Asymptotic Stability Conditions for Certain Classes of Mechanical Systems with Time Delay

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Abstract: The stability of the trivial equilibrium position of certain classes of mechanical systems with time-varying delay is studied. The considered systems are described by the second order differential equations in the Lagrange form. It is assumed that velocity forces are linear, whereas for positional forces both linear case and essentially nonlinear one are investigated. On the basis of the decomposition method, sufficient conditions of the asymptotic stability of the equilibrium position are found. It is shown that the proposed approach can be used for the stability analysis of hybrid mechanical systems with switched positional forces. Three examples are presented to demonstrate the effectiveness of the obtained results.

Key–Words: Mechanical systems, asymptotic stability, Lyapunov function, delay, hybrid system

1 Introduction

The stability analysis of mechanical systems is a fundamental research problem [1, 11, 27, 33]. In numerous applications, motions of mechanical systems are described by multivariate differential equations of the second order, and this essentially complicates the investigation of their dynamics [1, 17, 35].

A general and efficient approach to stability analysis for multidimensional systems (for large-scale or complex systems) is the decomposition method [1, 35]. It has been effectively applied to wide classes of mechanical systems, see, for example, [1, 2, 8, 14, 17, 21, 35, 38] and the references cited therein. Nevertheless, the problem of further development of decomposition method remains an actual one. Its importance is caused by the fact that stability conditions of complex systems obtained by the application of the method depend on the precision of estimation of a Lyapunov function derivative with respect to the considered system. Therefore, by means of appropriate choice of aggregation form, one can define more exactly the domain of system parameter values guaranteeing the stability of a programmed motion.

Furthermore, it is worth mentioning that realistic models of numerous mechanical systems must incorporate aftereffect phenomena in their dynamics [19, 20, 29, 30]. For this aim delay differential equations can be used. In particular, feedback control mechanical systems unavoidably involve delay because a certain time is needed for the system reaction on the input signal.

It is well known that the presence of time delay could cause instability, see [18, 29]. In some applications, it is not possible to assure that delay is sufficiently small, and even known. Therefore, it is important to obtain restrictions for delay values under which stability for the considered systems can be guaranteed [4, 10, 12, 18, 20, 26, 32]. This problem is especially difficult for systems with time-varying delay [18].

In the present paper, the stability of the trivial equilibrium position of certain classes of mechanical systems with time-varying delay is studied. The considered systems are described by the second order differential equations in the Lagrange form. It is assumed that velocity forces are linear, whereas for positional forces both linear case and essentially nonlinear one are investigated. On the basis of the decomposition method, sufficient conditions of the asymptotic stability of the equilibrium position are found. It is shown that the proposed approach can be used for the stability analysis of hybrid mechanical systems with switched positional forces. Three examples are presented to demonstrate the effectiveness of the obtained results.

2 Statement of the Problem

In the sequel, R denotes the field of real numbers, and R^n the n-dimensional Euclidean space.

Let motions of a mechanical system are described by the Lagrange equations of the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = -hB\dot{q}(t) \\ -C_1q(t) - C_2q(t-\tau(t)). \tag{1}$$

Here $q(t)$ and $\dot{q}(t)$ are n-dimensional vectors of generalized coordinates and generalized velocities respectively; the kinetic energy $T = T(q,\dot{q})$ of the system is of the form $T(q,\dot{q}) = \frac{1}{2}\dot{q}^TA(q)\dot{q}$, where $A(q)$ is a symmetric and continuously differentiable for $q \in R^n$ matrix; B, C_1, C_2 are constant matrices; h is a large positive parameter; the delay $\tau(t)$ is a continuous nonnegative and bounded for $t \in [0,+\infty)$ function. We assume that the matrix B is nonsingular.

Systems of the form (1) are widely used for the modelling of gyroscopic systems [1, 38]. The term $-C_2q(t-\tau(t))$ can be treated as a control vector, whereas the presence of delay $\tau(t)$ might be caused by a time lag between the moments of measuring of the state and the application of the corresponding control force.

Let $\tau_0 = \sup_{t\geq 0}\tau(t)$. We assume that initial functions for solutions of (1) belong to the space $C^1([-\tau_0,0],R^n)$ of continuously differentiable functions $\varphi(\theta) : [-\tau_0,0] \to R^n$ with the norm

$$\|\varphi\|_{\tau_0} = \max_{\theta\in[-\tau_0,0]}\left(\|\varphi(\theta)\| + \|\dot{\varphi}(\theta)\|\right),$$

and $\|\cdot\|$ denotes the Euclidean norm of a vector.

Moreover, assume that for the kinetic energy the estimates

$$k_1\|\dot{q}\|^2 \leq T(q,\dot{q}) \leq k_2\|\dot{q}\|^2,$$

$$\left\|\frac{\partial T(q,\dot{q})}{\partial \dot{q}}\right\| \leq k_3\|\dot{q}\|, \quad \left\|\frac{\partial T(q,\dot{q})}{\partial q}\right\| \leq k_4\|\dot{q}\|^2$$

hold for all $q,\ \dot{q} \in R^n$, where k_1,k_2,k_3,k_4 are positive constants.

System (1) admits the equilibrium position $q = \dot{q} = 0$. Let us investigate the stability of the equilibrium position.

