Motion Control of Human Bipedal Model in Sagittal Plane

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Abstract: - This paper discusses the construction of a mathematical model for a planar seven-link bipedal model which is comprised of an upper body and two legs (thigh, shank and feet in every leg) in the sagittal plane. Procedures of kinematic model and dynamic model constructions are presented in this paper. The mathematical model for dynamic equations of motion based on the absolute angle is obtained using Lagrange's equations. Then, a dimension transformation of mathematical model into relative angles was performed. New inertia matrix of the transformed equations was verified to be symmetric. Periodic cubic spline is used to obtain smooth walking trajectories of every joint in the biped model. Computed torque controller was applied in the trajectory tracking control of the proposed bipedal robot model. To investigate the performance of motion controller, a simulation study was conducted. The simulation results show that the performance of the purposed motion controller is superior with very minimal tracking error. In future, the controller will be extended and modified with various types of intelligent approaches to give continuous, automatic and online computation of required inertia matrix with certain constraint while the system is in motion.

Key-Words: - seven-link human biped model, Lagrange's Equation, computed torque control

1 Introduction

In this modern time, many research groups are actively involved in human bipedal robotics research including humanoid robots and exoskeletons. These types of robots are required to walk naturally to provide a sense of intimacy to human being. To achieve this, an accurate model of human gait has to be obtained to reflect natural human walking trajectories [1]. However, human walking pattern is a complex activity. It is hard for human's gait pattern to be directly incorporated into the bipedal robot model due to complex bipedal structure and excessive many degrees of freedom (DOF) in human gait [2].

There are 20 or more degrees of freedom (DOF) involved in human walking motion [3]. The inclusion of these high DOF into the mathematical model of human gait pattern could be problematic and may greatly restrict the implication of this model in the following engineering control tasks. Thus, the human model for the gait analysis has to be as simple as possible. Many research efforts have been devoted into the solution of this complicated control mechanism. They simplified the complex human biped model into various types of biped locomotion model which include two-link [4], three-link [5], five-link [6] and seven-link gait model [7] that constrained in sagittal plane.

This research study focused on the development of mathematical model for seven-link human bipedal model. Then, the mathematical model is applied in the MATLAB Simulink to observe the performances of the bipedal robot system.

This paper is structured as follows. Section 2 shows the kinetic model and dynamic equations of motion model for seven-link biped robot system using Lagrange's Equation. Analysis in Section 3 is presented the controller involved to observe the performances of bipedal robot system. This section also shows the simulation parameters setup and results from this study. Section 4 will be carried out the results and discussion. Finally, the conclusions of the study are made and some recommendations for future work is mentioned in Section 5.

2 Mathematical Modelling of Seven-Link Biped Model

In general, robot manipulators can be defined as a mechanical system that consists of links connected by joints. The links are numbered sequentially from the base (link 0) and up to the end-effector (link n)

[8]. The joints coincide to the contact points between two links. An actuator is usually placed at the joint. Therefore, every joint is controlled by an actuator independently and the joint movements give the relative movement of the links.

This section shows the kinematic model and dynamic equations of motion of the seven-link biped model briefly. To obtain the mathematical model of the biped system in this study, Lagrange's equations of motion has been used. The procedure of the derivation will be shown in this section briefly.

2.1 Kinematic Model

The human biped is modeled as the seven serial links mechanism in sagittal plane as shown in Fig. 1. The seven links consists torso (link 4) and three links in each leg which are thigh (link 3 and 5), shank (link 2 and 6) and feet (link 1 and 7). These links are connected via rotating joints which are two hip joints, two knee joints and two angle joints. The joints are assumed to be frictionless and every of them are driven by an independent DC motor. The derivation of mathematical model in this study was similar to that of S. Tzafestas et al. [3] [6] with the exception that this study concentrated more on the seven-link biped model instead of the initial simpler five-link biped model.

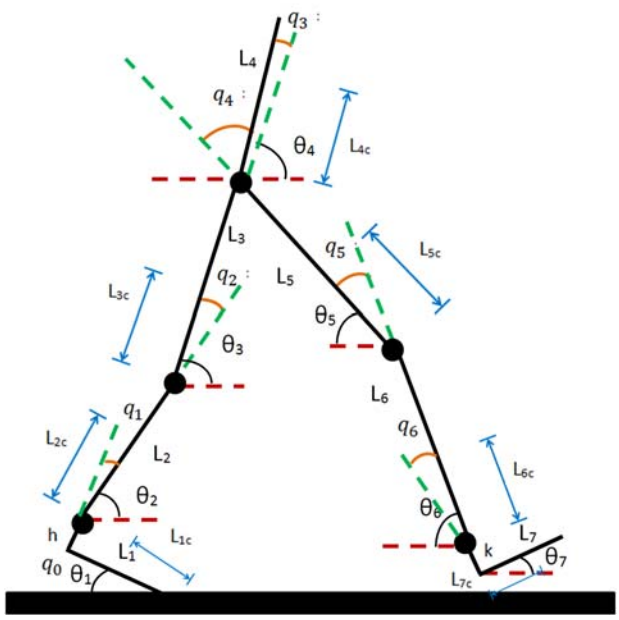

Fig. 1: Seven link planar biped model

To simplify the derivation, some assumptions have been made for example both left and right sides of the biped model are assumed to be symmetric; the biped locomotion is constrained to the sagittal plane only, and the friction of the ground is assumed to be large enough to avoid slippage at of the supporting end.

Walking dynamics usually takes place on the sagittal plane [9] which is defined as a plane that can be divided body parts into right and left sides [10]. The sagittal plane analysis drives the human biped gait pattern similar to the human gait pattern.

From Fig 1, the parameters of the biped model are shown as follows:

m_i : mass of link
L_i : length of link
L_{ic} : distance between the center of mass and the lower joint of link
I_i : moment of Inertia
θ_i : angle of link with respect to the horizontal axis (absolute angle)
q_i : relative angle deflections of the corresponding joints

(x_{ic}, y_{ic}) are the coordinates of center of mass of link i which is shown in (1) based from Fig. 1.

$$
\begin{aligned}
x_{1c} &= -L_{1c}cos\theta_1 \\
y_{1c} &= L_{1c}sin\theta_1 \\[6pt]
x_{2c} &= -L_1cos\theta_1 + hsin\theta_1 + L_{2c}cos\theta_2 \\
y_{2c} &= L_1sin\theta_1 + hcos\theta_1 + L_{2c}sin\theta_2 \\[6pt]
x_{3c} &= -L_1cos\theta_1 + hsin\theta_1 + L_2cos\theta_2 + L_{3c}cos\theta_3 \\
y_{3c} &= L_1sin\theta_1 + hcos\theta_1 + L_2sin\theta_2 + L_{3c}sin\theta_3 \\[6pt]
x_{4c} &= -L_1cos\theta_1 + hsin\theta_1 + L_2cos\theta_2 + L_3cos\theta_3 + L_{4c}cos\theta_4 \\
y_{4c} &= L_1sin\theta_1 + hcos\theta_1 + L_2sin\theta_2 + L_3sin\theta_3 + L_{4c}sin\theta_4 \\[6pt]
x_{5c} &= -L_1cos\theta_1 + hsin\theta_1 + L_2cos\theta_2 + L_3cos\theta_3 \\
&\quad +(L_5 - L_{5c})cos\theta_5 \\
y_{5c} &= L_1sin\theta_1 + hcos\theta_1 + L_2sin\theta_2 + L_3sin\theta_3 \\
&\quad -(L_5 - L_{5c})sin\theta_5 \\[6pt]
x_{6c} &= -L_1cos\theta_1 + hsin\theta_1 + L_2cos\theta_2 + L_3cos\theta_3 + L_5cos\theta_5 \\
&\quad +(L_6 - L_{6c})cos\theta_6 \\
y_{6c} &= L_1sin\theta_1 + hcos\theta_1 + L_2sin\theta_2 + L_3sin\theta_3 - L_5sin\theta_5 \\
&\quad -(L_6 - L_{6c})sin\theta_6 \\[6pt]
x_{7c} &= -L_1cos\theta_1 + hsin\theta_1 + L_2cos\theta_2 + L_3cos\theta_3 + L_5cos\theta_5 \\
&\quad +L_6cos\theta_6 + ksin\theta_7 + L_{7c}cos\theta_7 \\
y_{7c} &= L_1sin\theta_1 + hcos\theta_1 + L_2sin\theta_2 + L_3sin\theta_3 - L_5sin\theta_5 \\
&\quad -L_6sin\theta_6 - kcos\theta_7 + L_{7c}sin\theta_7
\end{aligned} \tag{1}
$$

2.2 Dynamic Model

The dynamic equations of a robot manipulator in closed form can be acquired by using Lagrange's Equations. It is one of the most common approaches used in the computation of robotic dynamic model.

In the past, Lagrange formulation has been used to derive the mathematical model of dynamic motion equations for the human biped model. The human biped model that can be generated from Lagrange's Equations [2] which are two-link [4], three-link [5], five-link [6] and seven-link gait model [7]. S. Tzafestas et al. [6] shown the calculations of five-link biped model using relative angles term. However, up to this date, the derivation of mathematical model for a seven-link biped model has still not been fully covered. The mathematical model as proposed by D. J. Braun et al. was considerably crude, and it lacked out of the essential information clarity needed in the comprehensive study of the biped robot dynamics [7][11].

In this study, the biped dynamic model is further simplified by considering only the single-leg support phase. During the single-leg support phase, one of the biped is in contact with the surface (support leg) carrying all of the biped weight, whilst the other leg which is freely swinging in the mid air in the forward walking direction [12] as shown in Fig. 1. The Lagrange's equation of motion can be written in the following form:

$$\frac{d}{dt}\left(\frac{dL}{d\dot{\theta}_i}\right) - \frac{dL}{d\theta_i} = \tau_i \tag{2}$$

where $L = K - P$

L : Lagrangian of n-DOF robot manipulator
K: kinetic energy
P: potential energy

The procedure of Lagrange's equations of motion formulation can be found in Appendix A and B. These equations can be arranged in the general form (The full equations can be referred in Appendix C) as

$$D(\theta)\ddot{\theta} + H(\theta,\dot{\theta}) + G(\theta) = T_\theta \tag{3}$$

where
$\theta = [\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7]^T$
$T_\theta = [T_{\theta 1}, T_{\theta 2}, T_{\theta 3}, T_{\theta 4}, T_{\theta 5}, T_{\theta 6}, T_{\theta 7}]^T$

$$H(\theta,\dot{\theta}) = \mathrm{col}\left[\sum_{j=1(j\neq i)}^{7} (h_{ij}(\dot{\theta}_j)^2\right]$$

$G(\theta) = \mathrm{col}[G_i(\theta)]$
$D(\theta) = [D_{ij}(\theta)] \qquad i,j = 1,\dots,7$

with

θ : joint angle vector
T_θ : generalized torque corresponds to θ_i
$D(\theta)$: 7×7 symmetric, positive-definite inertia matrix
$H(\theta,\dot{\theta})$: 7×1 vector of Coriolis and Centripetal torques
$G(\theta)$: 7×1 vector of gravitational torques

However, only six of seven DOF can be controlled directly by the driving torques at every joint. The angle θ_1 at the contact point with the walking surface which is known as hypothetical joint 0 is controlled indirectly using the gravitational effects [12]. The model in Equation (3) is transformed model using the relative angle for the control purpose.

