A second order constraint qualification for certain classes of optimal control problems

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Abstract: In this paper we introduce a second order constraint qualification which yields necessary conditions for optimal control problems with inequality and equality constraints in the time variable and the control functions. The second order conditions are shown to hold on a cone of critical directions which properly contains not only the cone obtained by considering only active constraints but also the usual one which depends on the sign of the corresponding Lagrange multipliers.

Key–Words: Optimal control, second order necessary conditions, extremals, normality

1 Introduction

In this paper we derive second order necessary conditions for certain optimal control problems posed over piecewise C^1 trajectories and piecewise continuous controls. For these problems, the integrand of the cost function is independent of the state variable and the constraints are expressed in terms of the dynamics, fixed endpoint conditions, and inequalities and equalities depending on the time variable and the control functions.

First order conditions for such problems are well established in the literature (see, for example, [6, 7, 11] and, more recently, [3] where mixed constraints with nonsmooth data are studied). Our aim in this paper is to derive in a simple way second order conditions which an extremal, if solving the problem, should satisfy. The main idea relies on linking up the properties that characterize the extremals with any solution to the underlying problem of minimizing the same functional over the set of equality and inequality constraints without dynamics or endpoint conditions.

Second order necessary conditions in terms of the accessory problem can be found, for example, in [14] for problems posed over controls in $L^\infty(T, \mathbf{R}^m)$. However, for the problem we shall deal with, the control functions are piecewise continuous and so do not form a Banach space. Thus, the technique used for L^∞ controls where results from abstract optimization theory on Banach spaces are applied to the optimal control problem, does not work for our problem.

There is an extensive literature on second order conditions for optimal control problems and how the theory can be applied to practical problems (see, for example, [1–7, 9–14] and references therein) but some fundamental questions remain unanswered. In particular, when dealing with such conditions, one usually faces two main features: (1) the normality assumptions imposed on the solution to the problem which imply, in particular, a positive cost multiplier, and (2) the set of critical directions where, under those normality assumptions, the second order conditions hold.

As we shall see, a weak notion of normality usually imposed for first order conditions is not enough to ensure the validity of the second order necessary conditions, and a stronger notion is needed. However, once this stronger assumption is imposed, the set of critical directions may be too restrictive. One is then, of course, interested in enlarging that set and, if possible, in weakening the normality assumptions. In this paper we show how, for the problem we shall deal with, these two aspects can be achieved. The results obtained here correspond to a generalization of second order conditions first derived in [13] (and with more detail in [4]) for optimal control problems where the constraints depend only on the control functions.

Let us point out that, in some of the references mentioned above, the approach followed to derive second order conditions is based precisely on that stronger notion of normality by taking into account only equality constraints for active indices and imposing normality assumptions with respect to the corresponding set of tangential constraints. The conditions one encounters in those cases will be said to be of a "weak type" since, as we shall see, the assumptions and the critical directions can be modified and, in the two senses mentioned above, improved.

2 Statement of the problem

Suppose we are given an interval $T := [t_0, t_1]$ in $\mathbf{R}$, two points ξ_0, ξ_1 in $\mathbf{R}^n$, and functions L and $\varphi = (\varphi_1, \dots, \varphi_q)$ mapping $T \times \mathbf{R}^m$ to $\mathbf{R}$ and $\mathbf{R}^q$ $(q \le m)$ respectively, and f mapping $T \times \mathbf{R}^n \times \mathbf{R}^m$ to $\mathbf{R}^n$.

