Discrete Time Sliding Mode Control of Nonlinear Uncertain Systems Based on Estimation of Uncertainties and External Disturbances

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Abstract: - In this paper, a new variable structure control algorithm for a class of discrete-time nonlinear uncertain systems is proposed. By using an estimator of uncertainties and external disturbances, the proposed algorithm ensures the stability of the closed loop system as well as the reference tracking.
The controller designed using the above technique is completely insensitive to the parametric uncertainty and the external disturbances. Simulations are carried out on an inverted pendulum benchmark and the yielded results confirm the effectiveness of our proposition.

Key-Words: - Discrete sliding mode control, uncertainty estimator, inverted pendulum

1 Introduction

The variable structure theory and its applications have formed one of the most attractive research areas of the last decades. It is principally characterized by its robustness with respect to the system's modeling uncertainties and external disturbances [1], [2], [3], [4]. Sliding mode control is a particular case of the variable structure control (VSC).

The sliding mode control (SMC) has been widely studied over several decades in continuous and discrete time-systems. The continuous-time SMC is known by its robustness with respect to uncertainties and external disturbances [5], [6]. It consists of two steps. The first is the design of a sliding surface along which the process can slide to find its desired final value. The second is the synthesis of the control law in such away that any state outside the sliding surface is forced to reach the desired sliding manifold in finite time and stay on it. However, the main disadvantage of the continuous time sliding mode control is the chattering phenomenon which appears as a source to excite unmodeled high frequency dynamics of the process. Many approaches have been proposed to solve this problem such as second and high order sliding mode control [7], [8].

With the advent of digital computers and their widespread use in control systems, considerable efforts have been considered in the study of discrete time sliding mode control technique (DSMC) [9, 10, 11, 12, 13].

The robustness property implies that the system state goes to a sliding surface in spite of parameter uncertainties and external disturbances.

However, in the discrete sliding mode the control input is constant over sampling period. Hence, when the states reach the switching surface, the subsequent control would be unable to keep the states to be confined to the surface. This leads the system to undergo only quasi-sliding mode, i.e., the system states would approach the sliding surface but would generally be unable to stay on it [9].

The condition under which the system states starting from any initial state, move towards the sliding surface and reach it in a finite time are called reaching condition or reaching law. The system trajectory under the reaching condition is called the reaching mode or the reaching phase.

In [14] Dote and Hoft are the first who have considered the discrete VSC problem and they have used an equivalent form of the continuous reaching condition ($S\dot{S} < 0$) to propose the following discrete reaching condition:

$$\left[S(k+1) - S(k)\right]S(k) < 0$$

The term quasi-sliding mode was introduced by Miloslavjevic [15] to express the insufficiency of the above condition and Sarpturk, and al., [16] propose the following reaching condition :

$$\left|S(k+1)\right| < \left|S(k)\right|$$

Furuta [17] used the equivalent form of a Lyapunov-type of continuous reaching condition

$$\dot{V} = \frac{1}{2}S^2 < 0 \text{ to give the discrete version:}$$

$$V(k+1)-V(k)<0 \quad \text{with} \quad V(k)=\frac{1}{2}\big[S(k)\big]^2$$

In this paper, a class of nonlinear discrete-time systems is considered involving both uncertainties and external disturbances. By employing an estimator of uncertainties and disturbances, the stability of the sliding mode is insured.

The paper is organized as follows. Section two describes the considered systems. In section three, the discrete sliding mode control is reviewed. The fourth section is devoted to simulation of the proposed DSMC on the inverted pendulum model. The final section concludes the paper.

2 Problem Formulation

Let's consider the single input single output nonlinear discrete-time system:

$$\begin{cases} x_i(k+1)=x_{i+1}(k), \quad i=1,2,...,n-1 \\ x_n(k+1)=F(x(k))+G(x(k))\,u(k)+w(k) \end{cases} \quad (1)$$

The model structure given in (1) is used to cope with the inverted pendulum model. The functions F and G are:

$$F(x(k))=f(x(k))+\Delta f(x(k)) \quad (2)$$

$$G(x(k))=g(x(k))+\Delta g(x(k)) \quad (3)$$

Where $x(k)=\big[x_1(k),x_2(k),...,x_n(k)\big]^T \in \Re^n$ is the state vector, f is a nonlinear function and g is a nonlinear function different from zero.