The dynamic motions of the biped model are calculated in the terms of relative angles the link for the control purpose which is

$$D(q)\ddot{q} + H(q,\dot{q}) + G(q) = T_q \tag{4}$$

Based on Fig. 1, $q_1, q_2, q_3, q_4, q_5, q_6$ and q_7 are the relative angle deflections of the corresponding joints and can be calculated as follows:

$$\begin{aligned} q_0 &= \theta_1 \\ q_1 &= \frac{\pi}{2} - \theta_1 - \theta_2 \\ q_2 &= \theta_3 - \theta_2 \\ q_3 &= \theta_4 - \theta_3 \\ q_4 &= \pi - \theta_4 - \theta_5 \\ q_5 &= \theta_6 - \theta_5 \\ q_6 &= \theta_6 + \theta_7 - \frac{\pi}{2} \end{aligned} \tag{5}$$

where
τ_0 : driving torque at toes of the supporting leg
τ_1 : driving torque at the ankle of the supporting leg
τ_2 : driving torque at knee of the supporting leg
τ_3 : driving torque at hip of the supporting leg
τ_4 : driving torque at hip of the free leg
τ_5 : driving torque at knee of the free leg
τ_6 : driving torque at ankle of the free leg

The seven-link model is using the q_i $(i = 0,1,\dots,6)$ instead of θ_i $(i = 1,2,\dots,7)$ of where corresponds to the hypothetical joint 0 at the contact point with $q_0 = \theta_1$.

The relationship between θ and q as follows:

$$T_{\theta i} = \sum_{j=1}^{6} \tau_j \frac{\partial q_j}{\partial \theta_i}, \qquad i = 1,2,\dots,7 \tag{6}$$

which gives

$$T_{\theta i} = E.\tau \tag{7}$$

where E is a 7×6 matrix and τ is a 6×1 matrix.

Therefore, the relation is formed as below:

$$T_\theta = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tau_{q1} \\ \tau_{q2} \\ \tau_{q3} \\ \tau_{q4} \\ \tau_{q5} \\ \tau_{q6} \end{bmatrix} \tag{8}$$

The generalized torques T_{qi} corresponds to the relative angle displacements which are:

$$T_{q0} = 0 \text{ and } T_{qi} = \tau_i \tag{9}$$

where τ_i is the actual driving torques at the joints of the model. $T_{q0} = 0$ shows that the angle q_0 of the hypothetical joint 0 is not directly controlled by a driving torque.

The angular displacement of every link can be written in terms of q_i which are:

$$\begin{aligned} \theta_1 &= q_0 \\ \theta_2 &= \frac{\pi}{2} - q_0 - q_1 \\ \theta_3 &= \frac{\pi}{2} - q_0 - q_1 + q_2 \\ \theta_4 &= \frac{\pi}{2} - q_0 - q_1 + q_2 + q_3 \\ \theta_5 &= \frac{\pi}{2} + q_0 + q_1 - q_2 - q_3 - q_4 \\ \theta_6 &= \frac{\pi}{2} + q_0 + q_1 - q_2 - q_3 - q_4 + q_5 \\ \theta_7 &= -q_0 - q_1 + q_2 + q_3 + q_4 - q_5 + q_5 \end{aligned} \tag{10}$$

From the relationship follows:

$$T_{qi} = \sum_{j=1}^{7} T_{\theta j} \frac{\partial q_j}{\partial q_i}, \qquad i = 0,1,\dots,6 \tag{11}$$

Thus, the generalized torques T_{qi} can be obtained as

$$\begin{aligned} T_{q0} &= -T_{\theta 1} + T_{\theta 2} + T_{\theta 3} + T_{\theta 4} - T_{\theta 5} - T_{\theta 6} + T_{\theta 7} \\ T_{q1} &= -T_{\theta 2} - T_{\theta 3} - T_{\theta 4} + T_{\theta 5} + T_{\theta 6} - T_{\theta 7} \\ T_{q2} &= T_{\theta 3} + T_{\theta 4} - T_{\theta 5} - T_{\theta 6} + T_{\theta 7} \\ T_{q3} &= T_{\theta 4} - T_{\theta 5} - T_{\theta 6} + T_{\theta 7} \\ T_{q4} &= -T_{\theta 5} - T_{\theta 6} + T_{\theta 7} \\ T_{q5} &= T_{\theta 6} - T_{\theta 7} \\ T_{q6} &= T_{\theta 7} \end{aligned} \tag{12}$$

Using the relationship in Equation (9), (10) and (12), the equations of motion are transformed into the following forms which are:

$$A_{11}\ddot{\theta}_1 + A\ddot{\theta}_2 + A_{13}\ddot{\theta}_3 + A_{14}\ddot{\theta}_4 + A_{15}\ddot{\theta}_5 + A_{16}\ddot{\theta}_6 + A_{17}\ddot{\theta}_7 + H_{q0} + G_{q0} = T_{q0} = 0$$

where $\quad A_{1j} = -D_{1j} + D_{2j} + D_{3j} + D_{4j} - D_{5j} - D_{6j} + D_{7j}, \ \ j = 1,2,3,4,5,6,7$
$\quad H_{q0} = -H_1 + H_2 + H_3 + H_4 - H_5 - H_6 + H_7$
$\quad G_{q0} = -G_1 + G_2 + G_3 + G_4 - G_5 - G_6 + G_7$

$$A_{21}\ddot{\theta}_1 + A_{22}\ddot{\theta}_2 + A_{23}\ddot{\theta}_3 + A_{24}\ddot{\theta}_4 + A_{25}\ddot{\theta}_5 + A_{26}\ddot{\theta}_6 + A_{27}\ddot{\theta}_7 + H_{q1} + G_{q1} = T_{q1}$$

where $\quad A_{2j} = -D_{2j} - D_{3j} - D_{4j} + D_{5j} + D_{6j} - D_{7j}, \quad j = 1,2,3,4,5,6,7$
$\quad H_{q1} = -H_2 - H_3 - H_4 + H_5 + H_6 - H_7$
$\quad G_{q1} = -G_2 - G_3 - G_4 + G_5 + G_6 - G_7$

$$A_{31}\ddot{\theta}_1 + A_{32}\ddot{\theta}_2 + A_{33}\ddot{\theta}_3 + A_{34}\ddot{\theta}_4 + A_{35}\ddot{\theta}_5 + A_{36}\ddot{\theta}_6 + A_{37}\ddot{\theta}_7 + H_{q2} + G_{q2} = T_{q2}$$

where $\quad A_{3j} = D_{3j} + D_{4j} - D_{5j} - D_{6j} + D_{7j}, \quad j = 1,2,3,4,5,6,7$
$\quad H_{q2} = H_3 + H_4 - H_5 - H_6 + H_7$
$\quad G_{q2} = G_3 + G_4 - G_5 - G_6 + G_7$

$$A_{41}\ddot{\theta}_1 + A_{42}\ddot{\theta}_2 + A_{43}\ddot{\theta}_3 + A_{44}\ddot{\theta}_4 + A_{45}\ddot{\theta}_5 + A_{46}\ddot{\theta}_6 + A_{47}\ddot{\theta}_7 + H_{q3} + G_{q3} = T_{q3}$$

where $\quad A_{4j} = D_{4j} - D_{5j} - D_{6j} + D_{7j}, \quad j = 1,2,3,4,5,6,7$
$\quad H_{q3} = H_4 - H_5 - H_6 + H_7$
$\quad G_{q3} = G_4 - G_5 - G_6 + G_7$

$$A_{51}\ddot{\theta}_1 + A_{52}\ddot{\theta}_2 + A_{53}\ddot{\theta}_3 + A_{54}\ddot{\theta}_4 + A_{55}\ddot{\theta}_5 + A_{56}\ddot{\theta}_6 + A_{57}\ddot{\theta}_7 + H_{q4} + G_{q4} = T_{q4}$$

where $\quad A_{5j} = -D_{5j} - D_{6j} + D_{7j}, \quad j = 1,2,3,4,5,6,7$
$\quad H_{q4} = -H_5 - H_6 + H_7$
$\quad G_{q4} = -G_5 - G_6 + G_7$

$$A_{61}\ddot{\theta}_1 + A_{62}\ddot{\theta}_2 + A_{63}\ddot{\theta}_3 + A_{64}\ddot{\theta}_4 + A_{65}\ddot{\theta}_5 + A_{66}\ddot{\theta}_6 + A_{67}\ddot{\theta}_7 + H_{q5} + G_{q5} = T_{q5}$$

where $\quad A_{6j} = D_{6j} - D_{7j}, \quad j = 1,2,3,4,5,6,7$
$\quad H_{q5} = H_6 - H_7$
$\quad G_{q5} = G_6 - G_7$

$$A_{71}\ddot{\theta}_1 + A_{72}\ddot{\theta}_2 + A_{73}\ddot{\theta}_3 + A_{74}\ddot{\theta}_4 + A_{75}\ddot{\theta}_5 + A_{76}\ddot{\theta}_6 + A_{77}\ddot{\theta}_7 + H_{q6} + G_{q6} = T_{q6}$$

where $\quad A_{7j} = D_{7j}, \quad j = 1,2,3,4,5,6,7$
$\quad H_{q6} = H_7$
$\quad G_{q6} = G_7$

(13)

The equation of motion is further modified and transformed into the equation using the relative angle (The full equations can be found in Appendix D) which is

$$D_q(q)\ddot{q} + H_q(q,\dot{q}) + G_q(q) = T_q \tag{14}$$

where

$D_q(j,1) = -A_{j1} + A_{j2} + A_{j3} + A_{j4} - A_{j5} - A_{j6} + A_{j7}$
$D_q(j,2) = -A_{j2} - A_{j3} - A_{j4} + A_{j5} + A_{j6} - A_{j7}$
$D_q(j,3) = A_{j3} + A_{j4} - A_{j5} - A_{j6} + A_{j7}$
$D_q(j,4) = A_{j4} - A_{j5} - A_{j6} + A_{j7}$
$D_q(j,5) = -A_{j5} - A_{j6} + A_{j7}$
$D_q(j,6) = A_{j6} - A_{j7}$
$D_q(j,7) = A_{j7}$ with $j = 1,2,3,4,5,6,7$

$H_q(q,\dot{q}) = [H_{q0}, H_{q1}, H_{q2}, H_{q3}, H_{q4}, H_{q5}, H_{q6}]^T$
$G_q(q) = [G_{q0}, G_{q1}, G_{q2}, G_{q3}, G_{q4}, G_{q5}, G_{q6}]^T$
$T_q = [T_{q0}, T_{q1}, T_{q2}, T_{q3}, T_{q4}, T_{q5}, T_{q6}]^T$

$D_q(q)$ is 7×7 symmetric, positive definite inertia matrix, $H_q(q,\dot{q})$ is the 7×1 vector of centripetal and Coriolis torques, $G_q(q)$ is the 7× 1 vector of gravitational torques and T_q is the vector of control torques applied at each joint. This mathematical model of human biped will be carried out using relative angles in term of absolute angles θ_i (i=1,2,3,4,5,6,7). To verify the new inertia matrix $D_q(q)$ from the obtained transformed human biped model, this matrix is symmetric.

3 Analysis

The performance of the proposed seven-link bipedal robot model with computed torque motion control scheme was investigated and verified in this study through simulation analysis. All simulation studies were performed using computation platform by Simulink®, MATLAB®. The following sections will be provide detailed descriptions on computed torque control scheme, trajectory planning, and the simulation setup.