Denote by X the space of piecewise C^1 functions mapping T to $\mathbf{R}^n$, by $\mathcal{U}_k$ the space of piecewise continuous functions mapping T to $\mathbf{R}^k$ $(k \in \mathbf{N})$, set $Z := X \times \mathcal{U}_m$,

$$D := \{(x, u) \in Z \mid \dot{x}(t) = f(t, x(t), u(t))\ (t \in T),$$

$$x(t_0) = \xi_0,\ x(t_1) = \xi_1\},$$

$$S := \{(x, u) \in D \mid \varphi_\alpha(t, u(t)) \le 0,$$

$$\varphi_\beta(t, u(t)) = 0\ (\alpha \in R,\ \beta \in Q,\ t \in T)\}$$

where $R = \{1, \dots, r\}$, $Q = \{r+1, \dots, q\}$, and consider the functional $I: Z \to \mathbf{R}$ given by

$$I(x, u) := \int_{t_0}^{t_1} L(t, u(t))dt \quad ((x, u) \in Z).$$

The problem we shall deal with, which we label (P), is that of minimizing I over S. Note that the functional I is independent of x but we use the notation $I(x, u)$ to emphasize the fact that we shall be concerned with elements of $Z = X \times \mathcal{U}_m$ satisfying the dynamics and endpoint constraints defining membership of D. In other words, the problem is posed over those $u \in \mathcal{U}_m$ satisfying the inequality and equality constraints given in S and for which, if x is the unique solution of the differential equation $\dot{x}(t) = f(t, x(t), u(t))$ $(t \in T)$ together with the initial condition $x(t_0) = \xi_0$, then $x(t_1) = \xi_1$.

Elements of Z will be called *processes* and of S *admissible processes*. We shall say that a process (x, u) *solves* (P) if (x, u) is admissible and $I(x, u) \le I(y, v)$ for all admissible process (y, v).

Given $(x, u) \in Z$ we shall find convenient to use the notation $(\tilde{x}(t))$ to represent $(t, x(t), u(t))$, and '$*$' will be used to denote transpose. We assume that L, f and φ are C^2 and the $q \times (m + r)$-dimensional matrix

$$\left(\frac{\partial \varphi_i}{\partial u^k} \quad \delta_{i\alpha}\varphi_\alpha\right)$$

$(i = 1, \dots, q;\ \alpha = 1, \dots, r;\ k = 1, \dots, m)$ has rank q on $\mathcal{A}$ (here $\delta_{\alpha\alpha} = 1$, $\delta_{\alpha\beta} = 0$ $(\alpha \ne \beta)$), where

$$\mathcal{A} := \{(t, u) \in T \times \mathbf{R}^m \mid \varphi_\alpha(t, u) \le 0\ (\alpha \in R),$$

$$\varphi_\beta(t, u) = 0\ (\beta \in Q)\}.$$

This condition (see [7, 10] for details) is equivalent to the condition that, at each point (t, u) in $\mathcal{A}$, the matrix

$$\left(\frac{\partial \varphi_i}{\partial u^k}\right) \qquad (i = i_1, \dots, i_p;\ k = 1, \dots, m)$$

has rank p, where $i_1, \dots, i_p$ are the indices $i \in \{1, \dots, q\}$ such that $\varphi_i(t, u) = 0$.

3 Necessity and normality

Usually first order conditions for this problem are established in terms of the Hamiltonian function (see, for example, [3, 6, 7, 11]), and one version can be written as follows. For all $(t, x, u, p, \mu, \lambda)$ in $T \times \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^n \times \mathbf{R}^q \times \mathbf{R}$ let

$$H(t, x, u, p, \mu, \lambda) := \langle p, f(t, x, u) - \lambda L(t, u) -$$

$$\langle \mu, \varphi(t, u)\rangle.$$

Theorem 3.1 *Suppose* (x_0, u_0) *solves* (P). *Then there exist* $\lambda_0 \ge 0$, $p \in X$, *and* $\mu \in \mathcal{U}_q$, *not vanishing simultaneously on* T, *such that*

a. $\mu_\alpha(t) \ge 0$ *and* $\mu_\alpha(t)\varphi_\alpha(t, u_0(t)) = 0$ $(\alpha \in R,\ t \in T)$.

b. *On every interval of continuity of* u_0, *we have* $\dot{p}(t) = -H_x^*(\tilde{x}_0(t), p(t), \mu(t), \lambda_0)$ *and* $H_u(\tilde{x}_0(t), p(t), \mu(t), \lambda_0) = 0$.

