Robust predictive control combined with an adaptive mechanism for constrained uncertain systems subject to disturbances

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Abstract: This paper proposes a discrete-time adaptive model predictive control (MPC) algorithm for a class of constrained linear time-invariant systems subject to state-dependent disturbances, which updates the estimate of uncertain system parameters on-line and produces the control input ensuring the constraint fulfillment. This method is based on an adaptive mechanism and a robust MPC algorithm using the comparison model which enables to estimate the future prediction error bound. First, the parameter estimation method for MPC based on the moving horizon estimation is introduced. It allows to predict explicitly the worst-case estimation error bound over the prediction horizon. Second, we propose an adaptive-type MPC strategy developed by combining an on-line parameter estimator with a robust MPC method based on the modified comparison model. The MPC controller designed in this way guarantees constraint fulfillment, closed-loop stability and feasibility in the presence of uncertain system parameters and state-dependent disturbances. A numerical example demonstrates the unique features of the proposed approach.

Key–Words: Constrained systems, model predictive control, parameter estimation, robust stability

1 Introduction

Model predictive control (MPC), often also referred to as receding horizon control (RHC) or moving horizon control, uses the pre-specified system models and iteratively determines the control input at each time instant through finite/infinite horizon optimization problem. This strategy is one of the most promising ways to handle control problems for systems having input and/or state constraints, and is by now a widely used in industry and well established technique (see e.g. [1, 2, 3, 4] and the references therein). In this control scheme, the prediction is achieved based on the system model, which implies the model quality plays a vital role in MPC, and in reality there always exist model uncertainties and these may cause a significantly large effect on the system performance. Although the nominal MPC has inherent robustness properties when no constraint exists, it cannot guarantee the robustness of general constrained uncertain systems [5]. Therefore, it has been an important issue to develop robust MPC methods which guarantee a certain control performance in the presence of model uncertainties. This type of MPC has been extensively studied for many years and the MPC approaches developed based on an explicit model uncertainty description such as polytopic model have been proposed. [6, 7, 8, 9, 10]. However, in this line of research, the system model is fixed though its uncertainties are taken explicitly into account. Therefore, its control performance is limited by the quality of the fixed (initial) model.

Another attractive way to handle model uncertainties in MPC is to update the system model on-line based on measurement data. Although the development of adaptive-type MPC strategy is one of the research issues for the control of constrained systems, there have been few reports on this topic so far [2]. One of the main reasons is the difficulty to guarantee the fulfillment of system constraints in the presence of an adaptive mechanism through the receding horizon strategy. In order to overcome this problem, we have to estimate the future behavior of the real system while updating the system parameters on-line. In addition, it seems extremely difficult to guarantee both feasibility and stability theoretically whenever an adaptive approaches is combined with MPC. For such control scheme, Fukushima et al. [11] have proposed a continuous-time adaptive-type MPC methodology for a class of linear uncertain systems subject to state and input constraints. Further, since the control input to be determined in MPC is generally piecewise constant, Kim and Sugie [12] have introduced its discrete-time formula from the viewpoint of implementability.

The purpose of this paper is to develop a discrete-time MPC algorithm combined with an adaptive mechanism for a class of uncertain linear time-invariant systems subject to state-dependent disturbances and constraints on state and control, which is one of the extensions of [11, 12]. This MPC controller updates the estimate of system parameters on-line and produces the control input satisfying the given state and input constraints in the presence of parameter estimation errors. The key idea is to combine the robust MPC method based on the extended comparison model [13] with a parameter estimation method suitable for MPC. First, we present such a parameter estimation method based on moving horizon estimation. This method allows to predict the worst-case estimation error bound over the prediction horizon, by which one can take account of the future model improvement. Then, it is shown that the proposed estimation method can be incorporated into robust MPC method [13], which can handle state-dependent disturbances, by modifying the comparison model to handle time-varying parameter estimation errors. Using such a comparison model, the original MPC problem based on an uncertain model can be transformed into a nominal one without uncertain parameters. In addition, we show that feasibility and stability are guaranteed under certain conditions. Finally, a numerical example is given to demonstrate the effectiveness of the proposed method.

The paper is organized as follows. In Section 2, the system is presented along with the constraints for states and input. Sections 3 and 4 present the main results, namely a new parameter estimation algorithm and a modified adaptive-type MPC algorithm which has a merit such that the future model improvement can be explicitly considered. Then, Section 5 shows feasibility and stability results. A numerical example is provided in Section 6. Finally, a conclusion is given in the end.

Nomenclature:

$\mathbb{R}$	set of real numbers
$\mathbb{R}^n$	set of $n-dimensional$ real vectors
$\mathbb{R}^{n\times m}$	set of $n\times m$-dimensional real matrices
x_i	i-th entry of the vector $x\in\mathbb{R}^n$
I_n	$n\times n$ identity matrix
$0_{m\times n}$	$m\times n$ null matrix
$\lvert a\rvert$	absolute value of the scalar a $\in\mathbb{R}$
$\lVert\cdot\rVert$	Euclidean norm of a vector or matrix
$\lVert\cdot\rVert_p$	p-norm of a vector or matrix
$\overline{\sigma}(M)$	largest singular value of the matrix M
$\overline{\lambda}(M)$	maximum eigenvalue of the matrix M
$\underline{\lambda}(M)$	minimum eigenvalue of the matrix M