3.1 Computed Torque Control

In this study, a computed torque control scheme with additional proportional-derivative (PD) control is used in trajectory tracking of the proposed biped model. Based on Fig. 2, the generalised computed torque equation is:

$$T_q = D_q(q)u + H_q(q,\dot{q}) + G_q(q) \tag{15}$$

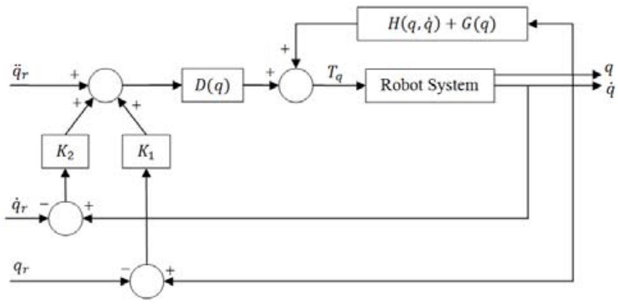

Fig. 2: PD-based computed torque control scheme [8]

The trajectory error, e is:

$$e = q - q_r \tag{16}$$

where

q : actual joint trajectory
q_r : reference joint trajectory

The control law based from PD control law is:

$$u = \ddot{q}_r + K_D\dot{e} + K_P e \tag{17}$$

where

$K_D = diag[k_{Di}]$
$K_P = diag[k_{Pi}]$
u: computed angular acceleration control signal

In order to attain the critical damped closed loop performance, the value of K_D and K_P are:

$K_D = diag[2\lambda]$
$K_P = diag[\lambda^2]$

where λ is the desired natural frequency of the closed-loop system

Due to the existence of leg contact with ground, the component u_0 cannot be computed with Equation (18) because it is chosen that $T_q(1) = 0$, and the biped model has pre-determined non-controllable joint q_0. Therefore, u_0 and u_{i+1} can be computed as follows:

$$u_0 = -\frac{1}{D_q(1,1)} \cdot \left\{ \sum_{j=1}^{6} [D_q(1,j+1) \cdot u_{j+1}] + H_q(1) + G_q(1) \right\} \tag{18}$$

$$u_{i+1} = \ddot{q}_{ri} + K_{Di}\dot{e}_i + K_{Pi}e_i \tag{19}$$

where $i = 0,1,2,3,4,5$

3.2 Trajectory Planning

The trajectory of revolute joints is divided into 100 segments throughout one gait cycle. There are many approaches that have been used in the interpolation and approximation in the trajectory planning. A good approach for linking the segments is required.

The gait cycle has to be continuous and periodically linked together in order to achieve a normal continuous motion. Therefore, Equation (20) generates the reference trajectory for link k [13]:

$$\theta_k(t) = a_i + b_i(t - t_k) + c_i(t - t_k)^2 + d_i(t - t_k)^3 \quad (20)$$

Human walking motion is in periodic function. Thus, a periodic cubic spline function can give the smoothness of velocity and acceleration of the biped model. The periodic boundary conditions have to be followed so that the trajectory will be continuous periodically. The boundary conditions are:

$$\theta_0 = \theta_n \quad (21)$$
$$\dot{\theta}_0 = \dot{\theta}_n \quad (22)$$
$$\ddot{\theta}_0 = \ddot{\theta}_n \quad (23)$$

From Equation (22), the coefficients of cubic spline are computed as follows:

$$a_i = \theta_i \quad (24)$$

$$c = A^{-1}z \quad (25)$$

where

$$A = \begin{bmatrix} 2(h_{n-1}+h_0) & h_0 & 0 & \cdots & 0 & h_{n-1} \\ h_0 & 2(h_0+h_1) & h_1 & 0 & \vdots & 0 \\ 0 & h_1 & 2(h_1+h_2) & h_2 & 0 & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \vdots & \ddots & \ddots & 2(h_{n-3}+h_{n-2}) & h_{n-2} \\ h_{n-1} & 0 & \cdots & 0 & h_{n-2} & 2(h_{n-2}+h_{n-1}) \end{bmatrix}$$

$$c = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-2} \\ c_{n-1} \end{bmatrix}$$

$$z = \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ \vdots \\ z_{n-2} \\ z_{n-1} \end{bmatrix}$$

with $h_i = t_{i+1} - t_i$

and $z_{i+1} = 3\left[\frac{\theta_{i+2}-\theta_{i+1}}{h_{i+1}} - \frac{\theta_{i+1}-\theta_i}{h_i}\right]$

Then, the value c_i is substituted into Equation (26) and Equation (27) to obtained the value of b_i and d_i.

$$b_i = \frac{\theta_{i+1} - \theta_i}{h_i} - \frac{(2c_i + c_{i+1})h_i}{3} \quad (26)$$

$$d_i = \frac{c_{i+1} - c_i}{3h_i} \quad (27)$$

Finally, all the values of a_i, b_i, c_i and d_i are substituted back into Equation (22) to obtain smooth walking trajectories of the joints. Fig. 3 shows the walking trajectories of every joint for biped robot model which are carried out by J. Perry and J. M. Burnfield [14].

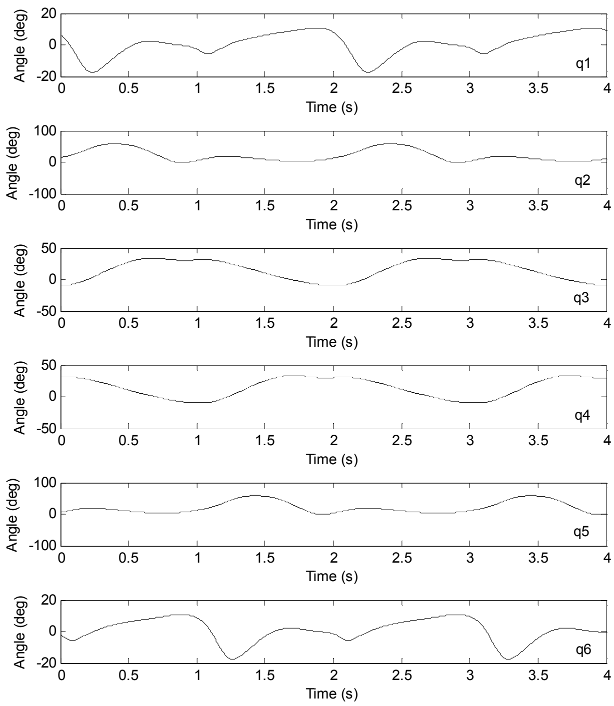

Fig. 3: Walking trajectories of every joint

3.3 Simulation Parameters Setup

The parameters of the biped robot model used in this study are shown in Table 1:

Table 1: Parameters of human biped model [15][16]

Link	Link Number	Mass, m (kg)	Length, L (m)	Loccation of Centre of Mass, L_c (m)	Moment of Inertia, I (kgm^2)
Torso	4	0.678M=44.07	0.75	0.375	0.990776
Thigh	3,5	0.1M=6.5	0.52	0.260	0.138952
Shank	2,6	0.046M=3.0225	0.37	0.185	0.065435
Foot	1	0.0145M=0.9425	0.27 with k,h=0.07	0.180	0.008774
	7			0.009	

The sampling time in this study is 0.001s. Through Heuristic Method, the λ was found to be 28rad/s. Therefore the controller gains for PD controller are:

k_P=784/s^2
k_D=56/s

4 Results and Discussion

This section will be presenting the results and discussion of this study. Fig. 4 until Fig. 9 show the tracking errors for each joints of the bipedal system. These results show that the system performs very well with considerably low trajectory tracking error. Therefore, the system is considerably stable and controllable.

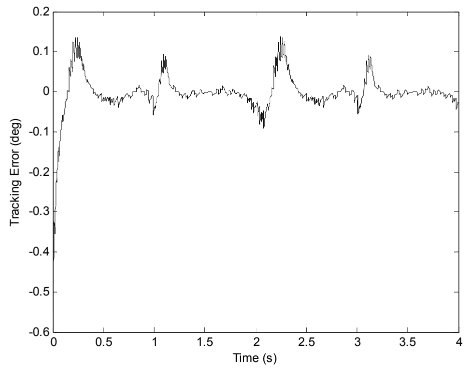

Fig. 4: Tracking error of joint 1

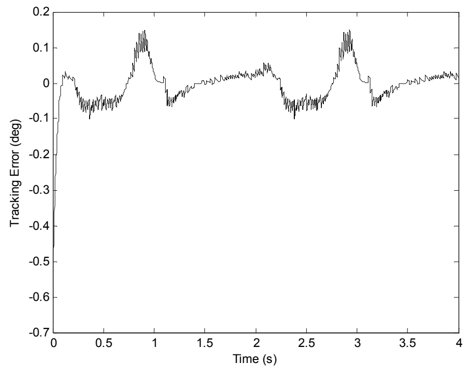

Fig. 5: Tracking error of joint 2

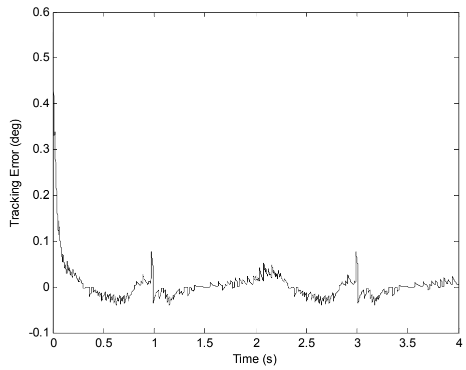

Fig. 6: Tracking error of joint 3

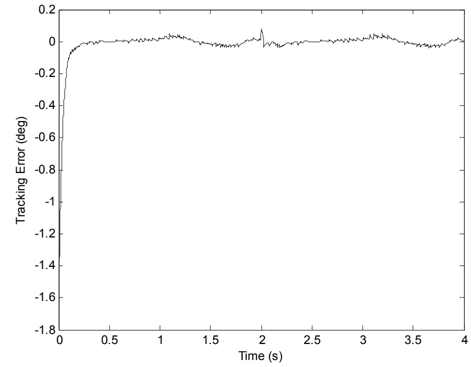

Fig. 7: Tracking error of joint 4

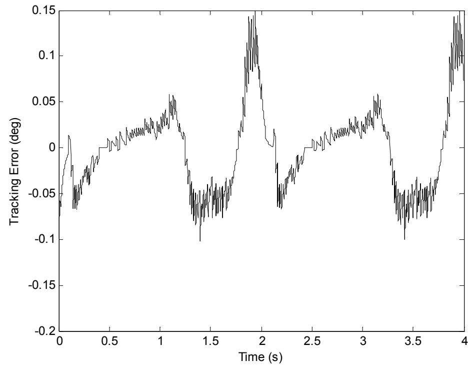

Fig. 8: Tracking error of joint 5

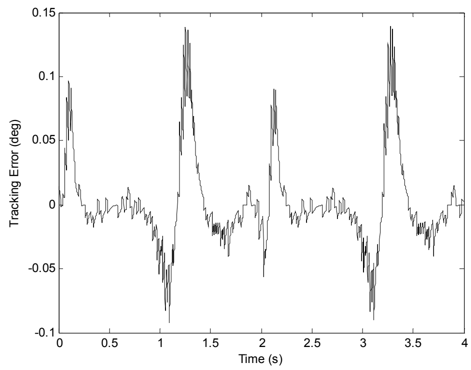

Fig. 9: Tracking error of joint 6

Table 2 summarizes the averaged tracking errors of each joints achieved in the bipedal robot model using PD-based computed torque control.

