Z[t_1,t_2]) < \theta \leq 1$ and then $\theta' = \varphi(Z[t_1,t_2]) < \theta$.

Now, if $\varphi(Y[t_1,t_3]) \geq \theta$, we have that $\varphi(Y[t_1,t_3]) \geq \theta > \theta'$. Then, $W^* \subseteq Z$. If $\varphi(Z[t_1,t_3]) \geq \theta$ then $W^* \subseteq Y$. Similarly, if $\varphi(Y[t_1,t_4]) \geq \theta$ then $Q^* \subseteq Z$. Finally, if $\varphi(Z[t_1,t_4]) \geq \theta$ then $Q^* \subseteq Y$. We obtain the following possibilities:

$$W^* \subseteq Z \text{ and } Q^* \subseteq Z,$$
$$W^* \subseteq Z \text{ and } Q^* \subseteq Y,$$
$$W^* \subseteq Y \text{ and } Q^* \subseteq Z,$$
$$W^* \subseteq Y \text{ and } Q^* \subseteq Y.$$

Suppose that $W^* \subseteq Z$ and $Q^* \subseteq Z$ hold true. Then, $\varphi(X[t_1,t_2]) = 1 \geq \theta$, $\varphi(Y[t_1,t_2]) = 1 \geq \theta$ and $\varphi(Z[t_1,t_2]) = \theta'$. Hence, by Theorem 1, the instance s actively satisfies the dependency $X \to\xrightarrow{\theta}_F Y$. This is a contradiction. Similarly, if $W^* \subseteq Y$ and $Q^* \subseteq Y$ hold true, then $\varphi(X[t_1,t_2]) = 1 \geq \theta$, $\varphi(Y[t_1,t_2]) = \theta'$ and $\varphi(Z[t_1,t_2]) = 1 \geq \theta$. Therefore, s actively satisfies the dependency $X \to\xrightarrow{\theta}_F Y$, i.e., a contradiction.

Suppose that $W^* \subseteq Z$ and $Q^* \subseteq Y$ hold true. Now,

$$\varphi(X[t_3,t_1]) = 1 \geq \min(\theta, \varphi(X[t_1,t_2])),$$
$$\varphi(Y[t_3,t_1]) = 1 \geq \min(\theta, \varphi(X[t_1,t_2])),$$
$$\varphi(Z[t_3,t_2]) = 1 \geq \min(\theta, \varphi(X[t_1,t_2])).$$

This contradicts the fact that the conditions (4) don't hold at the same time for any tuple $t_3' \in r^*$.

Similarly, if $W^* \subseteq Y$ and $Q^* \subseteq Z$ hold true, then

$$\varphi(X[t_4,t_1]) = 1 \geq \min(\theta, \varphi(X[t_1,t_2])),$$
$$\varphi(Y[t_4,t_1]) = 1 \geq \min(\theta, \varphi(X[t_1,t_2])),$$
$$\varphi(Z[t_4,t_2]) = 1 \geq \min(\theta, \varphi(X[t_1,t_2])).$$

Hence, a contradiction.

Since our assumption that s does not satisfy the dependency $V \to\xrightarrow{\theta_1}_F W$ always leads to contradiction, it cannot be true. Therefore, s satisfies all fuzzy multivalued dependences from the set $(\mathscr{F},\mathscr{G})^+$, i.e., s satisfies the claim of the theorem. This completes the proof. ◻

Corollary 6 *Let C be a set of fuzzy functional and fuzzy multivalued dependences on some universal set of attributes U. Suppose that c is some fuzzy functional or fuzzy multivalued dependency on U. The following statements are equivalent:*

(a) Any fuzzy relation instance on scheme $R(U)$ which satisfies all dependences in C, satisfies the dependency c.

(b) Any two-element, fuzzy relation instance on scheme $R(U)$ which satisfies all dependences in C, satisfies the dependency c.

Proof: ($\Rightarrow$) Suppose that (a) holds true. Let r be any two-element, fuzzy relation instance on $R(U)$ which satisfies all dependences in C. Since (a) is valid for all fuzzy relation instances on U satisfying all dependences in C, it follows that r satisfies c.

($\Leftarrow$) Suppose that (a) does not hold. Now, there exists a fuzzy relation instance r on $R(U)$ which satisfies all dependences in C and does not satisfy the dependency c.

Suppose that $c \in C^+$, where C^+ is the closure of C.