2 System description and problem formulation

Consider the following discrete-time linear time-invariant uncertain system described in controllable canonical form:

$$x(t+1)=A(\theta^*)x(t)+Bu(t)+B_d d(t),\ x(0)=x_0, \tag{1}$$

with

$$A(\theta^*)=\begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix},\ B=\begin{bmatrix}1\\0\\\vdots\\\vdots\\0\end{bmatrix}, \tag{2}$$

where $A(\theta^*)\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^n$, $B_d\in\mathbb{R}^{n\times d}$, $x(t)\in\mathbb{R}^n$ is the measurable state vector, x_0 is a given initial state, $u(t)\in\mathbb{R}$ is the control input, $d(t)\in\mathbb{R}^d$ is the bounded disturbance which satisfies

$$d(t)\in D,\ D:=\{d\in\mathbb{R}^d:\ \|d(t)\|\le 1\}. \tag{3}$$

It is assumed that the disturbance $d(t)\in D$ is measurable at the current time instant, but the future ones are uncertain [1, 14, 15, 16, 17]. We denote the uncertain parameter of $A(\theta^*)$ as

$$\theta^*:=\begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n\end{bmatrix}^T\in\mathbb{R}^n, \tag{4}$$

and use the notation $\theta(t)\in\mathbb{R}^n$ to denote the estimate of θ^* at time instant t. Then, we define the parameter estimation error as

$$\tilde{\theta}(t):=\theta(t)-\theta^* \tag{5}$$

which is unmeasurable. We assume that the initial estimate θ_0 of θ^* and initial estimation error bound ν_0, which satisfy the following condition

$$\|\theta_0-\theta^*\|\le\nu_0, \tag{6}$$

are given as a priori information. We introduce an additional variable $z(t)\in\mathbb{R}$ to denote that

$$z(t):=x_1(t+1)-u(t)-(B_d)_1\,d(t) \tag{7}$$

where $(B_d)_1$ denotes the first row of B_d. Notice that since $x_1(t+1)$ and $d(t)\in D$ are measurable and control $u(t)$ is a known value at time instant $t+1$, $z(t)$ is a measurable one. It is also important to note that (7) yields $z(t) = \theta^{*T}x(t)$, which is easily derived from (1). The system (1) is subject to the hard constraints on state $x(t)$ and control $u(t)$ as

$$\begin{aligned} &x(t) \in X, \quad X := \{x \in \mathbb{R}^n : \; |x_i(t)| \leq \psi_i, \; \forall i\}, \\ &u(t) \in U, \quad U := \{u \in \mathbb{R} : \; |u(t)| \leq \eta\}, \end{aligned} \tag{8}$$

with given $\psi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}$, which should be fulfilled at all time instants $t \geq 0$, and the following terminal constraint.

$$X_f := \{x \in \mathbb{R}^n : \; V(x) \leq 1\}, \; V(x) := \sqrt{x^T P x}, \tag{9}$$

where P is an $n \times n$ symmetric positive definite matrix. It is assumed that this matrix P and a state feedback gain $K_0 \in \mathbb{R}^n$, both of which are given in advance, satisfy the following assumptions [11, 12, 13].

Assumption 1: $1 \leq \sqrt{\underline{\lambda}(P)} \min_i \psi_i$ and $\|K_0\| \leq \sqrt{\underline{\lambda}(P)}\eta$

Assumption 2: $0 \leq \dfrac{b_0}{1 - a_0} < 1, \; a_0 < 1$ where

$$a_0 := \sqrt{1 - \frac{\underline{\lambda}(Q)}{\overline{\lambda}(P)}} + \frac{\overline{\sigma}(P^{\frac{1}{2}}B)\nu_0}{\sqrt{\underline{\lambda}(P)}}, \; b_0 := \|P^{\frac{1}{2}}B_d\|,$$
$$Q := P - F^T P F, \; F := A(\theta_0) + BK_0$$

Notice that Assumption 1 implies that the given feedback control $u(t) = K_0 x(t)$ always satisfies the constraints (8) in the terminal set (9). On the other hand, Assumption 2 implies that X_f is a robustly invariant set by $u(t) = K_0 x(t)$ [11, 12, 13, 18].

The goal of this paper is to develop a discrete-time robust MPC controller combined with an adaptive parameter estimator, which guarantees feasibility and stability in the presence of parameter estimation error. To that purpose, we first propose a new recursive adaptive parameter estimation algorithm suitable for MPC in the following section.

Remark 1: An example of systems described in (1) and (2) is a DC servo motor with uncertain parameters. Since the position and the velocity of the system can be measured (by a potentiometer and a tachogenerator, respectively), there is no loss of generality in the canonical form representation. A broad class of mechanical/electrical systems would satisfy similar conditions. Nevertheless, the authors aware that the representations, (1) and (2), may be restrictive in general. This is the price at the current status in order to incorporate an adaptive mechanism into MPC while ensuring feasibility and stability theoretically in the presence of the constraints on state and input.

3 Adaptive-type parameter estimator for robust predictive controller

In order to design an adaptive-type MPC controller, it is required to develop an appropriate adaptive parameter estimation method, by which the future model improvement can be taken into account explicitly. Moreover, it is important how to incorporate this method with robust MPC scheme in a less conservative manner. From this viewpoint, a novel recursive adaptive parameter estimation algorithm for MPC is proposed in the follows. It enables to predict the worst-case estimation error bound over the prediction horizon of MPC explicitly. This key feature allows us to develop an adaptive-type MPC controller based on a robust MPC method proposed by Fukushima and Bitmead [13], which will be described in the next section. To that purpose, we first introduce the following quantities.

$$\begin{aligned} F_1(t) &= \frac{\sum_{s=t}^{t-N_e+1} x(s)z(s)}{\alpha + \sum_{s=t}^{t-N_e+1} x^T(s)x(s)}, \\ F_2(t) &= \frac{\sum_{s=t}^{t-N_e+1} x(s)x^T(s)}{\alpha + \sum_{s=t}^{t-N_e+1} x^T(s)x(s)}, \end{aligned} \tag{10}$$

where N_e denotes the length of estimation horizon and $\alpha > 0$, both of which are chosen by the designer. We set $x(t) := 0$ and $z(t) := 0$ for $t < 0$. The proposed adaptive mechanism tries to estimate θ^* based on the above matrices $F_1(t)$ and $F_2(t)$. Then, the recursive adaptive parameter estimation algorithm for MPC is described as follows:

[Adaptive parameter estimation algorithm]

Step 0: At time $t = 0$, initialize $\overline{\gamma}(t) \in \mathbb{R}$, $f_1(t) \in \mathbb{R}^{n \times 1}$ and $f_2(t) \in \mathbb{R}^{n \times n}$ as follows:

$$\begin{aligned} &f_i(0) = F_i(0), \; i = 1, 2, \\ &\overline{\gamma}(0) = \sqrt{\overline{\lambda}((I_n - \kappa F_2(0))^T (I_n - \kappa F_2(0)))} \end{aligned} \tag{11}$$

where $\kappa > 0$ denotes an adaptive gain given by the designer. Then, go to Step 1.