This result assures the existence of multipliers (p, μ, λ_0) associated to a solution to the problem. Let us denote by $M(x, u)$ the set of such multipliers and define a set $\mathcal{E}$ of "extremals" as the set of all (x, u, p, μ) which have associated a nonzero cost multiplier normalized to one.

Definition 3.2 For all $(x, u) \in Z$ let $M(x, u)$ be the set of all $(p, \mu, \lambda_0) \in X \times \mathcal{U}_q \times \mathbf{R}$ with $\lambda_0 + |p| \ne 0$ satisfying

a. $\mu_\alpha(t) \ge 0$ and $\mu_\alpha(t)\varphi_\alpha(t, u(t)) = 0$ $(\alpha \in R,\ t \in T)$;

b. For $t \in T$, $\dot{p}(t) = -H_x^*(\tilde{x}(t), p(t), \mu(t), \lambda_0)$ and $H_u(\tilde{x}(t), p(t), \mu(t), \lambda_0) = 0$.

Denote by $\mathcal{E}$ the set of all $(x, u, p, \mu) \in Z \times X \times \mathcal{U}_q$ such that $(p, \mu, 1) \in M(x, u)$, that is,

a. $\mu_\alpha(t) \ge 0$ and $\mu_\alpha(t)\varphi_\alpha(t, u(t)) = 0$ $(\alpha \in R,\ t \in T)$;

b. $\dot{p}(t) = -f_x^*(\tilde{x}(t))p(t)$ and $f_u^*(\tilde{x}(t))p(t) = L_u^*(\tilde{x}(t)) + \varphi_u^*(t, u(t))\mu(t)$ $(t \in T)$.

The notion of "normality" is introduced so that the non-vanishing of the cost multiplier can be assured. This is accomplished by having zero as the unique solution to the adjoint equation whenever λ_0 is equal to 0.

Definition 3.3 A process $(x, u) \in S$ will be said to be *normal relative to* S if, given $p \in X$ and $\mu \in \mathcal{U}_q$ satisfying

i. $\mu_\alpha(t) \geq 0$ and $\mu_\alpha(t)\varphi_\alpha(t, u(t)) = 0$ ($\alpha \in R,\ t \in T$);

ii. $\dot{p}(t) = -f_x^*(\tilde{x}(t))p(t)$ ($t \in T$);

iii. $0 = f_u^*(\tilde{x}(t))p(t) - \varphi_u^*(t, u(t))\mu(t)$ ($t \in T$),

then $p \equiv 0$. In this event, clearly, also $\mu \equiv 0$.

From Theorem 3.1 and the above definitions it follows that, if (x_0, u_0) solves (P) and is normal relative to S, then there exists $(p, \mu) \in X \times \mathcal{U}_q$ such that $(x_0, u_0, p, \mu) \in \mathcal{E}$. Note also that uniqueness of the pair (p, μ) cannot be assured and, for that purpose, a stronger notion of normality is required or, as we shall see below, the definition of normality given before but applied to a set S_0 involving only equality constraints.

Denote the set of *active indices at* $(t, u) \in T \times \mathbf{R}^m$ by

$$I_a(t, u) := \{\alpha \in R \mid \varphi_\alpha(t, u) = 0\}$$

and, given $(x_0, u_0) \in S$, consider the set

$$S_0 := \{(x, u) \in D \mid \varphi_\gamma(t, u(t)) = 0$$

$$(\gamma \in I_a(t, u_0(t)) \cup Q,\ t \in T)\}.$$

Note that (x_0, u_0) is normal relative to S_0 if, given $(p, \mu) \in X \times \mathcal{U}_q$ satisfying

i. $\mu_\alpha(t)\varphi_\alpha(t, u_0(t)) = 0$ ($\alpha \in R,\ t \in T$);

ii. $\dot{p}(t) = -f_x^*(\tilde{x}_0(t))p(t)$ and $f_u^*(\tilde{x}_0(t))p(t) = \varphi_u^*(t, u_0(t))\mu(t)$ ($t \in T$),

then $p \equiv 0$.