Consider the auxiliary delay free subsystems

$$B\dot{y}(t) + (C_1 + C_2)y(t) = 0, \tag{2}$$

$$A(0)\dot{z}(t) + Bz(t) = 0. \tag{3}$$

We look for conditions under which the asymptotic stability of subsystems (2) and (3) implies that for the equilibrium position $q = \dot{q} = 0$ of system (1).

In the case when $\tau(t) \equiv 0$, such conditions have been obtained in [38]. In [22], results of [38] were extended to system (1) with a constant delay. However, it is worth mentioning that approaches proposed in [38] and [22] are based on the Lyapunov first method, and they are inapplicable to systems with time-varying delay.

Another approach to stability analysis of system (1) without delay has been proposed in [21]. It is based on the Lyapunov direct method. Let us note that, unlike the results of [38], this approach permits to obtain stability conditions for some types of time-varying systems. In particular, in [5], it was used for the stability investigation of hybrid linear mechanical systems with switched positional forces.

In this paper, we will show that the Kosov approach is applicable to system (1) with time-varying delay.

Furthermore, we will study asymptotic stability conditions for the essentially nonlinear system of the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = -B\dot{q}(t) \\ -Q_1(q(t)) - Q_2(q(t-\tau(t))), \tag{4}$$

where components of the vectors $Q_1(q)$ and $Q_2(q)$ are continuously differentiable for $q \in R^n$ homogeneous functions of the order $\mu > 1$, and the rest notation is the same as in (1).

Finally, we shall consider systems of the forms (1) and (4) with switched positional forces. Based on the developed approaches, the conditions guaranteeing the asymptotic stability of the trivial equilibrium position for an arbitrary admissible switching signal will be obtained.

3 Decomposition of Systems with Linear Positional Forces

Let the time-delay system (1) be given.

Theorem 1 *Assume that the isolated subsystems* (2) *and* (3) *are asymptotically stable. Then, for any continuous nonnegative and bounded for $t \in [0,+\infty)$ delay $\tau(t)$, there exists a number $h_0 > 0$ such that the equilibrium position $q = \dot{q} = 0$ of system* (1) *is asymptotically stable for all $h \geq h_0$.*

Proof: According to the approach suggested in [21], let us define new variables by the formulae

$$\dot{q} = z, \quad \frac{\partial T(q,\dot{q})}{\partial \dot{q}} + hBq = hBy. \tag{5}$$

With the aid of the Implicit Function Theorem, we obtain that there exists a number $\delta_0 > 0$ such that

$$q = y - \frac{1}{h}B^{-1}A(0)z + \Psi(h,y,z)$$

for $\|y\|+\|z\|<\delta_0$, and, for any $h>0$, the condition $\|\Psi(h,y,z)\|/(\|y\|+\|z\|)\to 0$ as $\|y\|+\|z\|\to 0$ holds.

Substitution (5) transforms system (1) to the form

$$\begin{aligned}
&B\dot y(t)=-\tfrac{1}{h}\left(C_1+C_2\right)y(t)\\
&-\tfrac{1}{h}C_2(y(t-\tau(t))-y(t))\\
&+\tfrac{1}{h^2}C_2B^{-1}A(0)z(t-\tau(t))\\
&+\tfrac{1}{h^2}C_1B^{-1}A(0)z(t)\\
&+L_1(h,y(t),z(t),y(t-\tau(t)),z(t-\tau(t))), \qquad (6)\\
&A(0)\dot z(t)=-hBz(t)-C_1y(t)\\
&-C_2y(t-\tau(t))+\tfrac{1}{h}C_1B^{-1}A(0)z(t)\\
&+\tfrac{1}{h}C_2B^{-1}A(0)z(t-\tau(t))\\
&+L_2(h,y(t),z(t),y(t-\tau(t)),z(t-\tau(t))),
\end{aligned}$$

where the vector functions $L_1(h,y,z,u,v)$ and $L_2(h,y,z,u,v)$ satisfy the conditions $\|L_i(h,y,z,u,v)\|/(\|y\|+\|z\|+\|u\|+\|v\|)\to 0$ as $\|y\|+\|z\|+\|u\|+\|v\|\to 0$, $i=1,2$.

It is known, see [33, 37], that from the asymptotic stability of isolated subsystems (2) and (3), it follows the existence of quadratic forms $V_1(y)$ and $V_2(z)$ such that the inequalities

$$a_{11}\|y\|^2\le V_1(y)\le a_{12}\|y\|^2,$$

$$a_{21}\|z\|^2\le V_2(z)\le a_{22}\|z\|^2,$$

$$\left\|\frac{\partial V_1}{\partial y}\right\|\le a_{13}\|y\|, \qquad \left\|\frac{\partial V_2}{\partial z}\right\|\le a_{23}\|z\|,$$

$$\dot V_1|_{(2)}\le -a_{14}\|y\|^2, \qquad \dot V_2|_{(3)}\le -a_{24}\|z\|^2$$

are valid for all $y,z\in R^n$. Here a_{ij} are positive constants, $i=1,2$, $j=1,2,3,4$.

Choose a Lyapunov function for system (6) in the form

$$V(y,z)=\varepsilon h^2V_1(y)+V_2(z), \qquad (7)$$

where ε is a positive parameter. Differentiating $V(y,z)$ with respect to system (6), we obtain

$$\dot V|_{(6)}\le -a_{14}\varepsilon h\|y(t)\|^2-ha_{24}\|z(t)\|^2+$$

$$+\varepsilon b_1\|y(t)\|\Big(h\|y(t-\tau(t))-y(t)\|+\|z(t)\|$$

$$+\|z(t-\tau(t))\|+h^2\|L_1\|\Big)$$

$$+b_2\|z(t)\|\Big(\|y(t-\tau(t))\|+\|y(t)\|$$

$$+\frac{1}{h}\|z(t)\|+\frac{1}{h}\|z(t-\tau(t))\|+\|L_2\|\Big).$$

Here $b_1=\mathrm{const}>0$, $b_2=\mathrm{const}>0$.