Step 1: Apply the following parameter update law.

$$\theta(t) = \theta(t-1) + \kappa(f_1(t) - f_2(t)\theta(t-1)). \tag{12}$$

Then, go to Step 2.

Step 2: At the next sampling instant, let $t \leftarrow t + 1$. Then, update $f_i(t)$ and $\overline{\gamma}(t)$ as

$$\begin{aligned} &f_i(t) = F_i(t), \; i = 1, 2, \\ &\overline{\gamma}(t) = \sqrt{\overline{\lambda}((I_n - \kappa F_2(t))^T (I_n - \kappa F_2(t)))}\,, \end{aligned} \tag{13}$$

if the following condition is satisfied.

$$\sqrt{\overline{\lambda}((I_n - \kappa F_2(t))^T (I_n - \kappa F_2(t)))} \leq \overline{\gamma}(t-1). \tag{14}$$

Otherwise, perform the following update.

$$\begin{aligned} f_i(t) &= f_i(t-1),\ i = 1, 2, \\ \overline{\gamma}(t) &= \overline{\gamma}(t-1). \end{aligned} \tag{15}$$

Then, go to Step 1.

It is important to note that one of the differences from the conventional parameter update law (see (20) given later) is to use the summation of measured data over the estimation horizon N_e as shown in (10). Another difference is that the condition (14) in Step 2 aims at choosing the best data set for parameter estimation in terms of the excitation of $x(t)$ over the horizon, which will be shown in the following. Notice that the value of $\overline{\gamma}(t)$ determined in the above algorithm means

$$\overline{\gamma}(t) := \min_{0 \leq t_i \leq t} \sqrt{\overline{\lambda}((I_n - \kappa F_2(t_i))^T (I_n - \kappa F_2(t_i)))} \tag{16}$$

where t_i denotes the sampling instant. Now we introduce the notation $\nu_{t+k|t}$ to denote the k-step ahead prediction of upper bound of parameter estimation error $\tilde{\theta}$ from t onwards; i.e., $\nu_{t+k|t} \geq \|\tilde{\theta}(t+k)\|$ where $k = 0, 1, \cdots, N$, and N denotes the length of prediction horizon of MPC. Then, the following key result can be obtained based on the proposed adaptive parameter estimation algorithm.

Lemma 1: The worst-case upper bound $\nu_{t+k|t}$ ($k = 0, 1, \cdots, N$) of the future parameter estimation error satisfying

$$\|\tilde{\theta}(t+k)\| \leq \nu_{t+k|t},\ k = 0, 1, \cdots, N, \tag{17}$$

can be predicted by

$$\begin{aligned} \nu_{t+k+1|t} &= \overline{\gamma}(t)\nu_{t+k|t},\ k = 0, 1, \cdots, N-1, \\ \nu_{t|t} &= \begin{cases} \nu_0 & \text{for } t = 0 \\ \nu_{t|t-1} & \text{for } t \geq 1 \end{cases} \end{aligned} \tag{18}$$

Proof: See Appendix A □

This result shows that the future worst-case estimation error bound can be predicted explicitly and, therefore, one can take into account the future improvement of $\theta(t)$ by using $\nu_{t+k|t}$ in the robust MPC method. Furthermore, (16) and (18) show that the proposed algorithm tries, at each time instant, to choose the "best" data set in the sense that the predicted estimation error bound $\nu_{t+k|t}$ is minimized more rapidly by minimizing $\overline{\gamma}$ than that of the previous time instant. The above feature results in the following lemma.

Lemma 2: Given a scalar system (18), the following holds.

$$\nu_{t+k|t+1} \leq \nu_{t+k|t},\ \ k = 1, 2, \cdots, N,\ \ \forall t \geq 0. \tag{19}$$

Proof: See Appendix B □

In the next section, we describe how this adaptive estimation approach can be incorporated into a robust MPC method for the development of a discrete-time adaptive-type MPC scheme.

Remark 2: The conventional parameter estimation method [19, 20, 21] which is described as

$$\begin{aligned} \theta(t) = \theta(t-1) \\ + \frac{\kappa x(t)}{\alpha + x^T(t)x(t)}(z(t) - x^T(t)\theta(t-1)) \end{aligned} \tag{20}$$

could be incorporated into a MPC method to develop an adaptive-type MPC algorithm. However, this method makes the MPC too conservative in the sense that the future value of error bound $\nu_{t+k|t}$ is fixed. That is, since $\overline{\gamma}(t) = 1$ in (16) at all time instants as long as (20) is used, $\nu_{t+k|t} = \nu_0$ for $k = 1, \cdots, N$. Therefore, although the estimation error bound $\|\tilde{\theta}(t+k)\|$ could be decreased, it cannot be considered and evaluated explicitly over the prediction horizon by the conventional methods.

Remark 3: The proposed parameter estimation algorithm has the additional computational burden including the determination of maximum eigenvalue compared with the conventional estimation algorithm [19, 20, 21]. However, as shown in Step 2, one can determine the required values such as f_1, f_2 and $\overline{\gamma}$ in (13) by checking the condition (14) only once at each time instant. It is probably not so much time-consuming procedure. Also, it is nothing but a simple mathematical computation to predict $\nu_{t+k|t}$ in (18). From this viewpoint, the computational burden due to the parameter estimation is much less than that of MPC.

4 Robust predictive control algorithm for constrained systems subject to disturbances

In this section, we describe how to combine the adaptive parameter estimation method proposed in Section

3 with a robust MPC method. To that purpose, the following notations will be adopted hereafter: $\hat{v}_{t+k|t}$ and $\tilde{v}_{t+k|t}$ denote the k-step ahead prediction of a variable v from t onwards; $\hat{\theta}_{t+k|t}$ denotes the k-step ahead estimation of variable θ from t onwards, which is calculated by

$$\begin{aligned} &\hat{\theta}_{t+k+1|t} = \hat{\theta}_{t+k|t} + \kappa(f_1(t) - f_2(t)\hat{\theta}_{t+k|t}), \\ &\hat{\theta}_{t|t} := \theta(t),\ k = 0, 1, \cdots, N-1, \end{aligned} \tag{21}$$

which is derived from (12). Notice that $\theta(t+1) = \hat{\theta}_{t+1|t}$ from (12) and (21).