To be consistent with other references (see, for example, [4–6, 12, 13]), we shall refer to normality relative to S and S_0 as "weak" and "strong normality" respectively. The following proposition is crucial. It is a simple consequence of Theorem 3.1 and the definitions given above (see also [12]).

Proposition 3.4 *If* (x_0, u_0) *solves* (P) *then* $M(x_0, u_0)$ *is not empty. If also* (x_0, u_0) *is strongly normal then there exists a unique* $(p, \mu) \in X \times \mathcal{U}_q$ *such that* $(x_0, u_0, p, \mu) \in \mathcal{E}$.

For second order conditions let us consider, for all (x, u, p, μ) in $Z \times X \times \mathcal{U}_q$ and (y, v) in Z, the quadratic form defined as

$$J((x, u, p, \mu); (y, v)) := \int_{t_0}^{t_1} 2\Omega(t, y(t), v(t))dt$$

where, for all $(t, y, v) \in T \times \mathbf{R}^n \times \mathbf{R}^m$,

$$2\Omega(t, y, v) := -[\langle y, H_{xx}(t)y\rangle\ + 2\langle y, H_{xu}(t)v\rangle +$$

$$\langle v, H_{uu}(t)v\rangle]$$

and $H(t)$ denotes $H(\tilde{x}(t), p(t), \mu(t), 1)$.

The following result was derived in [6] by reducing the original problem into a problem involving only equality constraints in the control. In what follows, the notation $\varphi_{\gamma u}(t, u_0(t))$ is short for $\dfrac{\partial \varphi_\gamma}{\partial u}(t, u_0(t))$.

Theorem 3.5 *Let* (x_0, u_0) *be an admissible process for which there exists* $(p, \mu) \in X \times \mathcal{U}_q$ *such that* $(x_0, u_0, p, \mu) \in \mathcal{E}$. *If* (x_0, u_0) *is a strongly normal solution to* (P) *then*

$$J((x_0, u_0, p, \mu); (y, v)) \geq 0$$

for all $(y, v) \in Z$ *satisfying*

i. $\dot{y}(t) = f_x(\tilde{x}_0(t))y(t) + f_u(\tilde{x}_0(t))v(t)$ ($t \in T$), *and* $y(t_0) = y(t_1) = 0$;

ii. $\varphi_{\gamma u}(t, u_0(t))v(t) = 0$ ($\gamma \in I_a(t, u_0(t)) \cup Q,\ t \in T$).

It is of interest to see if, in Theorem 3.5, the assumption of strong normality can be weakened and the set of critical directions where the second order condition holds can be enlarged.

4 Second order conditions

For any $(t, u) \in T \times \mathbf{R}^m$ define

$$\tau(t, u) := \{h \in \mathbf{R}^m \mid \varphi_{\alpha u}(t, u)h \leq 0\ (\alpha \in I_a(t, u)),$$

$$\varphi_{\beta u}(t, u)h = 0\ (\beta \in Q)\}.$$

Definition 4.1 Let $(x, u) \in Z$ and set $A(t) := f_x(\tilde{x}(t))$, $B(t) := f_u(\tilde{x}(t))$ ($t \in T$). We shall say that (x, u) is *τ-regular* if there is no nonnull solution $z \in X$ to the system

$$\dot{z}(t) = -A^*(t)z(t),$$

$$z^*(t)B(t)h \leq 0 \quad \text{for all } h \in \tau(t, u(t)) \quad (t \in T).$$