Table 2: Averaged tracking error of every joint obtained using computed torque control

Joint	Averaged Tracking Error (deg)
1	0.0261
2	0.0377
3	0.0190
4	0.0295
5	0.0347
6	0.0219

5 Conclusion

This paper presents a complete mathematical derivation workout for a generalized seven-link biped robot model walking on a flat horizontal surface. The equations of motion for the single support phase were constructed by using the biped model with one support leg is in contact with the surface carrying all of the biped weight, while the other leg which is freely swinging in the mid air in the forward walking direction. These equations were also developed using Lagrange's Equations using relative angles. The symmetrical matrix of new inertia matrix $D_q(q)$ shows that the obtained transformed human biped model is verified.

Trajectory planning is conducted by using periodic cubic spline to get smooth walking trajectories of every joint in the human biped model. Simulation study is done to explore the motion control performances of the seven-link biped robot using the obtained mathematical model using Simulink®, MATLAB® is successful in tracking the reference trajectories by giving small error value of every joint in the human bipedal model. For the future time, the controller of the human biped model will be extended and modified with different types of intelligent techniques to provide continuous, automatic and online computation of required inertia matrix under constraint while the system is in motion.

7 Acknowledgement

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APPENDIX

APPENDIX A

A. Kinetic Energy

$$K_1=\frac{1}{2}m_1L_{1c}^2\dot\theta_1^2+\frac{1}{2}I_1\dot\theta_1^2$$

$$\begin{aligned}K_2=&\frac{1}{2}m_2L_1^2\dot\theta_1^2+\frac{1}{2}m_2h^2\dot\theta_1^2+\frac{1}{2}m_2L_{2c}^2\dot\theta_2^2-m_2hL_{2c}\dot\theta_1\dot\theta_2\,sin(\theta_1+\theta_2)\\&+m_2L_1L_{2c}\dot\theta_1\dot\theta_2\,cos(\theta_1+\theta_2)+\tfrac{1}{2}I_2\dot\theta_2^2\end{aligned}$$

$$\begin{aligned}K_3=&\frac{1}{2}m_3L_1^2\dot\theta_1^2+\frac{1}{2}m_3h^2\dot\theta_1^2+\frac{1}{2}m_3L_2^2\dot\theta_2^2+\frac{1}{2}m_3L_{3c}^2\dot\theta_3^2\\&+m_3L_1L_2\dot\theta_1\dot\theta_2 cos(\theta_1+\theta_2)\\&+m_3L_1L_{3c}\dot\theta_1\dot\theta_3 cos(\theta_1+\theta_3)-m_3hL_2\dot\theta_1\dot\theta_2\,sin(\theta_1+\theta_2)\\&-m_3hL_{3c}\dot\theta_1\dot\theta_3\,sin(\theta_1+\theta_3)+m_3L_2L_{3c}\dot\theta_2\dot\theta_3 cos(\theta_2-\theta_3)+\tfrac{1}{2}I_3\dot\theta_3^2\end{aligned}$$

$$\begin{aligned}K_4=&\frac{1}{2}m_4L_1^2\dot\theta_1^2+\frac{1}{2}m_4h^2\dot\theta_1^2+\frac{1}{2}m_4L_2^2\dot\theta_2^2+\frac{1}{2}m_4L_3^2\dot\theta_3^2+\frac{1}{2}m_4L_{4c}^2\dot\theta_4^2\\&+m_4L_1L_2\dot\theta_1\dot\theta_2\,cos(\theta_1+\theta_2)+m_4L_1L_3\dot\theta_1\dot\theta_3\,cos(\theta_1+\theta_3)\\&+m_4L_1L_{4c}\dot\theta_1\dot\theta_4\,cos(\theta_1+\theta_4)-m_4hL_2\dot\theta_1\dot\theta_2\,sin(\theta_1+\theta_2)\\&-m_4hL_3\dot\theta_1\dot\theta_3\,sin(\theta_1+\theta_3)-m_4hL_{4c}\dot\theta_1\dot\theta_4\,sin(\theta_1+\theta_4)\\&+m_4L_2L_3\dot\theta_2\dot\theta_3\,cos(\theta_2-\theta_3)+m_4L_2L_{4c}\dot\theta_2\dot\theta_4\,cos(\theta_2-\theta_4)\\&+m_4L_3L_{4c}\dot\theta_3\dot\theta_4\,cos(\theta_3-\theta_4)+\tfrac{1}{2}I_4\dot\theta_4^2\end{aligned}$$

$$\begin{aligned}K_5=&\frac{1}{2}m_5L_1^2\dot\theta_1^2+\frac{1}{2}m_5h^2\dot\theta_1^2+\frac{1}{2}m_5L_2^2\dot\theta_2^2+\frac{1}{2}m_5L_3^2\dot\theta_3^2+\frac{1}{2}m_5L_5^2\dot\theta_5^2+\frac{1}{2}m_5L_{5c}^2\dot\theta_5^2\\&+m_5L_1L_2\dot\theta_1\dot\theta_2\,cos(\theta_1+\theta_2)+m_5L_1L_3\dot\theta_1\dot\theta_3\,cos(\theta_1+\theta_3)\\&-m_5L_1L_5\dot\theta_1\dot\theta_5\,cos(\theta_1-\theta_5)+m_5L_1L_{5c}\dot\theta_1\dot\theta_5\,cos(\theta_1-\theta_5)\\&-m_5hL_2\dot\theta_1\dot\theta_2\,sin(\theta_1+\theta_2)-m_5hL_3\dot\theta_1\dot\theta_3\,sin(\theta_1+\theta_3)\\&+m_5hL_5\dot\theta_1\dot\theta_5\,sin(\theta_1-\theta_5)-m_5hL_{5c}\dot\theta_1\dot\theta_5\,sin(\theta_1-\theta_5)\\&+m_5L_2L_3\dot\theta_2\dot\theta_3\,cos(\theta_2-\theta_3)-m_5L_2L_5\dot\theta_2\dot\theta_5\,cos(\theta_2+\theta_5)\\&+m_5L_2L_{5c}\dot\theta_2\dot\theta_5\,cos(\theta_2+\theta_5)-m_5L_3L_5\dot\theta_3\dot\theta_5\,cos(\theta_3+\theta_5)\\&+m_5L_3L_{5c}\dot\theta_3\dot\theta_5\,cos(\theta_3+\theta_5)-m_5L_5L_{5c}\dot\theta_5^2+\tfrac{1}{2}I_5\dot\theta_5^2\end{aligned}$$

$$\begin{aligned}K_6=&\tfrac{1}{2}m_6L_1^2\dot\theta_1^2+\tfrac{1}{2}m_6h^2\dot\theta_1^2+\tfrac{1}{2}m_6L_2^2\dot\theta_2^2+\tfrac{1}{2}m_6L_3^2\dot\theta_3^2+\tfrac{1}{2}m_6L_5^2\dot\theta_5^2+\tfrac{1}{2}m_6L_6^2\dot\theta_6^2\\&+\tfrac{1}{2}m_6L_{6c}^2\dot\theta_6^2+m_6L_1L_2\dot\theta_1\dot\theta_2\,cos(\theta_1+\theta_2)+m_6L_1L_3\dot\theta_1\dot\theta_3\,cos(\theta_1+\theta_3)\\&-m_6L_1L_5\dot\theta_1\dot\theta_5\,cos(\theta_1-\theta_5)-m_6L_1L_6\dot\theta_1\dot\theta_6\,cos(\theta_1-\theta_6)\\&+m_6L_1L_{6c}\dot\theta_1\dot\theta_6\,cos(\theta_1-\theta_6)-m_6hL_2\dot\theta_1\dot\theta_2\,sin(\theta_1+\theta_2)\\&-m_6hL_3\dot\theta_1\dot\theta_3\,sin(\theta_1+\theta_3)+m_6hL_5\dot\theta_1\dot\theta_5\,sin(\theta_1-\theta_5)\\&+m_6hL_6\dot\theta_1\dot\theta_6\,sin(\theta_1-\theta_6)-m_6hL_{6c}\dot\theta_1\dot\theta_6\,sin(\theta_1-\theta_6)\\&+m_6L_2L_3\dot\theta_2\dot\theta_3\,cos(\theta_2-\theta_3)-m_6L_2L_5\dot\theta_2\dot\theta_5\,cos(\theta_2+\theta_5)\\&-m_6L_2L_6\dot\theta_2\dot\theta_6\,cos(\theta_2+\theta_6)+m_6L_2L_{6c}\dot\theta_2\dot\theta_6\,cos(\theta_2+\theta_6)\\&-m_6L_3L_5\dot\theta_3\dot\theta_5\,cos(\theta_3+\theta_5)-m_6L_3L_6\dot\theta_3\dot\theta_6\,cos(\theta_3+\theta_6)\\&+m_6L_3L_{6c}\dot\theta_3\dot\theta_6\,cos(\theta_3+\theta_6)+m_6L_5L_6\dot\theta_5\dot\theta_6\,cos(\theta_5-\theta_6)\\&-m_6L_5L_{6c}\dot\theta_5\dot\theta_6\,cos(\theta_5-\theta_6)-m_6L_6L_{6c}\dot\theta_6^2+\tfrac{1}{2}I_6\dot\theta_6^2\end{aligned}$$

$$\begin{aligned}K_7=&\frac{1}{2}m_7L_1^2\dot\theta_1^2+\frac{1}{2}m_7h^2\dot\theta_1^2+\frac{1}{2}m_7L_2^2\dot\theta_2^2+\frac{1}{2}m_7L_3^2\dot\theta_3^2+\frac{1}{2}m_7L_5^2\dot\theta_5^2+\frac{1}{2}m_7L_6^2\dot\theta_6^2\\&+\frac{1}{2}m_7k^2\dot\theta_7^2\\&+\tfrac{1}{2}m_7L_{7c}^2\dot\theta_7^2+m_7L_1L_2\dot\theta_1\dot\theta_2\,cos(\theta_1+\theta_2)+m_7L_1L_3\dot\theta_1\dot\theta_3\,cos(\theta_1+\theta_3)\\&-m_7L_1L_5\dot\theta_1\dot\theta_5\,cos(\theta_1-\theta_5)-m_7L_1L_6\dot\theta_1\dot\theta_6\,cos(\theta_1-\theta_6)\\&+m_7L_1k\dot\theta_1\dot\theta_7\,sin(\theta_1+\theta_7)+m_7L_1L_{7c}\dot\theta_1\dot\theta_7\,cos(\theta_1+\theta_7)\\&-m_7hL_2\dot\theta_1\dot\theta_2\,sin(\theta_1+\theta_2)-m_7hL_3\dot\theta_1\dot\theta_3\,sin(\theta_1+\theta_3)\\&+m_7hL_5\dot\theta_1\dot\theta_5\,sin(\theta_1-\theta_5)+m_7hL_6\dot\theta_1\dot\theta_6\,sin(\theta_1-\theta_6)\\&+m_7hk\dot\theta_1\dot\theta_7\,cos(\theta_1+\theta_7)-m_7hL_{7c}\dot\theta_1\dot\theta_7\,sin(\theta_1+\theta_7)\\&+m_7L_2L_3\dot\theta_2\dot\theta_3\,cos(\theta_2-\theta_3)-m_7L_2L_5\dot\theta_2\dot\theta_5\,cos(\theta_2+\theta_5)\\&-m_7L_2L_6\dot\theta_2\dot\theta_6\,cos(\theta_2+\theta_6)-m_7L_2k\dot\theta_2\dot\theta_7\,sin(\theta_2-\theta_7)\\&+m_7L_2L_{7c}\dot\theta_2\dot\theta_7\,cos(\theta_2-\theta_7)-m_7L_3L_5\dot\theta_3\dot\theta_5\,cos(\theta_3+\theta_5)\\&-m_7L_3L_6\dot\theta_3\dot\theta_6\,cos(\theta_3+\theta_6)-m_7L_3k\dot\theta_3\dot\theta_7\,sin(\theta_3-\theta_7)\\&+m_7L_3L_{7c}\dot\theta_3\dot\theta_7\,cos(\theta_3-\theta_7)+m_7L_5L_6\dot\theta_5\dot\theta_6\,cos(\theta_5-\theta_6)\\&-m_7L_5k\dot\theta_5\dot\theta_7\,sin(\theta_5+\theta_7)-m_7L_5L_{7c}\dot\theta_5\dot\theta_7\,cos(\theta_5+\theta_7)\\&-m_7L_6k\dot\theta_6\dot\theta_7\,sin(\theta_6+\theta_7)-m_7L_6L_{7c}\dot\theta_6\dot\theta_7\,cos(\theta_6+\theta_7)+\tfrac{1}{2}I_7\dot\theta_7^2\end{aligned}$$