In order to reduce conservatism in MPC scheme, we adopt the following control parameterized in terms of the system state and a new variable $\tilde{u} \in \mathbb{R}$ [7, 9, 13, 22].

$$\begin{aligned} &u(t) = K(\theta(t))x(t) + \tilde{u}(t), \\ &K(\theta(t)) := -\theta^T(t) + \theta_0^T + K_0 \end{aligned} \tag{22}$$

where $K(\theta(t))$ is a feedback gain and $\tilde{u}(t)$, a feedforward control, is a decision variable. The key difference from existing other MPC methodologies is that $K(\theta(t))$ depends on the estimated parameter vector $\theta(t)$ which is updated based on (12) at each time instant t. Therefore, during the on-line operation, the feedback gain $K(\theta(t))$ is updated by (12) and the free variable $\tilde{u}(t)$ is computed by solving the constrained optimization problem of MPC shown later so that (8) is guaranteed. Then, under (22), the real system (1) can accordingly be rewritten as

$$\begin{aligned} &x(t+1) = Fx(t) + B\tilde{u}(t) - Be(t) + B_d d(t), \\ &e(t) := \tilde{\theta}^T(t)x(t). \end{aligned} \tag{23}$$

Notice that $e(t)$ implies the disturbance caused by the uncertain system parameters θ^* and depends on not only $\tilde{\theta}(t)$ but also $x(t)$ of the real system. Here, since we know the future worst-case upper bound $\nu_{t+k|t}$ of $\|\tilde{\theta}(t+k)\|$ as shown in Lemma 1, the future upper bound of $e(t+k)$ $(k = 0, 1, \cdots, N-1)$ can be described as follows:

$$\begin{aligned} &e(t+k) \in E,\ \ k = 0, 1, \cdots, N-1, \\ &E := \{e \in \mathbb{R} : \|e(t+k)\| \le \nu_{t+k|t}\|x(t+k)\|\}. \end{aligned} \tag{24}$$

However, the future upper bound of $e(t+k)$ can never be predicted in its present stage because $x(t+k)$ is unknown at the current time instant t.

On the other hand, for the design of MPC controller, the following nominal model is adopted.

$$\begin{aligned} &\hat{x}_{t+k+1|t} = A(\hat{\theta}_{t+k|t})\hat{x}_{t+k|t} + B\hat{u}_{t+k|t} + B_d f_k, \\ &f_k = \begin{cases} d(t) & \text{for } k = 0 \\ 0 & \text{for } k = 1, 2, \cdots, N-1, \end{cases} \end{aligned} \tag{25}$$

where $\hat{x}_{t|t} = x(t)$. The control sequence $\hat{u}_{t+k|t}$ to be determined is parameterized similarly to (22) as

$$\begin{aligned} &\hat{u}_{t+k|t} = K(\hat{\theta}_{t+k|t})\hat{x}_{t+k|t} + \tilde{u}_{t+k|t}, \\ &K(\hat{\theta}_{t+k|t}) := -\hat{\theta}^T_{t+k|t} + \theta_0^T + K_0 \end{aligned} \tag{26}$$

where $\hat{\theta}_{t+k|t}$ is obtained from (21) and thus only $\tilde{u}_{t+k|t}$ is computed by MPC scheme as mentioned in the above. Substituting (26) into (25) results in the following equation.

$$\begin{aligned} &\hat{x}_{t+k+1|t} = F\hat{x}_{t+k|t} + B\tilde{u}_{t+k|t} + B_d f_k, \\ &\hat{x}_{t|t} = x(t),\ \ k = 0, 1, \cdots, N-1. \end{aligned} \tag{27}$$

In MPC algorithm, the k-step ahead prediction of state x in (23) is obtained by using the model (27). In this case, the state prediction error due to the disregard of $d(t+k) \in D$ and $e(t+k) \in E$ for $k = 0, 1, \cdots, N-1$ is unavoidable and, therefore, the evaluation of this prediction error is necessary for the robustness analysis of the real system (23). Moreover, if the robust MPC approach using the optimization problem with nominal performance index is introduced, the suitable tightening of constraint is necessary to ensure that the control input which guarantees (8) in the presence of disturbances $e(t)$ and $d(t)$ can be computed at all time instants [11, 12]. This should be performed based on the consideration of the effects of both $d(t+k)$ and $e(t+k)$. However, since e depends on x as well as $\tilde{\theta}$ as mentioned in the above, it is not so easy to evaluate the effect of $e(t+k)$ for $k = 0, 1, \cdots, N-1$ upon the state prediction error.

In order to overcome this difficulty, we now introduce the following scalar system into the MPC formulation.

$$\begin{aligned} &w_{t+k+1|t} = a_{t+k|t}w_{t+k|t} + b_0 g_k + b_1|\tilde{u}_{t+k|t}|, \\ &w_{t|t} = V(x(t)),\ \ k = 0, 1, \cdots, N-1 \end{aligned} \tag{28}$$

where

$$\begin{aligned} &a_{t+k|t} := \sqrt{1 - \frac{\underline{\lambda}(Q)}{\overline{\lambda}(P)}} + \frac{\overline{\sigma}(P^{\frac{1}{2}}B)\nu_{t+k|t}}{\sqrt{\underline{\lambda}(P)}}, \\ &b_1 := \|P^{\frac{1}{2}}B\|, \\ &g_k = \begin{cases} \|d(t)\| & \text{for } k = 0 \\ 1 & \text{for } k = 1, 2, \cdots, N-1 \end{cases} \end{aligned}$$

It is constructed based on a priori information about the future upper bound of parameter estimation error $\nu_{t+k|t}$ in (18) [11, 12]. The system (28) enables to obtain an upper bound of the k-step ahead unknown future value $V(x(t+k))$, $k = 1, 2, \cdots, N$, as described in the following lemma.

Lemma 3: For any $\tilde{u}_{t+k|t}$ $(k = 0, 1, \cdots, N-1)$ and a given $x(t)$, the states of real system in (23) and scalar system in (28) satisfy

$$V(x(t+k)) \le w_{t+k|t}, \quad k = 0, 1, \cdots, N. \tag{29}$$

Proof: See Appendix C □

This lemma implies that the upper bound of the future state $x(t+k)$ of system (23) can be predicted by (28). Therefore, the k-step ahead upper bound of $e(t+k) \in E$ can be predicted based on (18) and (28).