Let us prove that, for sufficiently large values of h, the Lyapunov function (7) satisfies all the conditions of Theorem 4.2 from [19].

Choose a number $\delta\in(0,\delta_0)$. Assume that, for a solution $\left(y^T(t),z^T(t)\right)^T$ of (6), the inequality $\|y(\xi)\|+\|z(\xi)\|<\delta$, and the Razumikhin condition $V(y(\xi),z(\xi))\le 2V(y(t),z(t))$ are fulfilled for $\xi\in[t-2\tau_0,t]$. Then there exist positive numbers c_1 and c_2 such that the esimates

$$\|y(\xi)\|\le c_1\left(\|y(t)\|+\frac{1}{h\sqrt{\varepsilon}}\|z(t)\|\right),$$

$$\|z(\xi)\|\le c_2\left(h\sqrt{\varepsilon}\,\|y(t)\|+\|z(t)\|\right)$$

hold for $\xi\in[t-2\tau_0,t]$.

With the aid of these estimates and the Mean Value Theorem, we obtain

$$\|y(t-\tau(t))-y(t)\|\le\frac{\tau_0c_3}{h}\left(\|y(t)\|+\frac{1}{h\sqrt{\varepsilon}}\|z(t)\|\right.$$

$$\left.+\sqrt{\varepsilon}\,\|y(t)\|+\frac{1}{h}\|z(t)\|+\eta_1(\varepsilon,h,y(t),z(t))\right),$$

where $c_3=\mathrm{const}>0$, and $\eta_1(\varepsilon,h,y,z)/(\|y\|+\|z\|)\to 0$ as $\|y\|+\|z\|\to 0$.

Thus, we arrive at the inequality

$$\dot V(y(t),z(t))\le -a_{14}\varepsilon h\|y(t)\|^2-ha_{24}\|z(t)\|^2$$

$$+\varepsilon\tilde b_1\|y(t)\|\Big((\tau_0+\tau_0\sqrt{\varepsilon}+h\sqrt{\varepsilon})\|y(t)\|$$

$$+\left(1+\frac{\tau_0}{h}+\frac{\tau_0}{h\sqrt{\varepsilon}}\right)\|z(t)\|\Big)$$

$$+\tilde b_2\|z(t)\|\Big((1+\sqrt{\varepsilon})\|y(t)\|$$

$$+\left(\frac{1}{h}+\frac{1}{h\sqrt{\varepsilon}}\right)\|z(t)\|\Big)+\eta_2(\varepsilon,h,y(t),z(t)).$$

Here $\tilde b_1$ and $\tilde b_2$ are positive constants, whereas the function $\eta_2(\varepsilon,h,y,z)$ possesses the property $\eta_2(\varepsilon,h,y,z)/(\|y\|^2+\|z\|^2)\to 0$ as $\|y\|+\|z\|\to 0$.

Let the parameter ε satisfy the condition $4\tilde b_1\sqrt{\varepsilon}<a_{14}$. Then, for chosen ε, and for sufficiently large values of h and sufficiently small values of δ, the inequality

$$\dot V(y(t),z(t))\le -\frac{1}{2}\left(a_{14}\varepsilon h\|y(t)\|^2+ha_{24}\|z(t)\|^2\right)$$

holds. Hence [19], the zero solution of system (6) is asymptotically stable, and this implies the asymptotic stability of the equilibrium position $q=\dot q=0$ of the original system (1). □

Remark 2 *The proof of Theorem 1 contains a constructive procedure for the finding of the estimate of the set of large parameter values for which the asymptotic stability can be guaranteed.*

Remark 3 *Theorem 1 remains valid for the case when τ_0 is a function of h; and it can be even unbounded for $h \in (0, +\infty)$. The only restriction on it is the condition $\tau_0(h)/h \to 0$ as $h \to +\infty$.*

Example 1: Let the linear control system

$$\ddot{q}(t) + h \begin{pmatrix} b & -g \\ g & b \end{pmatrix} \dot{q}(t) - c\, q(t) = u \qquad (8)$$

be given. Here $q(t), \dot{q}(t) \in R^2$; b, g, c are positive constants; $u = (u_1, u_2)^T$ is a control vector; h is a positive parameter.

It is known [33], that, in the case when $u = 0$, system (8) is unstable. Consider the problem of stabilization of this system under the following restrictions on the control law:

(i) control forces should be nonconservative;

(ii) there exists a delay in the control scheme.

We assume that delay might be unknown and time-varying, and only an upper bound $\tau_0 > 0$ for delay values is given.