$\Delta f(x(k))$ and $\Delta g(x(k))$ are the uncertainties on $f(x(k))$ and $g(x(k))$ respectively.

The signal $u(k)\in\Re$ and $w(k)$ are the control input and the external disturbance respectively.

The system (1) can be rewritten in the form:

$$\begin{cases} x_i(k+1)=x_{i+1}(k), \quad i=1,2,...,n-1 \\ x_n(k+1)=f(x(k))+g(x(k))\,u(k)+D(k) \end{cases} \quad (4)$$

Where $D(k)$ is the term that includes the uncertainties and the external disturbance.

$$D(k)=\Delta f(x(k))+\Delta g(x(k))\,u(k)+w(k) \quad (5)$$

3 Discrete sliding mode control

The sliding function relative to this system is taken in this linear form:

$$S(k)=C^T(x(k)-x_d(k)) \quad (6)$$

where x_d is the desired state vector and $C^T=\big[c_1,c_2,...,c_n\big]$ is the sliding vector, c_n is

assumed to be 1 and c_i $(i=1,...,n)$ are chosen so that the roots of the polynomial $r(z)=c_1+c_2 z+...+c_n z^{n-1}$ are inside the unit circle. The sliding surface is:

$$\sigma=\big\{x\mid S(x)=0\big\} \quad (7)$$

The sliding function can be rewritten as:

$$S(k)=\sum_{i=1}^{n-1}c_i e_i(k)+e_n(k) \quad (8)$$

where $e_i(k)$ is the state error, $e_i(k)=x_i(k)-x_d^i(k)$ and $x_d^i(k)$ is the ith component of $x_d(k)$.

According to the sliding function (8), the sliding function value at time instant $k+1$ can be obtained as:

$$S(k+1)=\sum_{i=1}^{n-1}c_i e_i(k+1)+e_n(k+1) \quad (9)$$

Replacing $e_n(k+1)$ by its expression given above:

$$S(k+1)=\sum_{i=1}^{n-1}c_i e_i(k+1)+x_n(k+1)-x_d^n(k+1) \quad (10)$$

and from (4) we have:

$$\begin{aligned} S(k+1)=&\sum_{i=1}^{n-1}c_i e_i(k+1)+f(x(k))+g(x(k))\,u(k) \\ &+D(k)-x_d^n(k+1) \end{aligned} \quad (11)$$

To obtain some desired performances, such as strong robustness, fast convergence and chattering elimination, we introduce a reaching law to ensure the convergence of the sliding function $S(k)$ to zero. To ensure a quasi-sliding mode, the sliding function must verify the following reaching law [18]:

$$S(k+1)=(1-qT)S(k)-\eta\,T\,sign(S(k)) \quad (12)$$

with $0<1-qT=\phi<1$ and $0<\eta T<1$, where $T>0$, $\eta>0$ and $q>0$ are the sampling period, the reaching rate and the approximation rate index. Using the equations (11) and (12), the control law based on SMC developed to control system (1) in discrete time is given by:

$$\begin{aligned} u(k)=&-\frac{1}{g(x(k))}\hat{D}(k) \\ &-\frac{1}{g(x(k))}\left[\sum_{i=1}^{n-1}c_i\,e_i\,(k+1)+f(x(k))-x_d^n\,(k+1)\right] \\ &+\frac{1}{g(x(k))}\big[\phi S(k)-\eta\,T\,sign(S(k))\big] \end{aligned} \quad (13)$$

With $\hat{D}(k)$ is a compensator of perturbation that takes the following form:

$$\hat{D}(k)=-\frac{1}{\varepsilon}(\hat{D}(k-1)-D(k-1)) \quad (14)$$

with ε is a small positive constant, the expression of $D(k-1)$ is given from the second equation of system (4) at time instant k-1:

$$D(k-1) = x_n(k) - f(x(k-1)) - g(x(k-1)\,u(k-1) \quad (15)$$

Replacing the control u by its expression (12), we get:

$$D(k-1) = x_n(k) - f(x(k-1)) + \hat{D}(k-1)$$
$$+ \left[\sum_{i=1}^{n-1} c_i\, e_i(k) + f(x(k-1)) - x_d^n(k) \right] \quad (16)$$
$$- \left[\phi S(k-1) - \eta T\, sign(S(k-1)) \right]$$

then $D(k-1)$ is given by:

$$D(k-1) = \hat{D}(k-1) + S(k) - \phi S(k-1)$$
$$+ \eta T sign(S(k-1)) \quad (17)$$