Once the prediction error due to $e(t+k)$ and $d(t+k)$ can be evaluated, we then modify the original constraint sets, X and U in (8), to more restricted ones in order to guarantee feasibility and robust stability in the presence of disturbances e and d. The modified constraint sets, $\hat{X}$ and $\hat{U}$, imposed on the prediction using the nominal model (27) are described as follows:

$$\begin{aligned} &\hat{X}(t+k+1, \nu_{t+k+1|t}, w_{t+k+1|t}) := \\ &\quad \{\hat{x} \in \mathbb{R}^n : |\hat{x}_{i,t+k+1|t}| \\ &\qquad \le \psi_i - \hat{\psi}_i(t+k+1, \nu_{t+k+1|t}, w_{t+k+1|t}), \forall i\}, \\ &\hat{U}(t+k, \nu_{t+k|t}, w_{t+k|t}) := \\ &\quad \{K_0\hat{x} + \tilde{u} \in \mathbb{R} : |K_0\hat{x}_{t+k|t} + \tilde{u}_{t+k|t}| \\ &\qquad \le \eta - \hat{\eta}(t+k, \nu_{t+k|t}, w_{t+k|t})\} \end{aligned} \tag{30}$$

where

$$\begin{aligned} &\hat{\psi}_i(t+k+1, \nu_{t+k+1|t}, w_{t+k+1|t}) \\ &\quad := \sum_{s=0}^{k} \left(|\zeta_{1i}(s)| \frac{\nu_{t+k-s|t} w_{t+k-s|t}}{\sqrt{\underline{\lambda}(P)}} + \|\zeta_{2i}(s)\| \right), \\ &\hat{\eta}(t+k, \nu_{t+k|t}, w_{t+k|t}) := 2(\nu_{t+k|t} + \nu_0) \\ &\quad \times \sum_{s=0}^{k} \left(\|\xi_1(s)\| \frac{\nu_{t+k-s|t} w_{t+k-s|t}}{\sqrt{\underline{\lambda}(P)}} + \|\xi_2(s)\| \right) \\ &\quad + \sum_{s=0}^{k} \left(|K_0\xi_1(s)| \frac{\nu_{t+k-s|t} w_{t+k-s|t}}{\sqrt{\underline{\lambda}(P)}} + \|K_0\xi_2(s)\| \right) \\ &\quad + \frac{(\nu_{t+k|t} + \nu_0) w_{t+k|t}}{\sqrt{\underline{\lambda}(P)}}, \\ &\zeta_1(s) := F^s B, \ \ \zeta_2(s) := \begin{cases} F^{s-1}B_d & \text{for } s \neq 0, \\ 0_{n\times d} & \text{for } s = 0, \end{cases} \\ &\xi_1(s) := \begin{cases} F^{s-1}B & \text{for } s \neq 0, \\ 0_{n\times 1} & \text{for } s = 0, \end{cases} \\ &\xi_2(s) := \begin{cases} F^{s-2}B_d & \text{for } s = 2, 3, \cdots, N-1, \\ 0_{n\times d} & \text{for } s = 0, 1, \end{cases} \end{aligned} \tag{31}$$

and $\zeta_{ji}(s)$ $(j = 1, 2)$ denotes the ith row of $\zeta_j(s)$. The above modified constraints satisfy the following important property.

Theorem 1: For the nominal model (27) with a given $x(t) \in \mathbb{R}^n$, any feedforward control $\tilde{u}_{t+k|t} \in \mathbb{R}$ $(k = 0, \cdots, N-1)$ satisfying

$$\begin{aligned} &\hat{x}_{t+k+1|t} \in \hat{X}(t+k+1, \nu_{t+k+1|t}, w_{t+k+1|t}), \\ &K_0\hat{x}_{t+k|t} + \tilde{u}_{t+k|t} \in \hat{U}(t+k, \nu_{t+k|t}, w_{t+k|t}), \end{aligned} \tag{32}$$

also ensures the robust constraint fulfillment for state and control of the real system (23); i.e.,

$$\begin{aligned} &x(t+k+1) \in X, \\ &K(\hat{\theta}_{t+k|t})x(t+k) + \tilde{u}_{t+k|t} \in U, \end{aligned} \tag{33}$$

are guaranteed for all possible $e(t+k) \in E$ and $d(t+k) \in D$.

Proof: See Appendix D □

The above theorem shows the sufficient conditions for $\tilde{u}$ under which the MPC controller is able to guarantee one of the control objectives; i.e., the original constraints (33) can be always satisfied by the feedforward control $\tilde{u}$ which satisfies the modified ones (32).

Remark 4: The idea of restricting the original state and control bounds by suitable quantities which take into account the perturbations is not new. This can be found in Chisci et al. [23] proposed for the constrained discrete-time systems with persistent bounded disturbances. However, it is crucial to deal with state-dependent (possibly unbounded) disturbances in order to incorporate the parameter update mechanism, because the system parameter uncertainty results in the form of state-dependent disturbances as shown in (23). Therefore, we introduce the robust MPC method proposed by Fukushima and Bitmead [13] and extend it to handle the state-dependent disturbance due to the time-varying parameter estimation error.

The optimal feedforward control sequence $\tilde{u}_{t+k|t}$ for $k = 0, 1, \cdots, N-1$ is determined during the online operation by minimizing the objective function such as

$$J(x(t), \tilde{u}_{t+k|t}) := \sum_{k=0}^{N-1} \tilde{u}_{t+k|t}^T R \tilde{u}_{t+k|t} \tag{34}$$

where $R > 0$ is the weighting constant. Therefore, the following optimization problem for the proposed adaptive-type MPC developed based on the above results and the given constant ω ($\geq \max\{V(x_0), 1\}$) is solved on-line at each time instant t.