Define the control vector by the formula

$$u = \begin{pmatrix} 0 & p \\ -p & 0 \end{pmatrix} q(t - \tau(t)),$$

where $p = \text{const} > 0$. For this control law, the closed-loop system takes the form

$$\begin{aligned} &\ddot{q}(t) + h \begin{pmatrix} b & -g \\ g & b \end{pmatrix} \dot{q}(t) - c\, q(t) \\ &\quad + \begin{pmatrix} 0 & -p \\ p & 0 \end{pmatrix} q(t - \tau(t)) = 0. \end{aligned} \qquad (9)$$

Consider subsystems (2) and (3) corresponding to system (9). We obtain

$$\dot{y}(t) = \omega \begin{pmatrix} bc - gp & bp + gc \\ -bp - gc & bc - gp \end{pmatrix} y(t), \qquad (10)$$

$$\dot{z}(t) = \begin{pmatrix} -b & g \\ -g & -b \end{pmatrix} z(t). \qquad (11)$$

Here $\omega = 1/(b^2 + g^2)$.

Subsystem (11) is asymptotically stable, and for the asymptotic stability of subsystem (10) it is necessary and sufficient the fulfilment of the inequality

$$p > \frac{bc}{g}. \qquad (12)$$

Applying Theorem 1, we obtain that, under the condition (12), for an arbitrary given number $\tau_0 > 0$, there exists $h_0 > 0$ such that the closed-loop system (9) is asymptotically stable for all $h \geq h_0$ and for any continuous delay $\tau(t)$ satisfying the inequalities $0 \leq \tau(t) \leq \tau_0$.

The results of numerical simulation are represented in Figs. 1 and 2, where the dependence of the coordinate q_1 on time is shown. It was assumed that $\tau = \text{const} > 0$, and the following values of parameters of the system and initial conditions were chosen: $b = 2$, $c = 1$, $g = 1$, $p = 5$, $t_0 = 0$, and $q_1(\theta) = 0.2$, $q_2(\theta) = -0.3$, $\dot{q}_1(\theta) = 0$, $\dot{q}_2(\theta) = 0$ for $\theta \in [-\tau, 0]$.

Fig. 1 corresponds to the case of stable system. Here $h = 7$, $\tau = 1$.

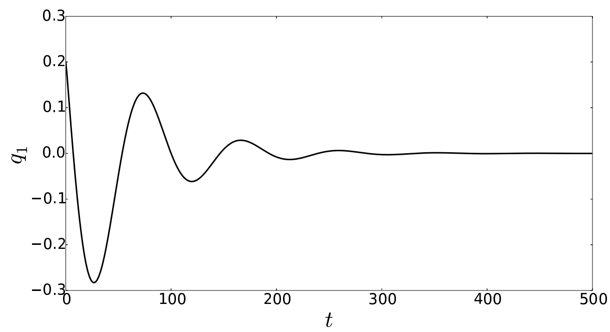

Figure 1. The case of stable system

On the other hand, Fig. 2 shows that the increasing of the value of delay and decreasing of the value of h might cause the instability. In this case, $h = 2.9$ and $\tau = 2$.

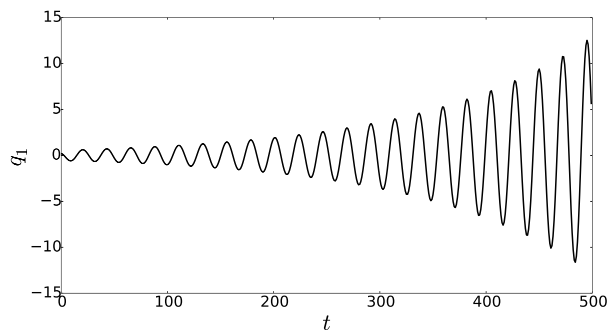

Figure 2. The case of unstable system

4 Decomposition of Systems with Nonlinear Positional Forces

Next, we turn to the case of system (4). The system admits the equilibrium position $q = \dot{q} = 0$ as well. We look for conditions of asymptotic stability for this equilibrium position.

Since $\mu > 1$, equations (4) are essentially nonlinear. Hence, stability analysis can not be carried out on the basis of linear approximation system. To solve the stated problem, let us apply again the decomposition approach.

Construct the isolated subsystems

$$B\dot{y}(t) = -Q_1(y(t)) - Q_2(y(t)), \qquad (13)$$

$$A(0)\dot{z}(t) = -Bz(t). \qquad (14)$$

Thus, instead of time-delay system (4) consisting of n nonlinear second order differential equations, we will consider two auxiliary first order delay free subsystems (13) and (14). Notice that subsystem (14) is linear, whereas (13) is a nonlinear system with homogeneous right-hand sides.

Theorem 4 *Let the zero solutions of isolated subsystems* (13) *and* (14) *be asymptotically stable. Then the equilibrium position* $q = \dot{q} = 0$ *of* (4) *is asymptotically stable for any continuous nonnegative and bounded for* $t \in [0, +\infty)$ *delay* $\tau(t)$.