Therefore the compensator is written as:

$$\hat{D}(k) = -\frac{1}{\varepsilon}(S(k) - \phi S(k-1) + \eta T\, sign(S(k-1)) \quad (18)$$

Theorem 1. If the control law (13), with the sliding function (8) and the disturbance compensator (18), is applied to the nonlinear uncertain system defined by (1), the state error converges to zero in finite time.

Proof. Let us consider the following positive definite function as a Lyapunov function candidate:

$$V(k) = \frac{1}{2}(S(k))^2 \quad (19)$$

$$\Delta V(k) = V(k+1) - V(k) = \frac{1}{2}((S(k+1))^2 - (S(k))^2)$$
$$= \frac{1}{2}[S(k+1) - S(k)][S(k+1) + S(k)] \quad (20)$$

The equation (12) gives:

$$S(k+1) - S(k) = -qT\,S(k) - \eta T\, sign(S(k))$$
$$= -S(k)(qT + \eta T\frac{1}{|S(k)|}) \quad (21)$$

then

$$\begin{cases} \text{if } S(k) > 0 \text{ then } S(k+1) - S(k) < 0 \\ \text{if } S(k) < 0 \text{ then } S(k+1) - S(k) > 0 \end{cases} \quad (22)$$

on the other hand

$$S(k+1) + S(k) = (1-q)S(k) - \eta T\, sign(S(k)) + S(k)$$
$$= S(k)((2-qT) - \eta T\frac{1}{|S(k)|}) \quad (23)$$

$$\begin{cases} \text{if } S(k) > 0 \text{ then } S(k+1) + S(k) > 0 \\ \text{if } S(k) < 0 \text{ then } S(k+1) + S(k) < 0 \end{cases} \quad (24)$$

The equations (21) and (23) lead to:

$$\begin{cases} \text{if } S(k) > 0 \text{ then } \Delta V(k) < 0 \\ \text{if } S(k) < 0 \text{ then } \Delta V(k) < 0 \end{cases} \quad (25)$$

and

$$\begin{cases} [S(k+1) + S(k)]\, sign(S(k)) > 0 \\ [S(k+1) - S(k)]\, sign(S(k)) < 0 \end{cases} \quad (26)$$

These inequalities assure the quasi-sliding motion and the convergence of the state trajectories on the hyperplane, which guarantees the output convergence.

4 Application on an inverted pendulum

The considered process consists of a cart and a pendulum attached to it.

The motion equation such process is obtained by applying the Euler-Lagrange equations given by:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\rho}} - \frac{\partial L}{\partial \rho} = u \quad (27)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \quad (28)$$

where θ is the angle between the pendulum and its upright position, $\dot{\theta}$ is the first derivative of θ with respect to t, ρ is the position of the cart, u is the force applied to the cart and L is the Lagrangian given by:

$$L = E_c - E_p \quad (29)$$

where E_c is the kinetic energy:

$$E_c = \frac{1}{2}(M+m)\dot{\rho}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}J\dot{\theta}^2 + ml\,\dot{x}\dot{\theta}\cos\theta \quad (30)$$

and E_p is the potential energy:

$$E_p = mgl\cos\theta \quad (31)$$

M is the cart mass, m is the pendulum mass, l is the length of the pendulum, $J = \frac{1}{3}ml^2$ is the moment of inertia of pendulum and g is the acceleration of gravity.