Now, as mentioned before, the fact that $c \in C^+$ and the fact that r is a fuzzy relation instance on $R(U)$ that obeys all dependences in C, yield that r satisfies c. Hence, a contradiction. We conclude, $c \notin C^+$.

Now, reasoning as in Section 3, we obtain that for $c \notin C^+$ there exists a fuzzy relation instance r^* on $R(U)$ which satisfies all dependences in C^+ and does not satisfy the dependency c. Therefore, by Theorem 5, there exists some two-element, fuzzy relation instance $s \subseteq r^*$ such that s satisfies all dependences in C^+ and does not satisfy the dependency c. Since $C \subseteq C^+$, it follows that s is a two-element, fuzzy relation instance on $R(U)$ which satisfies all dependences in C and does not satisfy the dependency c. In other words, (b) does not hold. This completes the proof. ◫

Theorem 7 *Let C be a set of fuzzy functional and fuzzy multivalued dependences on some universal set of attributes U. Suppose that c is some fuzzy functional or fuzzy multivalued dependency on U. Denote by C' resp. c' the set of fuzzy formulas resp. the fuzzy formula associated to C resp. c. Then, the following two conditions are equivalent:*

(a) Any two-element, fuzzy relation instance on scheme $R(U)$ which satisfies all dependences in C, satisfies the dependency c.

(b) $i_{r,\beta}\left(c'\right) > 0.5$ for every $i_{r,\beta}$ such that $i_{r,\beta}(\mathscr{K}) > 0.5$ for all $\mathscr{K} \in C'$.

Proof: Similar to the proof of [9, pp. 38-42, Th. 2.] in the case of Yager's fuzzy implication operator and the proof of [10, pp. 288-296, Th. 2.] in the case of Reichenbach's fuzzy implication operator. ◫

Corollary 8 *Let C be a set of fuzzy functional and fuzzy multivalued dependences on some universal set of attributes U. Suppose that c is some fuzzy functional or fuzzy multivalued dependency on U. Denote by C' resp. c' the set of fuzzy formulas resp. the fuzzy formula associated to C resp. c. Then, the following two conditions are equivalent:*

(a) Any fuzzy relation instance on scheme $R(U)$ which satisfies all dependences in C, satisfies the dependency c.

(b) $i_{r,\beta}\left(c'\right) > 0.5$ for every $i_{r,\beta}$ such that $i_{r,\beta}(\mathscr{K}) > 0.5$ for all $\mathscr{K} \in C'$.

Proof: An immediate consequence of Corollary 6 and Theorem 7. ◫

6 Conclusion

Problems like:

input: S a set of fuzzy functional and fuzzy multivalued dependences

input: s a fuzzy functional or a fuzzy multivalued dependency

output: $S \Rightarrow s$

can be efficiently solved using the results derived in this paper. Let us illustrate this in the following example.

Example 1. If the fuzzy functional and the fuzzy multivalued dependences:

$$A_1A_2 \overset{\theta_1}{\longrightarrow}_F A_3A_4,$$
$$A_2A_3 \overset{\theta_2}{\longrightarrow}_F A_6,$$
$$A_1A_3A_4 \to\overset{\theta_3}{\longrightarrow}_F A_5A_6A_7,$$
$$A_3A_4 \overset{\theta_4}{\longrightarrow}_F A_7$$

hold true, where $U = \{A_1, A_2, A_3, A_4, A_5, A_6, A_7\}$ is the universal set of attributes and $A_1 = Name$, $A_2 = Intelligence$, $A_3 = Abilities$, $A_4 = Activity$, $A_5 = Job$, $A_6 = Salary$, $A_7 = Success$, then the fuzzy functional dependency $A_1A_2A_3 \overset{\min(\theta_1,\theta_2,\theta_3,\theta_4)}{\to}_F A_6A_7$ holds also true.