Proof: By the usage of the substitution

$$\dot{q}(t) = z(t), \qquad \frac{\partial T(q, \dot{q})}{\partial \dot{q}} + Bq(t) = By(t),$$

we transform (4) to the system

$$\begin{aligned} &B\dot{y}(t) = -Q_1(y(t)) - Q_2(y(t)) \\ &+L_1(y(t), z(t), y(t-\tau(t)), z(t-\tau(t))), \\ &A(0)\dot{z}(t) = -Bz(t) \\ &+L_2(y(t), z(t), y(t-\tau(t)), z(t-\tau(t))). \end{aligned} \qquad (15)$$

Here vector functions $L_1(y, z, u, v)$ and $L_2(y, z, u, v)$ are continuous in the domain $\|y\|+\|z\|+\|u\|+\|v\| < \delta_0$ and satisfy the inequalities

$$\|L_1(y, z, u, v)\| \le d_1\Big(\|z\| \left(\|y\|^{\mu-1} + \|z\|^{\mu-1}\right)$$

$$+\|y-u\| \left(\|y\|^{\mu-1} + \|u\|^{\mu-1}\right) + \|z\|^2$$

$$+\|v\| \left(\|u\|^{\mu-1} + \|v\|^{\mu-1}\right)\Big),$$

$$\|L_2(y, z, u, v)\| \le d_2\Big(\|y\|^{\mu} + \eta(y, z)\|z\|$$

$$+\|u\|^{\mu} + \|v\|^{\mu}\Big),$$

where δ_0, d_1, d_2 are positive constants, and $\eta(y, z) \to 0$ as $\|y\| + \|z\| \to 0$.

The equilibrium position $q = \dot{q} = 0$ of the original system (4) is asymptotically stable if and only if the zero solution of (15) is asymptotically stable.

It is known, see [37], that from the asymptotic stability of the zero solutions of subsystems (13) and (14) it follows the existence of continuously differentiable for $y \in R^n$ and $z \in R^n$ homogeneous of orders γ_1 and γ_2 respectively Lyapunov functions $V_1(y)$ and $V_2(z)$ satisfying the assumptions of the Lyapunov asymptotic stability theorem. It is worth mentioning that, in the computation of these functions, one can take for γ_1 and γ_2 arbitrary numbers greater than 1.

Consider the function

$$V(y, z) = V_1(y) + V_2(z). \qquad (16)$$

For this function and its derivative with respect to system (15) the following estimates are valid

$$a_1 \left(\|y\|^{\gamma_1} + \|z\|^{\gamma_2}\right) \le V \le a_2 \left(\|y\|^{\gamma_1} + \|z\|^{\gamma_2}\right),$$

$$\dot{V}|_{(15)} \le -a_3 \left(\|y(t)\|^{\gamma_1+\mu-1} + \|z(t)\|^{\gamma_2}\right)$$

$$+a_4\|y(t)\|^{\gamma_1-1}\Big(\|z(t)\| \left(\|y(t)\|^{\mu-1} + \|z(t)\|^{\mu-1}\right)$$

$$+\|y(t) - y(t-\tau(t))\|\Big(\|y(t-\tau(t))\|^{\mu-1}$$

$$+\|y(t)\|^{\mu-1}\Big) + \|z(t-\tau(t))\| \Big(\|y(t-\tau(t))\|^{\mu-1}$$

$$+\|z(t-\tau(t))\|^{\mu-1}\Big) + \|z(t)\|^2\Big)$$

$$+a_5\|z(t)\|^{\gamma_2-1}\Big(\|y(t)\|^{\mu} + \eta(y(t), z(t))\|z(t)\|$$

$$+\|y(t-\tau(t))\|^{\mu} + \|z(t-\tau(t))\|^{\mu}\Big),$$

where $a_1, \dots, a_5$ are positive constants.

We will show that if for the orders of homogeneity γ_1 and γ_2 the inequalities

$$\max\left\{\gamma_2;\ \frac{\mu\gamma_2}{2} - \mu + 1\right\} < \gamma_1 < \mu\gamma_2 - \mu + 1 \qquad (17)$$

hold, then the Lyapunov function (16) satisfies all the conditions of Theorem 4.2 from [19].

Choose a number $\delta \in (0, \delta_0)$. Assume that, for a solution $(y^T(t), z^T(t))^T$ of (15), the inequality $\|y(\xi)\| + \|z(\xi)\| < \delta$, and the Razumikhin condition $V(y(\xi), z(\xi)) \le 2V(y(t), z(t))$ are fulfilled for $\xi \in [t - 2\tau_0, t]$. Then there exist positive numbers c_1 and c_2 such that the esimates

$$\begin{aligned} \|y(\xi)\| &\le c_1 \left(\|y(t)\| + \|z(t)\|^{\gamma_2/\gamma_1}\right), \\ \|z(\xi)\| &\le c_2 \left(\|y(t)\|^{\gamma_1/\gamma_2} + \|z(t)\|\right) \end{aligned} \qquad (18)$$

hold for $\xi \in [t - 2\tau_0, t]$.

Using estimates (18) and applying the Mean Value Theorem, we obtain

$$\|y(t-\tau(t)) - y(t)\| \le \tau_0 \|\dot{y}(t-\zeta\tau(t))\|$$

$$\le c_3\tau_0 \Big(\|y(t)\|^{\mu} + \|z(t)\|^{\mu} + \|z(t)\|^2$$

$$\|y(t)\|^{\mu\gamma_1/\gamma_2} + \|z(t)\|^{\mu\gamma_2/\gamma_1} \Big),$$

where $c_3 = \mathrm{const} > 0$, and $0 < \zeta < 1$.