Thus, the mathematical model of the inverted pendulum is given by:

$$(M+m)\ddot{\rho} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta + u = 0 \quad (32)$$

$$\frac{4}{3}ml^2\ddot{\theta} + ml\ddot{\rho}\cos\theta - mgl\sin\theta = 0 \quad (33)$$

Substituting (32) into (33) gives:

$$\ddot{\theta} = f(\theta,\dot{\theta}) + g(\theta,\dot{\theta})u \quad (34)$$

We define the state variables:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \quad (35)$$

Then f and g are given by:

$$f(x_1, x_2) = \frac{mlx_2^2 \sin(x_1(t)) \cos(x_1(t)) - (m + M) g \sin(x_1(t))}{ml \cos^2(x_1(t)) - (4/3)l(m + M)}$$
$$(36)$$

and

$$g(x_1, x_2) = \frac{-\cos(x_1(t))}{ml \cos^2(x_1(t)) - (4/3)l(m + M)} \quad (37)$$

The state model of the system described by (34-37) is written as:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(x_1, x_2) + g(x_1, x_2) u + w \\ y = x_1 \end{cases} \quad (38)$$

where w is the external disturbance and $y = x_1 = \theta$ is the pendulum position used as the system output. The discrete model of the inverted pendulum can be expressed as:

$$\begin{cases} x_1(k+1) = x_1(k) + T x_2(k) \\ x_2(k+1) = x_2(k) + T f(x_1, x_2) + T g(x_1, x_2) u + T w(k) \end{cases}$$
$$(39)$$

$$f(x_1, x_2) = \frac{mlx_2^2 \sin(x_1(k)) \cos(x_1(k)) - (m + M) g \sin(x_1(k))}{ml \cos^2(x_1(k)) - (4/3)l(m + M)}$$

$$g(x_1, x_2) = \frac{-\cos(x_1(k))}{ml \cos^2(x_1(k)) - (4/3)l(m + M)}$$

T is the sampling rate.

In this paper we take into consideration the uncertainties on the pendulum mass Δm and on the cart mass ΔM and the state model of the system can be written as:

$$\begin{cases} x_1(k+1) = x_1(k) + T x_2(k) \\ x_2(k+1) = x_2(k) + TF(x_1, x_2) + TG(x_1, x_2) u + T w(k) \end{cases}$$
$$(40)$$

$$F(x_1, x_2) = F_1(x_1, x_2) - F_2(x_1, x_2)$$

where:

$$F_1(x_1, x_2) = \frac{(m + \Delta m)l x_2^2 \sin(x_1(k)) \cos(x_1(k))}{(m + \Delta m)l \cos^2(x_1(k)) - (4/3)l(m + M + \Delta m + \Delta M)}$$

$$F_2(x_1, x_2) = \frac{(m + M + \Delta m + \Delta M) g \sin(x_1(k))}{(m + \Delta m)l \cos^2(x_1(k)) - (4/3) l(m + M + \Delta m + \Delta M)}$$

$$G(x_1, x_2) = \frac{-\cos(x_1(k))}{(m + \Delta m)l \cos^2(x_1(k)) - (4/3) l(m + M + \Delta m + \Delta M)}$$

$$w(k) = 0.1 \sin(0.1 k)$$

Once the model is determined we proceed to solve the stabilization problem by designing a controller to keep the pendulum in its stable equilibrium point in spite of disturbances.

The parameters of the inverted pendulum for simulation are $l = 0.5 m$, $m = 0.1 Kg$ and

$g = 9.81 m/s^2$. The initial conditions of the inverted pendulum are set for the simulation $x = [pi/12 \quad 0]$, $\eta = 0.05$ and the sampling rate $T = 0.1 s$.

By choosing the sliding function as:

$$S(k) = c_1(x_1(k) - x_d^1(k)) + (x_2(k) - x_d^2(k))$$

The control law is given by:

$$u(k) = -\frac{1}{g(x(k))} \hat{D}(k)$$

$$+ \frac{1}{g(x(k))} \left[-c_1(x_1(k+1) - x_d^1(k)) - x_d^2(k) - f(x(k)) \right]$$

$$+ \frac{1}{g(x(k))} \left[\phi S(k) - \eta T sign(S(k)) \right]$$

and the compensator is written as:

$$\hat{D}(k) = -\frac{1}{\varepsilon}(S(k) - \phi S(k-1) + \eta T sign(S(k-1)))$$