B. Potential Energy

$$\begin{aligned}P_1&=m_1gL_{1c}sin\theta_1\\P_2&=m_2gL_1sin\theta_1+m_2ghcos\theta_1+m_2gL_{2c}sin\theta_2\\P_3&=m_3gL_1sin\theta_1+m_3ghcos\theta_1+m_3gL_2sin\theta_2+m_3gL_{3c}sin\theta_3\\P_4&=m_4gL_1sin\theta_1+m_4ghcos\theta_1+m_4gL_2sin\theta_2+m_4gL_3sin\theta_3\\&\quad+m_4gL_{4c}sin\theta_4\\P_5&=m_5gL_1sin\theta_1+m_5ghcos\theta_1+m_5gL_2sin\theta_2+m_5gL_3sin\theta_3-m_5gL_5sin\theta_5\\&\quad+m_5gL_{5c}sin\theta_5\\P_6&=m_6gL_1sin\theta_1+m_6ghcos\theta_1+m_6gL_2sin\theta_2+m_6gL_3sin\theta_3-m_6gL_5sin\theta_5\\&\quad-m_6gL_6sin\theta_6+m_6gL_{6c}sin\theta_6\\P_7&=m_7gL_1sin\theta_1+m_7ghcos\theta_1+m_7gL_2sin\theta_2+m_7gL_3sin\theta_3-m_7gL_5sin\theta_5\\&\quad-m_7gL_6sin\theta_6-m_7gkcos\theta_7+m_7gL_{7c}sin\theta_7\end{aligned}$$

APPENDIX B

Link 1

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{dL}{d\dot{\theta}_1}\right)-\frac{dL}{d\theta_1}=\;&\ddot{\theta}_1\begin{bmatrix} m_1L_{1c}^2+I_1+m_2L_1^2+m_2h^2+m_3L_1^2+m_3h^2+m_4L_1^2 \\ +m_4h^2+m_5L_1^2+m_5h^2+m_6L_1^2+m_6h^2+m_7L_1^2+m_7h^2 \end{bmatrix}\\
&+\ddot{\theta}_2\left[\begin{pmatrix} m_2L_1L_{2c}\\ +m_3L_1L_2\\ +m_4L_1L_2\\ +m_5L_1L_2\\ +m_6L_1L_2\\ +m_7L_1L_2 \end{pmatrix}cos(\theta_1+\theta_2)+\begin{pmatrix} -m_2hL_{2c}\\ -m_3hL_2\\ -m_4hL_2\\ -m_5hL_2\\ -m_6hL_2\\ -m_7hL_2 \end{pmatrix}sin(\theta_1+\theta_2)\right]\\
&+\ddot{\theta}_3\left[\begin{pmatrix} m_3L_1L_{3c}\\ +m_4L_1L_3\\ +m_5L_1L_3\\ +m_6L_1L_3\\ +m_7L_1L_3 \end{pmatrix}cos(\theta_1+\theta_3)+\begin{pmatrix} -m_3hL_{3c}\\ -m_4hL_3\\ -m_5hL_3\\ -m_6hL_3\\ -m_7hL_3 \end{pmatrix}sin(\theta_1+\theta_3)\right]\\
&+\ddot{\theta}_4[m_4L_1L_{4c}\,cos(\theta_1+\theta_4)-m_4hL_{4c}\,sin(\theta_1+\theta_4)]\\
&+\ddot{\theta}_5\left[\begin{pmatrix} -m_5L_1L_5\\ +m_5L_1L_{5c}\\ -m_6L_1L_5\\ -m_7L_1L_5 \end{pmatrix}cos(\theta_1-\theta_5)+\begin{pmatrix} m_5hL_5\\ -m_5hL_{5c}\\ +m_6hL_5\\ +m_7hL_5 \end{pmatrix}sin(\theta_1-\theta_5)\right]\\
&+\ddot{\theta}_6\left[\begin{pmatrix} -m_6L_1L_6\\ +m_6L_1L_{6c}\\ -m_7L_1L_6 \end{pmatrix}cos(\theta_1-\theta_6)+\begin{pmatrix} m_6hL_6\\ -m_6hL_{6c}\\ +m_7hL_6 \end{pmatrix}sin(\theta_1-\theta_6)\right]\\
&+\ddot{\theta}_7\left[\begin{pmatrix} m_7L_1L_{7c}\\ +m_7hk \end{pmatrix}cos(\theta_1+\theta_7)+\begin{pmatrix} m_7L_1k\\ -m_7hL_{7c} \end{pmatrix}sin(\theta_1+\theta_7)\right]\\
&+\dot{\theta}_2^2\left[-\begin{pmatrix} m_2L_1L_{2c}\\ +m_3L_1L_2\\ +m_4L_1L_2\\ +m_5L_1L_2\\ +m_6L_1L_2\\ +m_7L_1L_2 \end{pmatrix}sin(\theta_1+\theta_2)+\begin{pmatrix} -m_2hL_{2c}\\ -m_3hL_2\\ -m_4hL_2\\ -m_5hL_2\\ -m_6hL_2\\ -m_7hL_2 \end{pmatrix}cos(\theta_1+\theta_2)\right]\\
&+\dot{\theta}_3^2\left[-\begin{pmatrix} m_3L_1L_{3c}\\ +m_4L_1L_3\\ +m_5L_1L_3\\ +m_6L_1L_3\\ +m_7L_1L_3 \end{pmatrix}sin(\theta_1+\theta_3)+\begin{pmatrix} -m_3hL_{3c}\\ -m_4hL_3\\ -m_5hL_3\\ -m_6hL_3\\ -m_7hL_3 \end{pmatrix}cos(\theta_1+\theta_3)\right]\\
&+\dot{\theta}_4^2[-m_4L_1L_{4c}\,sin(\theta_1+\theta_4)-m_4hL_{4c}\,cos(\theta_1+\theta_4)]\\
&+\dot{\theta}_5^2\left[\begin{pmatrix} -m_5L_1L_5\\ +m_5L_1L_{5c}\\ -m_6L_1L_5\\ -m_7L_1L_5 \end{pmatrix}sin(\theta_1-\theta_5)-\begin{pmatrix} m_5hL_5\\ -m_5hL_{5c}\\ +m_6hL_5\\ +m_7hL_5 \end{pmatrix}cos(\theta_1-\theta_5)\right]\\
&+\dot{\theta}_6^2\left[\begin{pmatrix} -m_6L_1L_6\\ +m_6L_1L_{6c}\\ -m_7L_1L_6 \end{pmatrix}sin(\theta_1-\theta_6)-\begin{pmatrix} m_6hL_6\\ -m_6hL_{6c}\\ +m_7hL_6 \end{pmatrix}cos(\theta_1-\theta_6)\right]\\
&+\dot{\theta}_7^2\left[\begin{pmatrix} m_7L_1k\\ -m_7hL_{7c} \end{pmatrix}cos(\theta_1+\theta_7)-\begin{pmatrix} m_7L_1L_{7c}\\ +m_7hk \end{pmatrix}sin(\theta_1+\theta_7)\right]\\
&+gcos\theta_1[m_1L_{1c}+m_2L_1+m_3L_1+m_4L_1+m_5L_1+m_6L_1+m_7L_1]\\
&-gsin\theta_1[m_2h+m_3h+m_4h+m_5h+m_6h+m_7h]
\end{aligned}
$$

Link 2

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{dL}{d\dot{\theta}_2}\right)-\frac{dL}{d\theta_2}=\;&\ddot{\theta}_1\left[\begin{pmatrix} m_2L_1L_{2c}\\ +m_3L_1L_2\\ +m_4L_1L_2\\ +m_5L_1L_2\\ +m_6L_1L_2\\ +m_7L_1L_2 \end{pmatrix}cos(\theta_1+\theta_2)+\begin{pmatrix} -m_2hL_{2c}\\ -m_3hL_2\\ -m_4hL_2\\ -m_5hL_2\\ -m_6hL_2\\ -m_7hL_2 \end{pmatrix}sin(\theta_1+\theta_2)\right]\\
&+\ddot{\theta}_2[m_2L_{2c}^2+I_2+m_3L_2^2+m_4L_2^2+m_5L_2^2+m_6L_2^2+m_7L_2^2]\\
&+\ddot{\theta}_3\left[\begin{pmatrix} m_3L_2L_{3c}+m_4L_2L_3\\ +m_5L_2L_3+m_6L_2L_3\\ +m_7L_2L_3 \end{pmatrix}cos(\theta_2-\theta_3)\right]\\
&+\ddot{\theta}_4[m_4L_2L_{4c}\,cos(\theta_2-\theta_4)]\\
&+\ddot{\theta}_5\left[\begin{pmatrix} -m_5L_2L_5+m_5L_2L_{5c}\\ -m_6L_2L_5-m_7L_2L_5 \end{pmatrix}cos(\theta_2+\theta_5)\right]\\
&+\ddot{\theta}_6\left[\begin{pmatrix} -m_6L_2L_6+m_6L_2L_{6c}\\ -m_7L_2L_6 \end{pmatrix}cos(\theta_2+\theta_6)\right]\\
&+\ddot{\theta}_7[m_7L_2L_{7c}\,cos(\theta_2-\theta_7)-m_7L_2k\,sin(\theta_2-\theta_7)]\\
&+\dot{\theta}_1^2\left[-\begin{pmatrix} m_2L_1L_{2c}\\ +m_3L_1L_2\\ +m_4L_1L_2\\ +m_5L_1L_2\\ +m_6L_1L_2\\ +m_7L_1L_2 \end{pmatrix}sin(\theta_1+\theta_2)+\begin{pmatrix} -m_2hL_{2c}\\ -m_3hL_2\\ -m_4hL_2\\ -m_5hL_2\\ -m_6hL_2\\ -m_7hL_2 \end{pmatrix}cos(\theta_1+\theta_2)\right]\\
&+\dot{\theta}_3^2\left[\begin{pmatrix} m_3L_2L_{3c}+m_4L_2L_3\\ +m_5L_2L_3+m_6L_2L_3\\ +m_7L_2L_3 \end{pmatrix}sin(\theta_2-\theta_3)\right]\\
&+\dot{\theta}_4^2[m_4L_2L_{4c}\,sin(\theta_2-\theta_4)]\\
&+\dot{\theta}_5^2\left[-\begin{pmatrix} -m_5L_2L_5+m_5L_2L_{5c}\\ -m_6L_2L_5-m_7L_2L_5 \end{pmatrix}sin(\theta_2+\theta_5)\right]\\
&+\dot{\theta}_6^2\left[-\begin{pmatrix} -m_6L_2L_6+m_6L_2L_{6c}\\ -m_7L_2L_6 \end{pmatrix}sin(\theta_2+\theta_6)\right]\\
&+\dot{\theta}_7^2[m_7L_2L_{7c}\,sin(\theta_2-\theta_7)-(-m_7L_2k)\,cos(\theta_2-\theta_7)]\\
&+gcos\theta_2[m_2L_{2c}+m_3L_2+m_4L_2+m_5L_2+m_6L_2+m_7L_2]
\end{aligned}
$$