By the use of homogeneous functions properties, see [37], it can be shown that if the parameters γ_1 and γ_2 satisfy the condition (17), then, for sufficiently small values of δ, the inequality

$$\dot{V}(y(t), z(t)) \le -\frac{a_3}{2}\Big(\|y(t)\|^{\gamma_1+\mu-1} + \|z(t)\|^{\gamma_2}\Big)$$

is valid. Thus, for the Lyapunov function (16), all the assumptions of Theorem 4.2 from [19] are fulfilled. So the zero solution of (15) is asymptotically stable. This implies that the equilibrium position $q = \dot{q} = 0$ of (4) is also asymptotically stable. □

Remark 5 *It is worth mentioning that mechanical systems with essentially nonlinear positional forces have been considered, for instance, in [9, 11, 15, 16, 36]. In particular, in [15, 16], they were applied for the developing of seismic mitigation devices.*

Remark 6 *Unlike Theorem 1, in Theorem 4 it is not required the presence in the considered equations of a large parameter.*

Remark 7 *The fulfilment of assumptions of Theorem 4 guarantees that the equilibrium position $q = \dot{q} = 0$ of* (4) *is asymptotically stable for any continuous nonnegative and bounded for $t \in [0, +\infty)$ delay $\tau(t)$. Thus, we obtained co-called delay-independent stability conditions, see [18].*

Example 2: Consider the control system

$$\ddot{q}(t) + b\,\dot{q}(t) + \|q(t)\|^2 \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix} q(t) = u, \quad (19)$$

where $q(t), \dot{q}(t) \in R^2$; b and c are positive constants; $u = (u_1, u_2)^T$ is a control vector. Equations of such type are used, for instance, for the modelling of rotor dynamics in a magnetic bearing system [31].

We are going to design a feedback control law to stabilize the equilibrium position $q = \dot{q} = 0$ of system (19). Assume that the control law depends on q, and is independent of $\dot{q}$. Furthermore, we consider the case when there exists a delay in the control scheme. The delay may be unknown and time-varying.

It is known, see [18, 29], that for the linear control law

$$u_1 = \alpha_{11}q_1(t-\tau(t)) + \alpha_{12}q_2(t-\tau(t)),$$

$$u_2 = \alpha_{21}q_1(t-\tau(t)) + \alpha_{22}q_2(t-\tau(t)),$$

where $\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}$ are constants, the presence of delay might result in instability of the equilibrium position.

Choose now functions u_1 and u_2 in the form

$$\begin{aligned} u_1 &= -\alpha_1 q_1^3(t-\tau(t)), \\ u_2 &= -\alpha_2 q_2^3(t-\tau(t)). \end{aligned} \quad (20)$$

Here α_1 and α_2 are positive constants.

Applying Theorem 4, we obtain that for the control law (20) the equilibrium position $q = \dot{q} = 0$ of the corresponding closed-loop system is asymptotically stable for any continuous nonnegative and bounded for $t \ge 0$ delay $\tau(t)$.

The results of numerical simulation are represented in Figs. 3 and 4, where the dependence of the coordinate q_1 on time is shown. The following values of parameters of the system were chosen: $b = 1$, $c = 8$, $a_1 = 2.5$, $a_2 = 1.5$, $\tau = 3$.

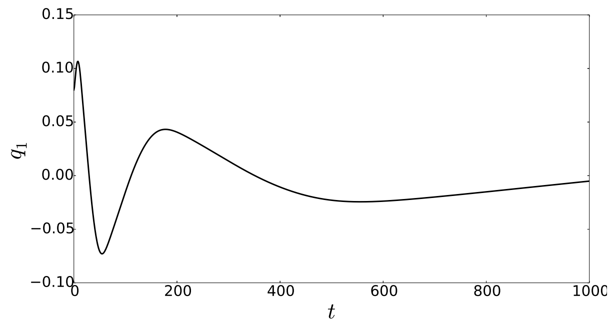

Figure 3. Simulation results ($q_1(\theta) = 0.08$, $q_2(\theta) = -0.08$)

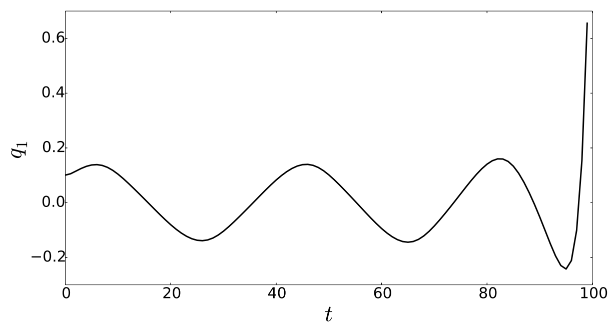

Figure 4. Simulation results ($q_1(\theta) = 0.1007$, $q_2(\theta) = -0.1007$)

Fig. 3 corresponds to the case when $t_0 = 0$, $q_1(\theta) = 0.08$, $q_2(\theta) = -0.08$, $\dot{q}_1(\theta) = 0$, $\dot{q}_2(\theta) = 0$

for $\theta \in [-3, 0]$, whereas for Fig.4 we have $t_0 = 0$, $q_1(\theta) = 0.1007$, $q_2(\theta) = -0.1007$, $\dot{q}_1(\theta) = 0$, $\dot{q}_2(\theta) = 0$ for $\theta \in [-3, 0]$.

These results show that, for nonlinear systems, we can guarantee only local asymptotic stability of the equilibrium position.

5 Decomposition of Systems with Switched Positional Forces

We will show now that the approaches proposed in the present paper can be used for the stability analysis of some classes of switched mechanical systems. A switched system is a particular kind of hybrid dynamical system that consists of a family of subsystems and a switching law determining at each time instant which subsystem is active [13, 23, 25].