Finite horizon constrained optimization problem for MPC:

$$\min_{\tilde{u}} J(x(t), \tilde{u}_{t+k|t}) \tag{35}$$

subject to (18), (27), (28), (32) and

$$\begin{aligned} &w_{t+k|t} \leq \omega, \quad k = 0, 1, \cdots, N-1, \\ &w_{t+N|t} \leq 1. \end{aligned} \tag{36}$$

The above optimization problem is referred to as *AMPC optimization problem* throughout the paper. The role of ω and the conditions which ω should satisfy will be mentioned in detail in Section 5. Notice that it is easily verified from Theorem 1 that, if the above optimization problem is feasible at each sampling instant, the original constraints (8) are satisfied for all $t \geq 0$. The proposed adaptive-type MPC algorithm based on the above optimization problem and the adaptive parameter estimation algorithm in Section 3 is summarized as follows:

[Adaptive-type MPC algorithm]

Step 1: Measure the current state $x(t)$ and then set $\hat{x}_{t|t} = x(t)$.

Step 2: Update the parameter $\theta(t)$ based on the adaptive parameter estimation algorithm in Section 3.

Step 3: Calculate $\nu_{t+k|t}$ and $\hat{\theta}_{t+k|t}$ using (18) and (21).

Step 4: Determine $\tilde{u}_{t+k|t}$ by solving AMPC optimization problem.

Step 5: Apply the control $u(t) = \hat{u}_{t|t}$ ($= K(\theta(t))x(t) + \tilde{u}_{t|t}$) to the real system.

Step 6: At the next time instant, let $t \leftarrow t+1$ and go to Step 1.

In the following section, feasibility and stability issues of the proposed MPC scheme are described.

Remark 5: The following constraint given in (28)

$$w_{t+k+1|t} = a_{t+k|t} w_{t+k|t} + b_0 g_k + b_1 |\tilde{u}_{t+k|t}|$$

is a nonlinear with respect to $\tilde{u}_{t+k|t}$. By introducing a new variable $\chi_{t+k|t} \in \mathbb{R}$, we can modify the above constraint to

$$\begin{aligned} &w_{t+k+1|t} = a_{t+k|t} w_{t+k|t} + b_0 g_k + b_1 \chi_{t+k|t}, \\ &|\tilde{u}_{t+k|t}| \leq \chi_{t+k|t}, \ w_{t|t} = V(x(t)), \end{aligned} \tag{37}$$

and the objective function $J(x(t), \tilde{u}_{t+k|t})$ to

$$J(x(t), \chi_{t+k|t}) := \sum_{k=0}^{N-1} \chi_{t+k|t}^T R \chi_{t+k|t}. \tag{38}$$

Then, the AMPC optimization problem is modified as follows:

$$\min_{\chi} J(x(t), \chi_{t+k|t})$$

subject to (18), (27), (32), (36) and (37). Notice that the above optimization problem has only linear constraints and can be solved by the standard quadratic programming (QP) method with free variables $\tilde{u}_{t+k|t}$ and $\chi_{t+k|t}$.

5 Further study on feasibility and stability

In the proposed AMPC optimization problem, we introduced the additional constraint (36) for $w_{t+k|t}$ in order to ensure the feasibility at all time instants. The constant ω is a number satisfying $\omega \geq \max\{V(x_0), 1\}$ and, on the other hand, the terminal constraint $w_{t+N|t} \leq 1$ is adopted to guarantee $x(t+T) \in X_f$ for the real system. Although ω is desired to be as large as possible for the feasibility at the current time instant, it should be bounded to guarantee feasibility at the next time instant, which is described in the following assumption [11, 12, 13].

Assumption 3: *The given* ω ($\geq \max\{V(x_0), 1\}$) *in (36) satisfies*

$$\begin{aligned} &\omega \nu_0 c_{\zeta_1} \leq \sqrt{\underline{\lambda}(P)} \left(\min_i \psi_i - c_{\zeta_2} \right) - 1, \\ &\omega \nu_0 (4\nu_0 c_{\xi_1} + c_{\xi_2} + 2) \leq \\ &\qquad \sqrt{\underline{\lambda}(P)} \left(\eta - 4\nu_0 c_{\xi_3} - c_{\xi_4} \right) - \|K_0\| \end{aligned}$$

where

$$\begin{aligned} &c_{\zeta_1} := \textstyle\sum_{s=0}^{N-1} \|\zeta_1(s)\|_\infty, \quad c_{\zeta_2} := \sum_{s=0}^{N-1} \|\zeta_2(s)\|_\infty, \\ &c_{\xi_1} := \textstyle\sum_{s=0}^{N-1} \|\xi_1(s)\|, \qquad c_{\xi_2} := \sum_{s=0}^{N-1} |K_0 \xi_1(s)|, \\ &c_{\xi_3} := \textstyle\sum_{s=0}^{N-1} \|\xi_2(s)\|, \qquad c_{\xi_4} := \sum_{s=0}^{N-1} \|K_0 \xi_2(s)\|. \end{aligned}$$

Notice that Assumption 3 is a sufficient condition for Assumption 1 [11, 12, 13]. If Assumption 3 cannot be satisfied for any $\omega \geq \max\{V(x_0), 1\}$, we need to consider a smaller terminal set X_f or modify the term K_0 of the feedback gain in (26). It is also important to notice that once the state is steered into the robustly invariant constraint set X_f, the control law (26) is completely switched to the feedback law $\hat{u} = K(\theta)\hat{x}$, since it is the optimal control in X_f in terms of the cost function in (35). That is, the control law of the proposed method converges to the given feedback law $\hat{u} = K(\theta)\hat{x}$.

In the following, we describe the feasibility and stability issues of adaptive-type MPC method. The next lemma is a key result to prove the ultimately bounded stability (see Theorem 2 given later) and means that if the proposed AMPC optimization problem is feasible at time instant t, then it is also feasible at the next time instant $t + 1$.

Lemma 4: Assume that Assumption 3 is satisfied and the AMPC optimization problem is feasible at the current time instant t; i.e., there exists $\tilde{u}$ which minimizing (35) subject to (18), (27), (28), (32) and (36) at time t. Then, the following control sequence

$$\tilde{u}_{t+k|t+1} = \begin{cases} \tilde{u}^*_{t+k|t}, & k = 1, 2, \cdots, N-1, \\ 0, & k = N, \end{cases} \quad (39)$$

where $\tilde{u}^*_{t+k|t}$ denotes the optimal solution determined at time instant t, is one of the feasible solutions of the AMPC optimization problem at the next time instant $t + 1$.