Link 3

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{dL}{d\dot{\theta}_3}\right)-\frac{dL}{d\theta_3}=\;&\ddot{\theta}_1\left[\begin{pmatrix} m_3L_1L_{3c}\\ +m_4L_1L_3\\ +m_5L_1L_3\\ +m_6L_1L_3\\ +m_7L_1L_3 \end{pmatrix}cos(\theta_1+\theta_3)+\begin{pmatrix} -m_3hL_{3c}\\ -m_4hL_3\\ -m_5hL_3\\ -m_6hL_3\\ -m_7hL_3 \end{pmatrix}sin(\theta_1+\theta_3)\right]\\
&+\ddot{\theta}_2\left[\begin{pmatrix} m_3L_2L_{3c}+m_4L_2L_3\\ +m_5L_2L_3+m_6L_2L_3\\ +m_7L_2L_3 \end{pmatrix}cos(\theta_2-\theta_3)\right]\\
&+\ddot{\theta}_3[m_3L_{3c}^2+I_3+m_4L_3^2+m_5L_3^2+m_6L_3^2+m_7L_3^2]\\
&+\ddot{\theta}_4[m_4L_3L_{4c}\,cos(\theta_3-\theta_4)]\\
&+\ddot{\theta}_5\left[\begin{pmatrix} -m_5L_3L_5+m_5L_3L_{5c}\\ -m_6L_3L_5-m_7L_3L_5 \end{pmatrix}cos(\theta_3+\theta_5)\right]\\
&+\ddot{\theta}_6\left[\begin{pmatrix} -m_6L_3L_6+m_6L_3L_{6c}\\ -m_7L_3L_6 \end{pmatrix}cos(\theta_3+\theta_6)\right]\\
&+\ddot{\theta}_7[m_7L_3L_{7c}\,cos(\theta_3-\theta_7)-m_7L_3k\,sin(\theta_3-\theta_7)]\\
&+\dot{\theta}_1^2\left[-\begin{pmatrix} m_3L_1L_{3c}\\ +m_4L_1L_3\\ +m_5L_1L_3\\ +m_6L_1L_3\\ +m_7L_1L_3 \end{pmatrix}sin(\theta_1+\theta_3)+\begin{pmatrix} -m_3hL_{3c}\\ -m_4hL_3\\ -m_5hL_3\\ -m_6hL_3\\ -m_7hL_3 \end{pmatrix}cos(\theta_1+\theta_3)\right]\\
&+\dot{\theta}_2^2\left[-\begin{pmatrix} m_3L_2L_{3c}+m_4L_2L_3\\ +m_5L_2L_3+m_6L_2L_3\\ +m_7L_2L_3 \end{pmatrix}sin(\theta_2-\theta_3)\right]\\
&+\dot{\theta}_4^2[m_4L_3L_{4c}\,sin(\theta_3-\theta_4)]\\
&+\dot{\theta}_5^2\left[-\begin{pmatrix} -m_5L_3L_5+m_5L_3L_{5c}\\ -m_6L_3L_5-m_7L_3L_5 \end{pmatrix}sin(\theta_3+\theta_5)\right]\\
&+\dot{\theta}_6^2\left[-\begin{pmatrix} -m_6L_3L_6+m_6L_3L_{6c}\\ -m_7L_3L_6 \end{pmatrix}sin(\theta_3+\theta_6)\right]\\
&+\dot{\theta}_7^2[m_7L_3L_{7c}\,sin(\theta_3-\theta_7)-(-m_7L_3k)\,cos(\theta_3-\theta_7)]\\
&+gcos\theta_3[m_3L_{3c}+m_4L_3+m_5L_3+m_6L_3+m_7L_3]
\end{aligned}
$$

Link 4

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{dL}{d\dot{\theta}_4}\right)-\frac{dL}{d\theta_4}=\;&\ddot{\theta}_1[m_4L_1L_{4c}\,cos(\theta_1+\theta_4)-m_4hL_{4c}\,sin(\theta_1+\theta_4)]\\
&+\ddot{\theta}_2[m_4L_2L_{4c}\,cos(\theta_2-\theta_4)]+\ddot{\theta}_3[m_4L_3L_{4c}\,cos(\theta_3-\theta_4)]\\
&+\ddot{\theta}_4[m_4L_{4c}^2+I_4]\\
&+\dot{\theta}_1^2[-m_4L_1L_{4c}\,sin(\theta_1+\theta_4)-m_4hL_{4c}\,cos(\theta_1+\theta_4)]\\
&+\dot{\theta}_2^2[-m_4L_2L_{4c}\,sin(\theta_2-\theta_4)]+\dot{\theta}_3^2[-m_4L_3L_{4c}\,sin(\theta_3-\theta_4)]\\
&+gcos\theta_4[m_4L_{4c}]
\end{aligned}
$$

Link 5

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{dL}{d\dot{\theta}_5}\right)-\frac{dL}{d\theta_5} = {} & \ddot{\theta}_1\left[\begin{pmatrix}-m_5L_1L_5\\+m_5L_1L_{5c}\\-m_6L_1L_5\\-m_7L_1L_5\end{pmatrix}cos(\theta_1-\theta_5)+\begin{pmatrix}m_5hL_5\\-m_5hL_{5c}\\+m_6hL_5\\+m_7hL_5\end{pmatrix}sin(\theta_1-\theta_5)\right]\\
&+\ddot{\theta}_2\left[\begin{pmatrix}-m_5L_2L_5\\+m_5L_2L_{5c}\\-m_6L_2L_5\\-m_7L_2L_5\end{pmatrix}cos(\theta_2+\theta_5)\right]\\
&+\ddot{\theta}_3\left[\begin{pmatrix}-m_5L_3L_5+m_5L_3L_{5c}\\-m_6L_3L_5-m_7L_3L_5\end{pmatrix}cos(\theta_3+\theta_5)\right]\\
&+\ddot{\theta}_5[m_5L_5^2+m_5L_{5c}^2-2m_5L_5L_{5c}+I_5+m_6L_5^2+m_7L_5^2]\\
&+\ddot{\theta}_6[(m_6L_5L_6-m_6L_5L_{6c}+m_7L_5L_6)\,cos(\theta_5-\theta_6)]\\
&+\ddot{\theta}_7[(-m_7L_5L_{7c})\,cos(\theta_5+\theta_7)+(-m_7L_5k)\,sin(\theta_5+\theta_7)]\\
&+\dot{\theta}_1^2\left[-\begin{pmatrix}-m_5L_1L_5\\+m_5L_1L_{5c}\\-m_6L_1L_5\\-m_7L_1L_5\end{pmatrix}sin(\theta_1-\theta_5)+\begin{pmatrix}m_5hL_5\\-m_5hL_{5c}\\+m_6hL_5\\+m_7hL_5\end{pmatrix}cos(\theta_1-\theta_5)\right]\\
&+\dot{\theta}_2^2\left[-\begin{pmatrix}-m_5L_2L_5\\+m_5L_2L_{5c}\\-m_6L_2L_5\\-m_7L_2L_5\end{pmatrix}sin(\theta_2+\theta_5)\right]\\
&+\dot{\theta}_3^2\left[-\begin{pmatrix}-m_5L_3L_5+m_5L_3L_{5c}\\-m_6L_3L_5-m_7L_3L_5\end{pmatrix}sin(\theta_3+\theta_5)\right]\\
&+\dot{\theta}_6^2[(m_6L_5L_6-m_6L_5L_{6c}+m_7L_5L_6)\,sin(\theta_5-\theta_6)]\\
&+\dot{\theta}_7^2[-(-m_7L_5L_{7c})\,sin(\theta_5+\theta_7)-m_7L_5k\,cos(\theta_5+\theta_7)]\\
&-gcos\theta_5[m_5L_5-m_5L_{5c}+m_6L_5+m_7L_5]
\end{aligned}
$$

Link 6

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{dL}{d\dot{\theta}_6}\right)-\frac{dL}{d\theta_6} = {} & \ddot{\theta}_1\left[\begin{pmatrix}-m_6L_1L_6\\+m_6L_1L_{6c}\\-m_7L_1L_6\end{pmatrix}cos(\theta_1-\theta_6)+\begin{pmatrix}m_6hL_6\\-m_6hL_{6c}\\+m_7hL_6\end{pmatrix}sin(\theta_1-\theta_6)\right]\\
&+\ddot{\theta}_2[(-m_6L_2L_6+m_6L_2L_{6c}-m_7L_2L_6)\,cos(\theta_2+\theta_6)]\\
&+\ddot{\theta}_3[(-m_6L_3L_6+m_6L_3L_{6c}-m_7L_3L_6)\,cos(\theta_3+\theta_6)]\\
&+\ddot{\theta}_5[(m_6L_5L_6-m_6L_5L_{6c}+m_7L_5L_6)\,cos(\theta_5-\theta_6)]\\
&+\ddot{\theta}_6[m_6L_6^2+m_6L_{6c}^2-2m_6L_6L_{6c}+I_6+m_7L_6^2]\\
&+\ddot{\theta}_7[-m_7L_6L_{7c}\,cos(\theta_6+\theta_7)-m_7L_6k\,sin(\theta_6+\theta_7)]\\
&+\dot{\theta}_1^2\left[-\begin{pmatrix}-m_6L_1L_6\\+m_6L_1L_{6c}\\-m_7L_1L_6\end{pmatrix}sin(\theta_1-\theta_6)+\begin{pmatrix}m_6hL_6\\-m_6hL_{6c}\\+m_7hL_6\end{pmatrix}cos(\theta_1-\theta_6)\right]\\
&+\dot{\theta}_2^2[-(-m_6L_2L_6+m_6L_2L_{6c}-m_7L_2L_6)\,sin(\theta_2+\theta_6)]\\
&+\dot{\theta}_3^2[-(-m_6L_3L_6+m_6L_3L_{6c}-m_7L_3L_6)\,sin(\theta_3+\theta_6)]\\
&+\dot{\theta}_5^2[-(m_6L_5L_6-m_6L_5L_{6c}+m_7L_5L_6)\,sin(\theta_5-\theta_6)]\\
&+\dot{\theta}_7^2[-(-m_7L_6L_{7c})\,sin(\theta_6+\theta_7)+(-m_7L_6k)\,cos(\theta_6+\theta_7)]\\
&-gcos\theta_6[m_6L_6-m_6L_{6c}+m_7L_6]
\end{aligned}
$$