As one can show, the notions of weak normality (normality relative to S) and τ-regularity are equivalent (for a simpler case we refer to [12]). The set $\tau(t, u)$ defined above is a cone of "critical directions" associated with S. In a similar way, for any $(t, u) \in T \times \mathbf{R}^m$, let

$$\tau_0(t, u) := \{h \in \mathbf{R}^m \mid \varphi_{\gamma u}(t, u)h = 0$$

$$(\gamma \in I_a(t,u) \cup Q)\}.$$

By Definition 4.1, $(x,u) \in Z$ is τ_0-regular if there is no nonnull solution $z \in X$ to the system

$$\dot{z}(t) = -A^*(t)z(t),$$

$$z^*(t)B(t)h = 0 \quad \text{for all } h \in \tau_0(t,u(t)) \quad (t \in T).$$

Recall that the notion of strong normality, or normality relative to the set S_0 depends on a given admissible process (x_0,u_0). Note that (x_0,u_0) is τ_0-regular $\Leftrightarrow$ (x_0,u_0) is strongly normal.

Suppose now that we are given $\mu \in \mathcal{U}_q$ with $\mu_\alpha(t) \geq 0$ $(\alpha \in R,\ t \in T)$, and we define

$$S_1 = \{(x,u) \in D \mid \ \varphi_\alpha(t,u(t)) \leq 0$$

$$(\alpha \in R,\ \mu_\alpha(t) = 0,\ t \in T),$$

$$\varphi_\beta(t,u(t)) = 0$$

$$(\beta \in R \text{ with } \mu_\beta(t) > 0, \text{ or } \beta \in Q,\ t \in T)\}.$$

Associate with this set, for all $(t,u) \in T \times \mathbf{R}^m$ and $\mu \in \mathbf{R}^q$, the cone of directions

$$\tau_1(t,u,\mu) := \{h \in \mathbf{R}^m \mid \varphi_{\alpha u}(t,u)h \leq 0$$

$$(\alpha \in I_a(t,u),\ \mu_\alpha = 0),$$

$$\varphi_{\beta u}(t,u)h = 0 \ (\beta \in R \text{ with } \mu_\beta > 0, \text{ or } \beta \in Q)\}.$$

If there is no nonnull solution $z \in X$ to the system

$$\dot{z}(t) = -A^*(t)z(t),$$

$$z^*(t)B(t)h \leq 0 \text{ for all } h \in \tau_1(t,u(t),\mu(t)) \ (t \in T)$$

then (x,u) is called τ_1-*regular*.

Note that, if $\mu \in \mathcal{U}_q$ is such that $\mu_\alpha(t) \geq 0$ $(\alpha \in R,\ t \in T)$ and $(x,u) \in S_1$, then (x,u) is τ_1-regular $\Leftrightarrow$ (x,u) is normal relative to S_1. Also, as one readily verifies, if (x_0,u_0) is normal relative to S_0 (strongly normal) then it is normal relative to S_1 (with μ as above) which in turn implies normality relative to S (weakly normal).

Now, in this section we shall derive second order necessary conditions for problem (P). Our main result will be proved with the help of an auxiliary result established in [4] and partially based on the theory presented in [8]. It is a consequence of the full rank assumption mentioned in Section 2.

Let us consider the problem, which we label (C), of minimizing $\int_{t_0}^{t_1} L(t,u(t))dt$ on the set

$$\mathcal{C} := \{u \in \mathcal{U}_m \mid (t,u(t)) \in \mathcal{A} \ (t \in T)\}.$$