Proof. See Appendix E □

It is important to note that the feasible solution $\tilde{u}$ of the AMPC optimization problem ensures that the constraints on state and control given in (8) are fulfilled, since the condition (32) in Theorem 1 is always satisfied. Then, the following key result is obtained based on the above lemma.

Theorem 2: Suppose that Assumption 3 is satisfied and the AMPC optimization for MPC is feasible at $t = 0$. Then, the proposed adaptive-type MPC strategy has the following properties.

(i) The AMPC optimization problem is feasible at any time instant $t > 0$.

(ii) For any μ satisfying

$$\mu > \frac{b_0}{1 - \beta_0}, \quad \beta_0 := \sqrt{1 - \frac{\underline{\lambda}(Q)}{\bar{\lambda}(P)}}$$

there exists t_c which guarantees

$$\|x(t)\| \leq \frac{\mu}{\sqrt{\underline{\lambda}(P)}}, \quad \forall t \geq t_c. \quad (40)$$

(iii) It is guaranteed that $x(t) \in X$ and $u(t) \in U$ for any $t \geq 0$.

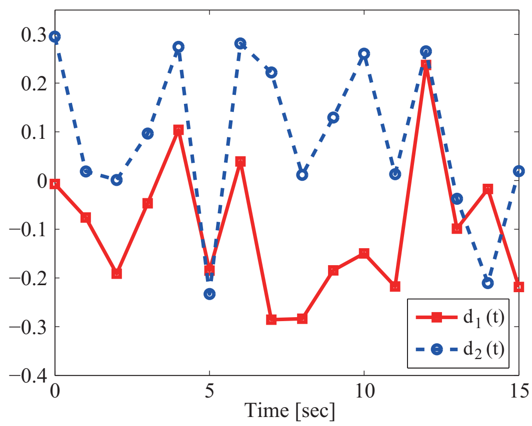

Figure 1: Time trajectory of disturbance $d(t)$

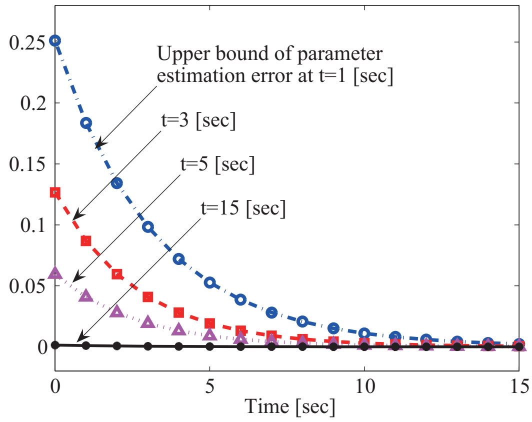

Figure 2: Upper bound $\nu_{t+k|t}$ over the prediction horizon

Proof. See Appendix F □

It shows that the proposed adaptive-type MPC scheme is feasible at all time instants, provided that it is feasible at $t = 0$, and is able to steer the state x into a ball around the origin with radius of $b_0/((1-\beta_0)\sqrt{\underline{\lambda}(P)})$ without violating the constraint (8). In case that the disturbance term $d(t)$ is zero, the state is steered to the origin [12].

6 Simulation example

In this section, we present a numerical example that illustrates the features of the proposed adaptive-type MPC scheme. Consider the following discrete-time linear time-invariant system in controllable canonical form.

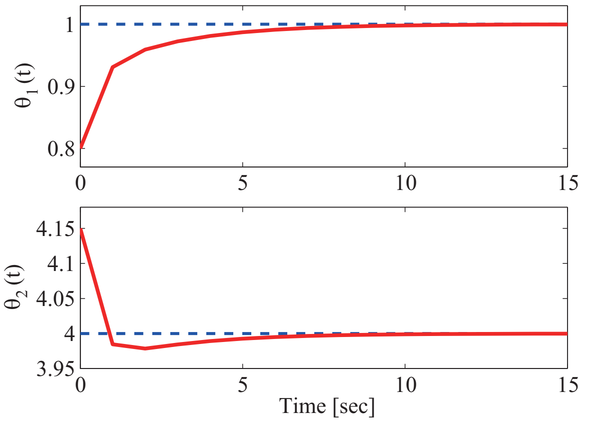

Figure 3: Time trajectory of the estimated parameter $\theta(t)$

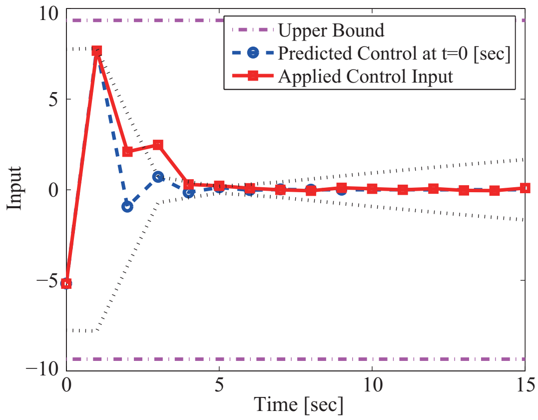

Figure 4: Time trajectory of control input $u(t)$

$$x(t+1) = \begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0.08 & 0 \\ 0 & 0.08 \end{bmatrix} d(t) \quad (41)$$