Proof: I (applying IR1–IR17)

We deduce:

1) $A_1A_2 \overset{\theta_1}{\longrightarrow}_F A_3A_4$ (input)

2) $A_1A_2 \overset{\theta_1}{\longrightarrow}_F A_1A_3A_4$ (from 1) and IR3)

3) $A_1A_2 \to\overset{\theta_1}{\longrightarrow}_F A_1A_3A_4$ (from 2) and IR9)

4) $A_1A_3A_4 \to\overset{\theta_3}{\longrightarrow}_F A_5A_6A_7$ (input)

5) $A_1A_2 \overset{\min(\theta_1,\theta_3)}{\to\to}_F A_5A_6A_7$ (from 3),4) and IR8)

6) $A_3A_4 \overset{\theta_4}{\longrightarrow}_F A_7$ (input)

7) $A_1A_2 \overset{\min(\theta_1,\theta_3,\theta_4)}{\to}_F A_7$ (from 5),6) and IR10)

8) $A_1A_2A_3 \overset{\min(\theta_1,\theta_3,\theta_4)}{\to}_F A_3A_7$ (from 7) and IR3)

9) $A_2A_3 \overset{\theta_2}{\longrightarrow}_F A_6$ (input)

10) $A_1A_2A_3 \overset{\theta_2}{\to}_F A_1A_6$ (from 9) and IR3)

11) $A_1A_2A_3 \overset{\min(\theta_1,\theta_2,\theta_3,\theta_4)}{\rightarrow}_F A_1A_3A_6A_7$ (from 8),10) and IR11)

12) $A_1A_3A_6A_7 \rightarrow_F A_6A_7$ (from IR2)

13) $A_1A_3A_6A_7 \overset{\min(\theta_1,\theta_2,\theta_3,\theta_4)}{\rightarrow}_F A_6A_7$ (from 12) and IR1)

14) $A_1A_2A_3 \overset{\min(\theta_1,\theta_2,\theta_3,\theta_4)}{\rightarrow}_F A_6A_7$ (from 11),13) and IR4)

Thus, the assertion follows. ◫

Proof: II (applying Corollary 8 and the resolution principle)

In order to apply Corollary 8, we associate the fuzzy formulas:

$$\begin{aligned}
&\mathscr{K}_1 \equiv (A_1 \wedge A_2) \Rightarrow (A_3 \wedge A_4),\\
&\mathscr{K}_2 \equiv (A_2 \wedge A_2) \Rightarrow A_6,\\
&\mathscr{K}_3 \equiv (A_1 \wedge A_3 \wedge A_4) \Rightarrow ((A_5 \wedge A_6 \wedge A_7) \vee A_2),\\
&\mathscr{K}_4 \equiv (A_3 \wedge A_4) \Rightarrow A_7,\\
&c' \equiv (A_1 \wedge A_2 \wedge A_3) \Rightarrow (A_6 \wedge A_7)
\end{aligned}$$

to the given set of fuzzy dependences.

We have four axioms: $\mathscr{K}_1$, $\mathscr{K}_2$, $\mathscr{K}_3$ and $\mathscr{K}_4$.

We shall prove c' by the refutation of its negation.

Let us find conjunctive normal forms of the formulas: $\mathscr{K}_1$, $\mathscr{K}_2$, $\mathscr{K}_3$, $\mathscr{K}_4$ and $\neg c'$. We have:

$$\begin{aligned}
\mathscr{K}_1 &\equiv (\neg A_1 \vee \neg A_2) \vee (A_3 \wedge A_4)\\
&\equiv (\neg A_1 \vee \neg A_2 \vee A_3) \wedge (\neg A_1 \vee \neg A_2 \vee A_4),\\
\mathscr{K}_2 &\equiv \neg A_2 \vee \neg A_3 \vee A_6,\\
\mathscr{K}_3 &\equiv (\neg A_1 \vee \neg A_3 \vee \neg A_4) \vee ((A_5 \wedge A_6 \wedge A_7) \vee A_2)\\
&\equiv (\neg A_1 \vee \neg A_3 \vee \neg A_4)\\
&\quad \vee ((A_5 \vee A_2) \wedge (A_6 \vee A_2) \wedge (A_7 \vee A_2))\\
&\equiv (\neg A_1 \vee A_2 \vee \neg A_3 \vee \neg A_4 \vee A_5) \wedge\\
&\quad (\neg A_1 \vee A_2 \vee \neg A_3 \vee \neg A_4 \vee A_6) \wedge\\
&\quad (\neg A_1 \vee A_2 \vee \neg A_3 \vee \neg A_4 \vee A_7),\\
\mathscr{K}_4 &\equiv \neg A_3 \vee \neg A_4 \vee A_7,\\
\neg c' &\equiv \neg(\neg(A_1 \wedge A_2 \wedge A_3) \vee (A_6 \wedge A_7))\\
&\equiv A_1 \wedge A_2 \wedge A_3 \wedge (\neg A_6 \vee \neg A_7).
\end{aligned}$$