Link 7

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{dL}{d\dot{\theta}_7}\right)-\frac{dL}{d\theta_7} = {} & \ddot{\theta}_1[(m_7L_1L_{7c}+m_7hk)\,cos(\theta_1+\theta_7)\\
&\quad+(m_7L_1k-m_7hL_{7c})\,sin(\theta_1+\theta_7)]\\
&+\ddot{\theta}_2[m_7L_2L_{7c}\,cos(\theta_2-\theta_7)-m_7L_2k\,sin(\theta_2-\theta_7)]\\
&+\ddot{\theta}_3[m_7L_3L_{7c}\,cos(\theta_3-\theta_7)-m_7L_3k\,sin(\theta_3-\theta_7)\\
&+\ddot{\theta}_5[-m_7L_5L_{7c}\,cos(\theta_5+\theta_7)-m_7L_5k\,sin(\theta_5-\theta_7)]\\
&+\ddot{\theta}_6[-m_7L_6L_{7c}\,cos(\theta_6+\theta_7)-m_7L_6k\,sin(\theta_6+\theta_7)]\\
&+\ddot{\theta}_7[m_7k^2+m_7L_{7c}^2+I_7]\\
&+\dot{\theta}_1^2\begin{bmatrix}(m_7L_1k-m_7hL_{7c})\,cos(\theta_1+\theta_7)\\-(m_7L_1L_{7c}+m_7hk)\,sin(\theta_1+\theta_7)\end{bmatrix}\\
&+\dot{\theta}_2^2[-m_7L_2k\,cos(\theta_2-\theta_7)-m_7L_2L_{7c}\,sin(\theta_2-\theta_7)]\\
&+\dot{\theta}_3^2[-m_7L_3k\,cos(\theta_3-\theta_7)-m_7L_3L_{7c}\,sin(\theta_3-\theta_7)]\\
&+\dot{\theta}_5^2[-m_7L_5k\,cos(\theta_5-\theta_7)+m_7L_5L_{7c}\,sin(\theta_5+\theta_7)]\\
&+\dot{\theta}_6^2[-m_7L_6k\,cos(\theta_6+\theta_7)+m_7L_6L_{7c}\,sin(\theta_6+\theta_7)]\\
&+gcos\theta_7[m_7L_{7c}]+gsin\theta_7[m_7k]
\end{aligned}
$$

APPENDIX C

$$
\begin{aligned}
D_{11} = {} & m_1L_{1c}^2+I_1+m_2L_1^2+m_2h^2+m_3L_1^2+m_3h^2+m_4L_1^2+m_4h^2+m_5L_1^2\\
&+m_5h^2+m_6L_1^2+m_6h^2+m_7L_1^2+m_7h^2\\
D_{12} = {} & \begin{pmatrix}m_2L_1L_{2c}+m_3L_1L_2+m_4L_1L_2\\+m_5L_1L_2+m_6L_1L_2+m_7L_1L_2\end{pmatrix}cos(\theta_1+\theta_2)\\
&+\begin{pmatrix}-m_2hL_{2c}-m_3hL_2-m_4hL_2\\-m_5hL_2-m_6hL_2-m_7hL_2\end{pmatrix}sin(\theta_1+\theta_2)\\
D_{13} = {} & (m_3L_1L_{3c}+m_4L_1L_3+m_5L_1L_3+m_6L_1L_3+m_7L_1L_3)\,cos(\theta_1+\theta_3)\\
&+(-m_3hL_{3c}-m_4hL_3-m_5hL_3-m_6hL_3-m_7hL_3)\,sin(\theta_1+\theta_3)\\
D_{14} = {} & m_4L_1L_{4c}\,cos(\theta_1+\theta_4)-m_4hL_{4c}\,sin(\theta_1+\theta_4)\\
D_{15} = {} & (-m_5L_1L_5+m_5L_1L_{5c}-m_6L_1L_5-m_7L_1L_5)\,cos(\theta_1-\theta_5)\\
&+(m_5hL_5-m_5hL_{5c}+m_6hL_5+m_7hL_5)sin(\theta_1-\theta_5)\\
D_{16} = {} & (-m_6L_1L_6+m_6L_1L_{6c}-m_7L_1L_6)\,cos(\theta_1-\theta_6)\\
&+(m_6hL_6-m_6hL_{6c}+m_7hL_6)sin(\theta_1-\theta_6)\\
D_{17} = {} & (m_7L_1k-m_7hL_{7c})\,sin(\theta_1+\theta_7)+(m_7L_1L_{7c}+m_7hk)cos(\theta_1+\theta_7)
\end{aligned}
$$

$$
\begin{aligned}
D_{21} &= D_{12}\\
D_{22} &= m_2L_{2c}^2+I_2+m_3L_2^2+m_4L_2^2+m_5L_2^2+m_6L_2^2+m_7L_2^2\\
D_{23} &= (m_3L_2L_{3c}+m_4L_2L_3+m_5L_2L_3+m_6L_2L_3+m_7L_2L_3)cos(\theta_2-\theta_3)\\
D_{24} &= m_4L_2L_{4c}\,cos(\theta_2-\theta_4)\\
D_{25} &= (-m_5L_2L_5+m_5L_2L_{5c}-m_6L_2L_5-m_7L_2L_5)\,cos(\theta_2+\theta_5)\\
D_{26} &= (-m_6L_2L_6+m_6L_2L_{6c}-m_7L_2L_6)cos(\theta_2+\theta_6)\\
D_{27} &= -m_7L_2ksin(\theta_2-\theta_7)+m_7L_2L_{7c}cos(\theta_2-\theta_7)
\end{aligned}
$$

$$
\begin{aligned}
D_{31} &= D_{13}\\
D_{32} &= D_{23}\\
D_{33} &= m_3L_{3c}^2+I_3+m_4L_3^2+m_5L_3^2+m_6L_3^2+m_7L_3^2\\
D_{34} &= m_4L_3L_{4c}\,cos(\theta_3-\theta_4)\\
D_{35} &= (-m_5L_3L_5+m_5L_3L_{5c}-m_6L_3L_5-m_7L_3L_5)\,cos(\theta_3+\theta_5)\\
D_{36} &= (-m_6L_3L_6+m_6L_3L_{6c}-m_7L_3L_6)\,cos(\theta_3+\theta_6)\\
D_{37} &= -m_7L_3ksin(\theta_3-\theta_7)+m_7L_3L_{7c}\,cos(\theta_3-\theta_7)
\end{aligned}
$$

$$
\begin{aligned}
D_{41} &= D_{14}\\
D_{42} &= D_{24}\\
D_{43} &= D_{34}\\
D_{44} &= m_4L_{4c}^2+I_4\\
D_{45} &= 0\\
D_{46} &= 0\\
D_{47} &= 0
\end{aligned}
$$

$$
\begin{aligned}
D_{51} &= D_{15}\\
D_{52} &= D_{25}\\
D_{53} &= D_{35}\\
D_{54} &= D_{45}\\
D_{55} &= m_5L_5^2+m_5L_{5c}^2-2m_5L_5L_{5c}+I_5+m_6L_5^2+m_7L_5^2\\
D_{56} &= (m_6L_5L_6-m_6L_6L_{6c}+m_7L_5L_6)cos(\theta_5-\theta_6)\\
D_{57} &= -m_7L_5k\,sin(\theta_5+\theta_7)-m_7L_5L_{7c}cos(\theta_5+\theta_7)
\end{aligned}
$$

$$
\begin{aligned}
D_{61} &= D_{16}\\
D_{62} &= D_{26}\\
D_{63} &= D_{36}\\
D_{64} &= D_{46}\\
D_{65} &= D_{56}\\
D_{66} &= m_6L_6^2+m_2L_{6c}^2-2m_6L_6L_{6c}+I_6+m_7L_6^2\\
D_{67} &= -m_7L_6ksin(\theta_6+\theta_7)-m_7L_6L_{7c}\,cos(\theta_6+\theta_7)
\end{aligned}
$$

$$
\begin{aligned}
D_{71} &= D_{17}\\
D_{72} &= D_{27}\\
D_{73} &= D_{37}\\
D_{74} &= D_{47}\\
D_{75} &= D_{57}\\
D_{76} &= D_{67}\\
D_{77} &= m_7k^2+m_7L_{7c}^2+I_7
\end{aligned}
$$

APPENDIX D

A. Inertia Matrix

$$
\begin{aligned}
D_q(1,1) = {} & D_{11}-2D_{12}-2D_{13}-2D_{14}+2D_{15}+2D_{16}-2D_{17}+D_{22}+2D_{23}\\
&+2D_{24}-2D_{25}-2D_{26}+2D_{27}+D_{33}+2D_{34}-2D_{35}-2D_{36}+2D_{37}\\
&+D_{44}-2D_{45}-2D_{46}+2D_{47}+D_{55}+2D_{56}-2D_{57}+D_{66}-2D_{67}\\
&+D_{77}\\
D_q(1,2) = {} & D_{12}+D_{13}+D_{14}-D_{15}-D_{16}+D_{17}-D_{22}-2D_{23}-2D_{24}+2D_{25}\\
&+2D_{26}-2D_{27}-D_{33}-2D_{34}+2D_{35}+2D_{36}-2D_{37}-D_{44}+2D_{45}\\
&+2D_{46}-2D_{47}-D_{55}-2D_{56}+2D_{57}-D_{66}+2D_{67}-D_{77}\\
D_q(1,3) = {} & -D_{13}-D_{14}+D_{15}+D_{16}-D_{17}+D_{23}+D_{24}-D_{25}-D_{26}+D_{27}\\
&+D_{33}+2D_{34}-2D_{35}-2D_{36}+2D_{37}+D_{44}-2D_{45}-2D_{46}+2D_{47}\\
&+D_{55}+2D_{56}-2D_{57}+D_{66}-2D_{67}+D_{77}\\
D_q(1,4) = {} & -D_{14}+D_{15}+D_{16}-D_{17}+D_{24}-D_{25}-D_{26}+D_{27}+D_{34}-D_{35}\\
&-D_{36}+D_{37}+D_{44}-2D_{45}-2D_{46}+2D_{47}+D_{55}+2D_{56}-2D_{57}\\
&+D_{66}-2D_{67}+D_{77}\\
D_q(1,5) = {} & D_{15}+D_{16}-D_{17}-D_{25}-D_{26}+D_{27}-D_{35}-D_{36}+D_{37}-D_{45}-D_{46}\\
&+D_{47}+D_{55}+2D_{56}-2D_{57}+D_{66}-2D_{67}+D_{77}\\
D_q(1,6) = {} & -D_{16}+D_{17}+D_{26}-D_{27}+D_{36}-D_{37}+D_{46}-D_{47}-D_{56}+D_{57}\\
&-D_{66}+2D_{67}-D_{77}\\
D_q(1,7) = {} & -D_{17}+D_{27}+D_{37}+D_{47}-D_{57}-D_{67}+D_{77}
\end{aligned}
$$