Figure 1 gives the evolutions of the states $x_1(k)$ and $x_d^1(k)$. Figure 2 provides the evolution of the state $x_2(k)$. Figure 3 presents the evolution of the sliding surface. The evolution of the control is presented in Figure 4.
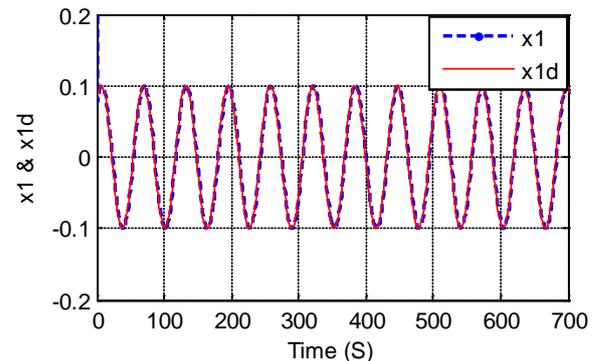
Fig.1 Evoluations of states $x_1(k)$ and $x_d^1(k)$
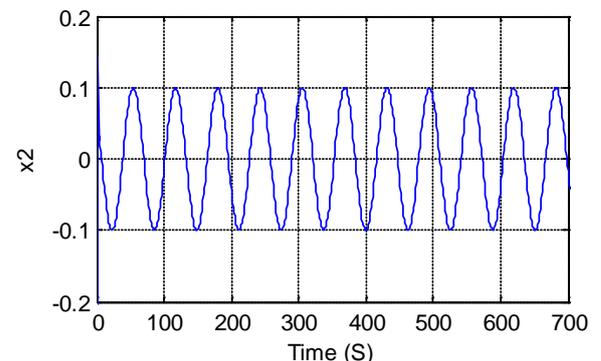
Fig.2 Evoluation of state $x_2(k)$
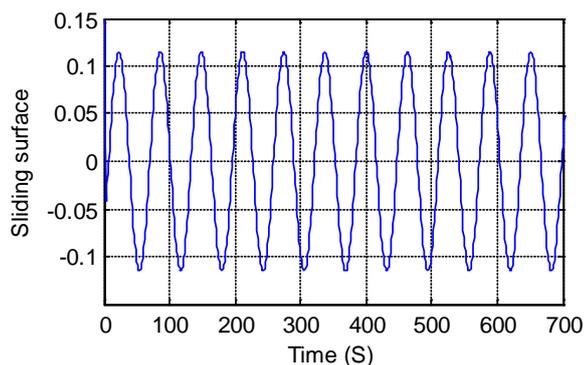
Fig.3 Evolution of the sliding surface
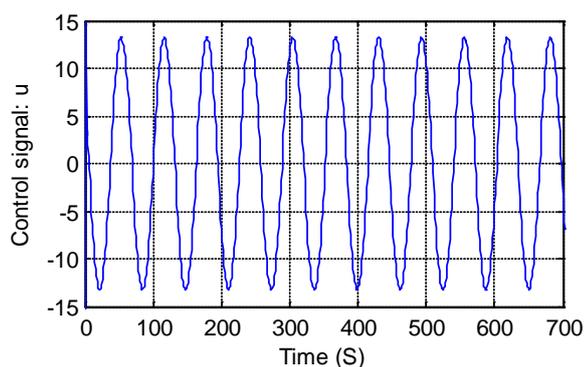
Fig.4 Evolution of the control signal

These Figures prove that relatively satisfactory performances are recorded in terms of rejecting disturbances. We note that the control law used has allowed the stabilization and the traking of the desired trajectory. Also we grantees the convergence of the sliding surface in the neighborhood of zero.

By using an estimator of uncertainties in the control law, the proposed approach gives good results. It proves the robustness of the control against uncertainties and disturbances.

4 Conclusion

In this work, a new discrete-time quasi-sliding mode control law for a non-linear uncertain system, based on the estimation of uncertainties and disturbance is developed. The application of this control law on an inverted pendulum has given satisfactory results for the stabilization and the trajectory tracking while overcoming the chatter problems of sliding mode control. The robustness performance of our strategy is verified both theoretically and by means of a simulation example.

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