Let M be the set of all conjunctive terms that appear in conjunctive normal forms of the formulas: $\mathscr{K}_1$, $\mathscr{K}_2$, $\mathscr{K}_3$, $\mathscr{K}_4$ and $\neg c'$. Therefore, the elements of the set M are: $\neg A_1 \vee \neg A_2 \vee A_3$, $\neg A_1 \vee \neg A_2 \vee A_4$, $\neg A_2 \vee \neg A_3 \vee A_6$, $\neg A_1 \vee A_2 \vee \neg A_3 \vee \neg A_4 \vee A_5$, $\neg A_1 \vee A_2 \vee \neg A_3 \vee \neg A_4 \vee A_6$, $\neg A_1 \vee A_2 \vee \neg A_3 \vee \neg A_4 \vee A_7$, $\neg A_3 \vee \neg A_4 \vee A_7$, A_1, A_2, A_3 and $\neg A_6 \vee \neg A_7$.

Applying resolution principle to the set M, we obtain:

1) $\neg A_1 \vee \neg A_2 \vee A_4$ (input)

2) A_1 (input)

3) $\neg A_2 \vee A_4$ (resolution from 1) and 2))

4) A_2 (input)

5) A_4 (resolvent from 3) and 4))

6) $\neg A_2 \vee \neg A_3 \vee A_6$ (input)

7) $\neg A_3 \vee A_6$ (resolvent from 4) and 6))

8) A_3 (input)

9) A_6 (resolvent from 7) and 8))

10) $\neg A_3 \vee \neg A_4 \vee A_7$ (input)

11) $\neg A_4 \vee A_7$ (resolvent from 8) and 10))

12) A_7 (resolvent from 5) and 11))

13) $\neg A_6 \vee \neg A_7$ (input)

14) $\neg A_7$ (resolvent from 9) and 13))

Resolving 12) and 14), we conclude that the formulas: $\mathscr{K}_1$, $\mathscr{K}_2$, $\mathscr{K}_3$, $\mathscr{K}_4$ and $\neg c'$ are not simultaneously valid. In other words, the assertion *(b)* of Corollary 8 holds true. Now, by the assertion *(a)* of Corollary 8, $A_1A_2A_3 \overset{\min(\theta_1,\theta_2,\theta_3,\theta_4)}{\rightarrow}_F A_6A_7$ follows from the given dependences. ◫

Thus, Corollary 8 gives us an opportunity to avoid the disadvantages of the classical approach to these problems (disadvantages like: number of inference rules, matter of their choice, uncertainty of outcome, etc.) and to position ourselves into equivalent, fuzzy logic environment, where our steps can be fully automated.

In view of Example 1 (proof II), it is clear that the translation of fuzzy dependences to fuzzy formulas, finding of conjunctive normal forms of fuzzy formulas, an application of resolution principle represent these automated steps.

Once, a refutation within resolution principle is achieved, our task is done.

Note that the results derived in this paper do not offer a code of an application that would practically support our work. Such automatization has yet to be done.

Also, note that the fuzzy logic operators applied in this paper are fixed. It would be worth to determine if the set of applicable operators to our results could be widen to include not only the operators applied here and in [9], [10].

Finally, we point out that our research in itself is original in many aspects. Hence, the contribution of the used literature to the results obtained in the paper is only limited and partial. However, we highlight the following:

Our background and preliminary material is mainly founded on the similarity-based fuzzy relational database approach [18]. It assumes the use of fuzzy relation instances, similarity relations, conformances, the formal definitions of fuzzy functional and fuzzy multivalued dependences, linguistic strengths and limit strengths of dependences, inference rules, closures, dependency basis, soundness and completeness. [18], however, is based on [4]-[6]. Some of the classical relational database model [8] definitions, we adopt from [7] (entities for example, attributes, attribute values, tuples, etc.). The definition of active fuzzy multivalued dependency fulfillment is our own. It plays the key role in a large part of the paper, especially in the proofs of the main results of Sections 4 and 5. Motivated by the satisfy (falsify) interpretation definition [11], we introduce the valuation definition, i.e., the valuation joined to some two-element, fuzzy relation instance. This definition plays the key role in the proof of Theorem 7, Section 5 of the paper (see the corresponding proof [9] in the case of Yager's fuzzy implication operator and the corresponding proof [10] in the case of Reichenbach's fuzzy implication operator). The references [3] and [17] provides detailed study on fuzzy implication operators. The remaining references deal with various types of fuzzy implication operators and are of secondary importance to this paper.

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