Lemma 4.1 *Let* $u_0 \in \mathcal{C}$. *Then* u_0 *solves* (C) $\Leftrightarrow$ $L(t,u) \geq L(t,u_0(t))$ $(t \in T)$ *whenever* $(t,u) \in \mathcal{A}$. *In this event, there exists a unique* $\mu \in \mathcal{U}_q$ *such that* $F_u(t,u_0(t),\mu(t)) = 0$ $(t \in T)$ *where*

$$F(t,u,\mu) := L(t,u) + \langle \mu, \varphi(t,u) \rangle.$$

Moreover, $\mu_\alpha(t) \geq 0$ *and* $\mu_\alpha(t)\varphi_\alpha(t,u_0(t)) = 0$ $(\alpha \in R,\ t \in T)$, *and*

$$\langle h, F_{uu}(t,u_0(t),\mu(t))h \rangle \geq 0$$

for all $h \in \tau_1(t,u_0(t),\mu(t))$ $(t \in T)$.

Note that, if (x_0,u_0) is admissible for (P) and u_0 solves (C), then (x_0,u_0) solves (P). However, we may have a solution (x_0,u_0) to the problem (P) but u_0 does not solve (C). In fact, it may even afford a maximum to I on $\mathcal{C}$. A simple example illustrates this fact.

Example 4.2 Consider the problem of minimizing $I(x,u) = \int_0^1 u(t)dt$ subject to $(x,u) \in Z$ and

$$\dot{x}(t) = u^2(t),\ x(0) = x(1) = 0,\ u(t) \leq 0.$$

Then $(x_0,u_0) \equiv (0,0)$ is a solution to (P), being the only admissible process, but $u_0 \equiv 0$ does not solve (C), that is, it does not minimize $\int_0^1 u(t)dt$ over the set

$$\mathcal{C} = \{u \in \mathcal{U}_1 \mid u(t) \leq 0 \ (t \in T)\}.$$

In this example, u_0 maximizes I on $\mathcal{C}$. Note also that (x_0,u_0) is not normal with respect to S since $B(t) = f_u(\tilde{x}_0(t)) = 0$ $(t \in [0,1])$.

The following example illustrates an opposite situation where, though one can exhibit a solution to the problem (C), the original problem (P) may fail to have a solution.

Example 4.3 Consider the problem of minimizing $I(x,u) = \int_0^1 u_2^2(t)dt$ subject to $(x,u) \in Z$ and

$$\begin{cases} \dot{x}(t) = u_3(t) - u_1(t) + u_1^2(t)u_2(t) \ (t \in [0,1]); \\ x(0) = x(1) = 0; \\ u_3(t) - u_1(t) \leq -2,\ -u_3(t) \leq 1, \\ u_1^2(t)u_2(t) \leq 2 \ (t \in [0,1]). \end{cases}$$

Clearly $u_0 \equiv (1,0,-1)$ is a solution to the problem (C) of minimizing $\int_0^1 u_2^2(t)dt$ on the set

$$\mathcal{C} = \{(u_1,u_2,u_3) \in \mathcal{U}_3 \mid u_3(t) - u_1(t) \leq -2,$$

$$-u_3(t) \leq 1,\ u_1^2(t)u_2(t) \leq 2 \ (t \in [0,1])\}.$$

However, if (x,u) is admissible, then necessarily $0 < u_2(t) \leq 2$ $(t \in [0,1])$. This can be easily seen since, by the constraints, we have $\dot{x}(t) \leq 0$ and $x(0) = x(1) = 0$ implying that $x \equiv 0$ and so

$u_3 - u_1 + u_1^2 u_2 \equiv 0$. Therefore $u_1^2 u_2 \equiv 2$ and, since $-1 \leq u_3(t) \leq u_1(t) - 2$, we have $u_1(t) \geq 1$. This proves the claim.