In various cases, it is necessary to design a control system in such a way that it remains stable for any admissible switching law [23, 34]. A general approach to the problem is based on the construction of a common Lyapunov function for family of subsystems corresponding to switched system. This approach has been effectively used in many papers, see, for instance, [3, 5–7, 13, 23–25, 28, 34], and the references cited therein. However, the problem of the existence of a common Lyapunov function has not got a constructive solution even for the case of family of linear time-invariant systems [23, 34]. This problem is especially difficult for mechanical systems with switched force fields. Such systems possess a special structure. Therefore, well known approaches developed for switched systems of general form may be inefficient or even inapplicable for mechanical systems, see [5].

In this section, we consider time-delay mechanical systems of the forms (1) and (4) with switched positional forces. We will look for conditions under which the trivial equilibrium position of these systems is asymptotically stable for an arbitrary admissible switching law.

Let the family of time-delay mechanical systems

$$\begin{aligned}\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} &= -hB\dot{q}(t) - C_1^{(s)}q(t) \\ &-C_2^{(s)}q(t-\tau(t)),\ \ s = 1,\dots,N,\end{aligned} \tag{21}$$

be given. Here $C_1^{(s)}$ and $C_2^{(s)}$ are constant matrices, and the rest notation is the same as in (1).

Switched system generated by the family (21) and a switching law σ is

$$\begin{aligned}\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} &= -hB\dot{q}(t) \\ &-C_1^{(\sigma)}q(t) - C_2^{(\sigma)}q(t-\tau(t)).\end{aligned} \tag{22}$$

Hereinafter, a switching law is defined as a piecewise constant function $\sigma = \sigma(t) : [0, +\infty) \to S = \{1, \dots, N\}$. We assume that the function $\sigma(t)$ is right-continuous, and on every bounded time interval the function has a finite number of discontinuities. This kind of switching law is called admissible one.

Consider subsystem (3) and the family of subsystems

$$\begin{gathered}B\dot{y}(t) + \left(C_1^{(s)} + C_2^{(s)}\right)y(t) = 0, \\ s = 1,\dots,N.\end{gathered} \tag{23}$$

Theorem 8 *Let subsystem* (3) *and all subsystems from the family* (23) *be asymptotically stable, and family* (23) *admit a continuously differentiable for $y \in R^n$ homogeneous of the second order common Lyapunov function $V_1(y)$ satisfying the assumptions of the Lyapunov asymptotic stability theorem. Then, for any continuous nonnegative and bounded for $t \in [0, +\infty)$ delay $\tau(t)$, there exists a number $h_0 > 0$ such that the equilibrium position $q = \dot{q} = 0$ of system* (22) *is asymptotically stable for all $h \geq h_0$ and for an arbitrary switching law.*

Proof: Substitution (5) transforms (21) to the family

$$\begin{aligned}&B\dot{y}(t) = -\tfrac{1}{h}\left(C_1^{(s)} + C_2^{(s)}\right)y(t) \\ &-\tfrac{1}{h}C_2^{(s)}(y(t-\tau(t)) - y(t)) \\ &+\tfrac{1}{h^2}C_2^{(s)}B^{-1}A(0)z(t-\tau(t)) \\ &+\tfrac{1}{h^2}C_1^{(s)}B^{-1}A(0)z(t) \\ &+L_1^{(s)}(h, y(t), z(t), y(t-\tau(t)), z(t-\tau(t))), \\ &A(0)\dot{z}(t) = -hBz(t) - C_1^{(s)}y(t) \\ &-C_2^{(s)}y(t-\tau(t)) + \tfrac{1}{h}C_1^{(s)}B^{-1}A(0)z(t) \\ &+\tfrac{1}{h}C_2^{(s)}B^{-1}A(0)z(t-\tau(t)) \\ &+L_2^{(s)}(h, y(t), z(t), y(t-\tau(t)), z(t-\tau(t))),\end{aligned} \tag{24}$$

where the vector functions $L_i^{(s)}(h, y, z, u, v)$ satisfy the conditions $\|L_i^{(s)}(h, y, z, u, v)\|/(\|y\| + \|z\| + \|u\| + \|v\|) \to 0$ as $\|y\| + \|z\| + \|u\| + \|v\| \to 0$, $i = 1, 2$; $s = 1, \dots, N$.

Let $V_1(y)$ be a common Lyapunov function for subsystems (23) possessing the properties specified in the theorem. From the asymptotic stability of subsystem (3), it follows the existence of a quadratic form $V_2(z)$ satisfying the assumptions of the Lyapunov asymptotic stability theorem.

Consider the Lyapunov function $V(y,z)$ defined by the formula (7). Similar to the proof of Theorem 1, it can be shown that, for sufficiently small values of ε and sufficiently large values of h, $V(y,z)$ is a common Lyapunov function for family (24) satisfying all the assumptions of Theorem 4.2 from [19]. $\square$

Remark 9 *Theorem 8 permits to reduce, for sufficiently large values of h, the problem of finding of a common Lyapunov function for family* (21) *consisting of second order systems to that for family* (23) *of first order systems. Unlike systems from* (21)*, systems from the family* (23) *are delay free, and, generally, do not possess a special structure. Therefore, some known conditions of the existence of a common Lyapunov function can be applied to family* (23)*.*

Next, cosider the family of systems

$$\frac{d}{dt}\frac{\partial T}{\partial \dot q}-\frac{\partial T}{\partial q}=-hB\dot q(t)-Q_1^{(s)}(q(t))$$
$$-Q_2^{(s)}(q(t-\tau(t))),\ \ s=1,\dots,N,$$

and the corresponding switched system

$$\begin{aligned}\frac{d}{dt}\frac{\partial T}{\partial \dot q}-\frac{\partial T}{\partial q}&=-hB\dot q(t)\\&-Q_1^{(\sigma)}(q(t))-Q_2^{(\sigma)}(q(t-\tau(t))).\end{aligned}\tag{25}$$

Here components of the vectors $Q_1^{(s)}(q)$ and $Q_2^{(s)}(q)$ are continuously differentiable for $q\in R^n$ homogeneous functions of the order $\mu>1$, and the rest notation is the same as in (4).