$D_q(2,1) = D_q(1,2)$
$D_q(2,2) = D_{22} + 2D_{23} + 2D_{24} - 2D_{25} - 2D_{26} + 2D_{27} + D_{33} + 2D_{34} - 2D_{35} - 2D_{36} + 2D_{37} + D_{44} - 2D_{45} - 2D_{46} + 2D_{47} + D_{55} + 2D_{56} - 2D_{57} + D_{66} - 2D_{67} + D_{77}$
$D_q(2,3) = -D_{23} - D_{24} + D_{25} + D_{26} - D_{27} - D_{33} - 2D_{34} + 2D_{35} + 2D_{36} - 2D_{37} - D_{44} + 2D_{45} + 2D_{46} - 2D_{47} - D_{55} - 2D_{56} + 2D_{57} - D_{66} + 2D_{67} - D_{77}$
$D_q(2,4) = -D_{24} + D_{25} + D_{26} - D_{27} - D_{34} + D_{35} + D_{36} - D_{37} - D_{44} + 2D_{45} + 2D_{46} - 2D_{47} - D_{55} - 2D_{56} + 2D_{57} - D_{66} + 2D_{67} - D_{77}$
$D_q(2,5) = D_{25} + D_{26} - D_{27} + D_{35} + D_{36} - D_{37} + D_{45} + D_{46} - D_{47} - D_{55} - 2D_{56} + 2D_{57} - D_{66} + 2D_{67} - D_{77}$
$D_q(2,6) = -D_{26} + D_{27} - D_{36} + D_{37} - D_{46} + D_{47} + D_{56} - D_{57} + D_{66} - 2D_{67} + D_{77}$
$D_q(2,7) = -D_{27} - D_{37} - D_{47} + D_{57} + D_{67} - D_{77}$

$D_q(3,1) = D_q(1,3)$
$D_q(3,2) = D_q(2,3)$
$D_q(3,3) = D_{33} + 2D_{34} - 2D_{35} - 2D_{36} + 2D_{37} + D_{44} - 2D_{45} - 2D_{46} + 2D_{47} + D_{55} + 2D_{56} - 2D_{57} + D_{66} - 2D_{67} + D_{77}$
$D_q(3,4) = D_{34} - D_{35} - D_{36} + D_{37} + D_{44} - 2D_{45} - 2D_{46} + 2D_{47} + D_{55} + 2D_{56} - 2D_{57} + D_{66} - 2D_{67} + D_{77}$
$D_q(3,5) = -D_{35} - D_{36} + D_{37} - D_{45} - D_{46} + D_{47} + D_{55} + 2D_{56} - 2D_{57} + D_{66} - 2D_{67} + D_{77}$
$D_q(3,6) = D_{36} - D_{37} + D_{46} - D_{47} - D_{56} + D_{57} - D_{66} + 2D_{67} - D_{77}$
$D_q(3,7) = D_{37} + D_{47} - D_{57} - D_{67} + D_{77}$

$D_q(4,1) = D_q(1,4)$
$D_q(4,2) = D_q(2,4)$
$D_q(4,3) = D_q(3,4)$
$D_q(4,4) = D_{44} - 2D_{45} - 2D_{46} + 2D_{47} + D_{55} + 2D_{56} - 2D_{57} + D_{66} - 2D_{67} + D_{77}$
$D_q(4,5) = -D_{45} - D_{46} + D_{47} + D_{55} + 2D_{56} - 2D_{57} + D_{66} - 2D_{67} + D_{77}$
$D_q(4,6) = D_{46} - D_{47} - D_{56} + D_{57} - D_{66} + 2D_{67} - D_{77}$
$D_q(4,7) = D_{47} - D_{57} - D_{67} + D_{77}$

$D_q(5,1) = D_q(1,5)$
$D_q(5,2) = D_q(2,5)$
$D_q(5,3) = D_q(3,5)$
$D_q(5,4) = D_q(4,5)$
$D_q(5,5) = D_{55} + 2D_{56} - 2D_{57} + D_{66} - 2D_{67} + D_{77}$
$D_q(5,6) = -D_{56} + D_{57} - D_{66} + 2D_{67} - D_{77}$
$D_q(5,7) = -D_{57} - D_{67} + D_{77}$

$D_q(6,1) = D_q(1,6)$
$D_q(6,2) = D_q(2,6)$
$D_q(6,3) = D_q(3,6)$
$D_q(6,4) = D_q(4,6)$
$D_q(6,5) = D_q(5,6)$
$D_q(6,6) = D_{66} - 2D_{67} + D_{77}$
$D_q(6,7) = D_{67} - D_{77}$

$D_q(7,1) = D_q(1,7)$
$D_q(7,2) = D_q(2,7)$
$D_q(7,3) = D_q(3,7)$
$D_q(7,4) = D_q(4,7)$
$D_q(7,5) = D_q(5,7)$
$D_q(7,6) = D_q(6,7)$
$D_q(7,7) = D_{77}$

B. Coriolis and Centripetal Torques Matrix

$H_{11} = 0$
$$H_{12} = \begin{pmatrix} -m_2hL_{2c} - m_3hL_2 - m_4hL_2 \\ -m_5hL_2 - m_6hL_2 - m_7hL_2 \end{pmatrix} cos(\theta_1 + \theta_2) - \begin{pmatrix} m_2L_1L_{2c} + m_3L_1L_2 + m_4L_1L_2 \\ +m_5L_1L_2 + m_6L_1L_2 + m_7L_1L_2 \end{pmatrix} sin(\theta_1 + \theta_2)$$
$H_{13} = (-m_3hL_{3c} - m_4hL_3 - m_5hL_3 - m_6hL_3 - m_7hL_3) cos(\theta_1 + \theta_3) - (m_3L_1L_{3c} + m_4L_1L_3 + m_5L_1L_3 + m_6L_1L_3 + m_7L_1L_3) sin(\theta_1 + \theta_3)$
$H_{14} = (-m_4hL_{4c}) cos(\theta_1 + \theta_4) - (m_4L_1L_{4c}) sin(\theta_1 + \theta_4)$
$H_{15} = -(m_5hL_5 - m_5hL_{5c} + m_6hL_5 + m_7hL_5) cos(\theta_1 - \theta_5) + (-m_5L_1L_5 + m_5L_1L_{5c} - m_6L_1L_5 - m_7L_1L_5) sin(\theta_1 - \theta_5)$
$H_{16} = -(m_6hL_6 - m_6hL_{6c} + m_7hL_6) cos(\theta_1 - \theta_6) + (-m_6L_1L_6 + m_6L_1L_{6c} - m_7L_1L_{6c}) sin(\theta_1 - \theta_6)$
$H_{17} = (m_7L_1k - m_7hL_{7c}) cos(\theta_1 + \theta_7) - (m_7L_1L_{7c} + m_7hk) sin(\theta_1 + \theta_7)$

$H_{21} = H_{12}$
$H_{22} = 0$
$H_{23} = (m_3L_2L_{3c} + m_4L_2L_3 + m_5L_2L_3 + m_6L_2L_3 + m_7L_2L_3) sin(\theta_2 - \theta_3)$
$H_{24} = m_4L_2L_{4c} sin(\theta_2 - \theta_4)$
$H_{25} = -(-m_5L_2L_5 + m_5L_2L_{5c} - m_6L_2L_5 - m_7L_2L_5) sin(\theta_2 + \theta_5)$
$H_{26} = -(-m_6L_2L_6 + m_6L_2L_{6c} - m_7L_2L_6) sin(\theta_2 + \theta_6)$
$H_{27} = -(-m_7L_2k) cos(\theta_2 - \theta_7) + m_7L_2L_{7c} sin(\theta_2 - \theta_7)$

$H_{31} = H_{13}$
$H_{32} = -H_{23}$
$H_{33} = 0$
$H_{34} = m_4L_3L_{4c} sin(\theta_3 - \theta_4)$
$H_{35} = -(-m_5L_3L_5 + m_5L_3L_{5c} - m_6L_3L_5 - m_7L_3L_5) sin(\theta_3 + \theta_5)$
$H_{36} = -(-m_6L_3L_6 + m_6L_3L_{6c} - m_7L_3L_6) sin(\theta_3 + \theta_6)$
$H_{37} = -(-m_7L_3k) cos(\theta_3 - \theta_7) + m_7L_3L_{7c} sin(\theta_3 - \theta_7)$

$H_{41} = H_{14}$
$H_{42} = -H_{244}$
$H_{43} = -H_{34}$
$H_{44} = 0$
$H_{45} = 0$
$H_{46} = 0$
$H_{47} = 0$

$H_{51} = -H_{15}$
$H_{52} = H_{25}$
$H_{53} = H_{35}$
$H_{54} = H_{45} = 0$
$H_{55} = 0$
$H_{56} = (m_6L_5L_6 - m_6L_5L_{6c} + m_7L_5L_6) sin(\theta_5 - \theta_6)$
$H_{57} = (-m_7L_5k) cos(\theta_5 + \theta_7) - (-m_7L_5L_{7c}) sin(\theta_5 + \theta_7)$

$H_{61} = -H_{16}$
$H_{62} = H_{26}$
$H_{63} = H_{36}$
$H_{64} = H_{46} = 0$
$H_{65} = -H_{56}$
$H_{66} = 0$
$H_{67} = (-m_7L_6k) cos(\theta_6 + \theta_7) - (-m_7L_6L_{7c}) sin(\theta_6 + \theta_7)$

$H_{71} = H_{17}$
$H_{72} = -H_{27}$
$H_{73} = -H_{37}$
$H_{74} = H_{47} = 0$
$H_{75} = H_{57}$
$H_{76} = H_{67}$
$H_{77} = 0$

$H_i = H_{i1} + H_{i2} + H_{i3} + H_{i4} + H_{i5} + H_{i6} + H_{i7}$ where $i = 1,2,\dots,7$

$H_{q0} = -H_1 + H_2 + H_3 + H_4 - H_5 - H_6 + H_7$
$H_{q1} = -H_2 - H_3 - H_4 + H_5 + H_6 - H_7$
$H_{q2} = H_3 + H_4 - H_5 - H_6 + H_7$
$H_{q3} = H_4 - H_5 - H_6 + H_7$
$H_{q4} = -H_5 - H_6 + H_7$
$H_{q5} = H_6 - H_7$
$H_{q6} = H_7$

C. Gravitational Torques Matrix

$G_1 = gcos\theta_1(m_1L_{1c} + m_2L_1 + m_3L_1 + m_4L_1 + m_5L_1 + m_6L_1 + m_7L_1) - gsin\theta_1(m_2h + m_3h + m_4h + m_5h + m_6h + m_7h)$
$G_2 = gcos\theta_2(m_2L_{2c} + m_3L_2 + m_4L_2 + m_5L_2 + m_6L_2 + m_7L_2)$
$G_3 = gcos\theta_3(m_3L_{3c} + m_4L_3 + m_5L_3 + m_6L_3 + m_6L_3)$
$G_4 = gcos\theta_4(m_4L_{4c})$
$G_5 = -gcos\theta_5(m_5L_5 - m_5L_{5c} + m_6L_5 + m_7L_5)$
$G_6 = -gcos\theta_6(m_6L_6 - m_6L_{6c} + m_7L_6)$
$G_7 = gcos\theta_7(m_7L_{7c}) + gsin\theta_7(m_7k)$

$G_{q0} = -G_1 + G_2 + G_3 + G_4 - G_5 - G_6 + G_7$
$G_{q1} = -G_2 - G_3 - G_4 + G_5 + G_6 - G_7$
$G_{q2} = G_3 + G_4 - G_5 - G_6 + G_7$
$G_{q3} = G_4 - G_5 - G_6 + G_7$
$G_{q4} = -G_5 - G_6 + G_7$
$G_{q5} = G_6 - G_7$
$G_{q6} = G_7$