Theorem 4.4 *Suppose* (x_0, u_0) *solves* (P) *and there exists* $(p, \mu) \in X \times \mathcal{U}_q$ *satisfying*

a. $\mu_\alpha(t) \geq 0$ *and* $\mu_\alpha(t)\varphi_\alpha(t, u_0(t)) = 0$ $(\alpha \in R,\ t \in T)$*;*

b. $\dot{p}(t) = -f_x^*(\tilde{x}_0(t))p(t)$ *and* $p^*(t)f_u(\tilde{x}_0(t)) = L_u(t, u_0(t)) + \mu^*(t)\varphi_u(t, u_0(t))$ $(t \in T)$*.*

Suppose also that u_0 *solves* (C). *If* $p \equiv 0$ *then*

$$J((x_0, u_0, p, \mu); (y, v)) \geq 0$$

for all $(y, v) \in Z$ *with* $v(t) \in \tau_1(t, u_0(t), \mu(t))$ $(t \in T)$. *In particular,* $p \equiv 0$ *if* (x_0, u_0) *is normal relative to* S_1.

Proof: Let

$$F(t, u, \nu) := L(t, u) + \langle \nu, \varphi(t, u) \rangle.$$

By Lemma 4.1,

$$L(t, u) \geq L(t, u_0(t)) \quad (t \in T)$$

whenever $(t, u) \in \mathcal{A}$, and so there exists a unique $\nu \in \mathcal{U}_q$ such that

$$F_u(t, u_0(t), \nu(t)) =$$

$$L_u(t, u_0(t)) + \nu^*(t)\varphi_u(t, u_0(t)) = 0 \ (t \in T).$$

Moreover, $\nu_\alpha(t) \geq 0$ and $\nu_\alpha(t)\varphi_\alpha(t, u_0(t)) = 0$ $(\alpha \in R,\ t \in T)$, and

$$\langle h, F_{uu}(t, u_0(t), \nu(t))h \rangle \geq 0$$

for all $h \in \tau_1(t, u_0(t), \nu(t))$. Assume that $p \equiv 0$. By (b), we have

$$F_u(t, u_0(t), \mu(t)) =$$

$$L_u(t, u_0(t)) + \mu^*(t)\varphi_u(t, u_0(t)) = 0 \ (t \in T)$$

and so, by uniqueness, $\mu \equiv \nu$. Since

$$H(t, x, u, p, \mu, 1) =$$

$$\langle p, f(t, x, u) \rangle - L(t, u) - \langle \mu, \varphi(t, u) \rangle =$$

$$-F(t, u, \mu)$$

we have $2\Omega(t, y, v) = \langle v, F_{uu}(t, u_0(t), \mu(t))v \rangle$ and the first part follows.

To show that normality relative to S_1 implies that $p \equiv 0$ note first that

$$L_u(t, u_0(t)) = -\nu^*(t)\varphi_u(t, u_0(t)) \quad (t \in T)$$

and so, by (b),

$$p^*(t)B(t) = \sum_1^q (\mu_\alpha(t) - \nu_\alpha(t))\varphi_{\alpha u}(t, u_0(t)).$$

This implies that, for all $h \in \tau_1(t, u_0(t), \mu(t))$,

$$p^*(t)B(t)h = \sum_{\alpha \in N(t)} -\nu_\alpha(t)\varphi_{\alpha u}(t, u_0(t))h$$

where

$$N(t) = \{\alpha \in I_a(t, u_0(t)) \mid \mu_\alpha(t) = 0\}.$$

We conclude that $p^*(t)B(t)h \geq 0$ for all $h \in \tau_1(t, u_0(t), \mu(t))$ and therefore $-p$ is a solution to the system

$$\dot{z}(t) = -A^*(t)z(t),$$

$$z^*(t)B(t)h \leq 0 \text{ for all } h \in \tau_1(t, u_0(t), \mu(t)) \ (t \in T).$$

It follows that, if (x_0, u_0) is normal relative to S_1, then $p \equiv 0$. ∎

5 Conclusions

In this paper we provide a constraint qualification for second order necessary conditions which can be applied to certain classes of optimal control problems involving inequality and equality constraints in the time variable and the control functions. We show that the conditions hold on a cone of critical directions which contains the usual cone which depends on the sign of the corresponding Lagrange multipliers.

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