Construct the auxiliary family of delay free homogeneous subsystems

$$\begin{aligned}B\dot y(t)=-Q_1^{(s)}(y(t))-Q_2^{(s)}(y(t)),\\ s=1,\dots,N.\end{aligned}\tag{26}$$

Theorem 10 *Let the zero solutions of subsystem* (3) *and all subsystems from the family* (26) *be asymptotically stable, and family (26) admit a continuously differentiable for $y\in R^n$ common homogeneous Lyapunov function $V_1(y)$ satisfying the assumptions of the Lyapunov asymptotic stability theorem. Then the equilibrium position $q=\dot q=0$ of* (25) *is asymptotically stable for any continuous nonnegative and bounded for $t\in[0,+\infty)$ delay $\tau(t)$ and for an arbitrary switching law.*

The proof of the theorem is similar to that of Theorem 4.

Remark 11 *Sufficient conditions for the existence of a common homogeneous Lyapunov function for a family of homogeneous systems have been obtained in [7].*

Example 3: Let the family consisting of two control systems

$$\begin{cases}\ddot q_1(t)+\dot q_1(t)-q_2^3(t)=u_1^{(1)},\\ \ddot q_2(t)+\dot q_2(t)+q_1^3(t)=u_2^{(1)},\end{cases}\tag{27}$$

$$\begin{cases}\ddot q_1(t)+\dot q_1(t)+q_2^3(t)=u_1^{(2)},\\ \ddot q_2(t)+\dot q_2(t)-q_1^3(t)=u_2^{(2)}\end{cases}\tag{28}$$

be given. Here $q_1(t),q_2(t)\in R$; $u_1^{(1)}$, $u_2^{(1)}$, $u_1^{(2)}$, $u_2^{(2)}$ are control variables.

It is known [27], that, in the case when $u_1^{(1)}=u_2^{(1)}=u_1^{(2)}=u_2^{(2)}=0$, the equilibrium position $q_1=q_2=\dot q_1=\dot q_2=0$ for both systems (27) and (28) is unstable. Consider the problem of stabilization of the equilibrium position.

Assume that the control variables are chosen in the form

$$u_1^{(1)}=0,\quad u_2^{(1)}=-2q_2^3(t-\tau(t)),$$
$$u_1^{(2)}=-2q_1^3(t-\tau(t)),\quad u_2^{(2)}=0.$$

Then we obtain the family of closed-loop systems

$$\begin{cases}\ddot q_1(t)+\dot q_1(t)-q_2^3(t)=0,\\ \ddot q_2(t)+\dot q_2(t)+q_1^3(t)+2q_2^3(t-\tau(t))=0,\end{cases}\tag{29}$$

$$\begin{cases}\ddot q_1(t)+\dot q_1(t)+q_2^3(t)+2q_1^3(t-\tau(t))=0,\\ \ddot q_2(t)+\dot q_2(t)-q_1^3(t)=0.\end{cases}\tag{30}$$

In this case, the corresponding subsystem (3) is

$$\begin{cases}\dot z_1(t)+z_1(t)=0,\\ \dot z_2(t)+z_2(t)=0,\end{cases}\tag{31}$$

and the family (26) can be written as follows

$$\begin{cases}\dot y_1(t)-y_2^3(t)=0,\\ \dot y_2(t)+y_1^3(t)+2y_2^3(t)=0,\end{cases}\tag{32}$$

$$\begin{cases}\dot y_1(t)+2y_1^3(t)+y_2^3(t)=0,\\ \dot y_2(t)-y_1^3(t)=0.\end{cases}\tag{33}$$

It is easily verified that the zero solutions of subsystems (31), (32), (33) are asymptotically stable. Moreover, in [7], it was proved that the family consisting of subsystems (32) and (33) admits the common homogeneous Lyapunov function

$$V(y_1, y_2) = 5y_1^2 + 4y_1y_2 + 5y_2^2.$$

Applying Theorem 10, we obtain that the equilibrium position $q_1 = q_2 = \dot{q}_1 = \dot{q}_2 = 0$ of the switched system generated by the family (29), (30) and a switching signal $\sigma(t)$ is asymptotically stable for any continuous nonnegative and bounded for $t \in [0, +\infty)$ delay $\tau(t)$ and for an arbitrary switching law.

6 Conclusion

In the present paper, by the usage of the decomposition method, sufficient conditions of the asymptotic stability of equilibrium positions for some classes of time-delay mechanical systems are obtained. It should be noted that, to provide the decomposition, in the case of linear positional forces the presence of a large parameter in the considered equations is required, whereas in the case of nonlinear positional forces the decomposition is ensured by orders of their homogeneity.

The results of the paper can be extended to mechanical systems with essentially nonlinear velocity forces. Moreover, the proposed approaches permit to obtain stability conditions for systems with delay both in positional forces and in velocity ones.

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