The initial state has been chosen equal to $x_0 = [-2\ 1.7]^T$. We denote the uncertain parameter vector as $\theta^* = [1\ 4]^T$ and assume that the initial estimate θ_0 and the estimation error bound ν_0 are given as follows:

$$\theta_0 = [0.8\ \ 4.15]^T, \ \ \nu_0 = 0.251, \quad (42)$$

which satisfy (6). The system (41) is affected by randomly generated disturbance $d(t)$ as shown in Fig 1 and the bound on the disturbance $\|d(t)\| \leq 0.37$ is given as a priori information. The constraints on control and state are

$$|u(t)| \leq 9.35, \ \ |x_i(t)| \leq 2, \ \ i = 1, 2, \quad (43)$$

and the terminal constraint is

$$X_f = \{x(t) \in \mathbb{R}^n : \ V(x(t)) \leq 1\}. \quad (44)$$

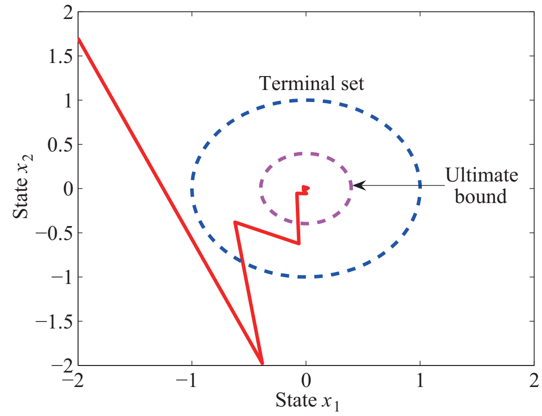

Figure 5: Trajectory of system state $x(t)$

During the simulation, we have used K_0 and P chosen as

$$K_0 = \begin{bmatrix} -0.8445 & -4.0418 \end{bmatrix}, \quad P = \begin{bmatrix} 0.1026 & -0.0051 \\ -0.0051 & 0.2026 \end{bmatrix}. \quad (45)$$

We choose the length of prediction horizon as $N = 20$. For an adaptive parameter estimation algorithm, we choose the estimation horizon as $N_e = 5$ and other design parameters as $\alpha = 0.3$ and $\kappa = 1.2$.

The simulation results illustrated hereafter were performed on a Pentium IV 3.2GHz machine running Matlab 7.0. The average time required to determine a control $u(t)$ was 0.04[sec]. Fig 2 shows that since the decay rate $\bar{\gamma}$ is minimized at each time instant as shown in Section 3, the predicted error bound $\nu_{\cdot|t}$ based on (18) is minimized more rapidly than that of the previous time instant, which has been mentioned in Lemma 2. The convergence of the estimated parameters to their true values by the proposed method in Section 3 is shown in Fig 3. In Fig 4, the dotted line shows the upper bound for control used at time instant $t = 0$[sec], which is calculated based on the modified constraint set (30). Therefore, at time instant $t = 0$, the future control sequence $K_0\hat{x}_{t+k|t} + \hat{u}_{t+k|t}$, which is denoted as the dashed line, is predicted by solving the AMPC optimization problem using the above modified upper bound. Then, $u(0) = K(\theta(0))x(0) + \tilde{u}(0)$ is applied to the plant and the same procedure is repeated at the next time instant as mentioned in Section 4. In this figure, the solid line shows the control trajectory $u(t)$ applied to the real system and it verifies that the control input obtained by the proposed adaptive-type MPC strategy satisfies the given constraints. When this control is used, we obtain the state trajectory as shown in Fig 5. It verifies that the state trajectory goes into a set smaller than the terminal set as mentioned in Section 5.

7 Conclusion

In this paper, we have proposed a discrete-time adaptive-type model predictive control algorithm for a class of linear systems with uncertain parameters subject to both state-dependent disturbances and hard constraints on state and control input. It updates the estimate of system parameters on-line and produces a control input satisfying the given constraints. In order to construct such an adaptive-type MPC scheme, we first introduce a parameter update algorithm based on the moving horizon estimation method. It allows to predict the worst-case estimation error bound over the given prediction horizon. We then have incorporated the estimation algorithm with a robust MPC method based on the modified comparison models which are extended to be applicable to the time-varying case. Furthermore, we have shown that the proposed algorithm guarantees feasibility and stability of the closed-loop system in the presence of input/state constraints, state-dependent disturbances and parameter estimation error. As a future work, the assumption on the system description in canonical form should be relaxed. Also, it is important to extend the results to the output feedback case, though it is a difficult problem even in the existing robust MPC framework.

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APPENDIX

Appendix A: Proof of Lemma 1

We first define $T_{\min}$ as

$$T_{\min} := \arg \min_{0\le t_i \le t} \sqrt{\overline{\lambda}((I_n - \kappa F_2(t_i))^T (I_n - \kappa F_2(t_i)))} \tag{46}$$

Then, $f_1(t)$ in the proposed estimation algorithm implies that

$$f_1(t) = \left(\frac{\displaystyle\sum_{s=T_{min}}^{T_{min}-N_e+1} x(s)x^T(s)}{\alpha + \displaystyle\sum_{s=T_{min}}^{T_{min}-N_e+1} x^T(s)x(s)} \right) \theta^* \tag{47}$$

which also means that

$$f_1(t) = f_2(t)\theta^*. \tag{48}$$

Therefore, we have from (12)

$$\tilde{\theta}(t) = (I_n - \kappa f_2(t))\tilde{\theta}(t-1). \tag{49}$$

Then, for $\varpi(\tilde{\theta}(t)) := \|\tilde{\theta}(t)\|$, we obtain

$$\begin{aligned}\varpi(\tilde{\theta}(t)) &= \sqrt{\tilde{\theta}^T(t-1)(I_n - \kappa f_2(t))^T (I_n - \kappa f_2(t))\tilde{\theta}(t-1)} \\ &\le \sqrt{\overline{\lambda}((I_n - \kappa f_2(t))^T (I_n - \kappa f_2(t)))}\, \|\tilde{\theta}(t-1)\|.\end{aligned} \tag{50}$$

It follows from (16), (46) and (50) that

$$\varpi(\tilde{\theta}(t)) \le \overline{\gamma}(t)\varpi(\tilde{\theta}(t-1)). \tag{51}$$

Thus, (17) is proved by induction.

Appendix B: Proof of Lemma 2

It follows from (18) that

$$\begin{aligned}&\nu_{t+k+1|t+1} - \nu_{t+k+1|t} \\ &\quad = \overline{\gamma}^k(t+1)\nu_{t+1|t+1} - \overline{\gamma}^k(t)\nu_{t+1|t}.\end{aligned} \tag{52}$$

Also, from (18), we have

$$\nu_{t+1|t+1} = \varpi(\tilde{\theta}(t+1)) \le \nu_{t+1|t}. \tag{53}$$

Since $\overline{\gamma}(t+1) \le \overline{\gamma}(t)$ from (16), it follows from (53) and (52) that

$$\nu_{t+k+1|t+1} - \nu_{t+k+1|t} \le 0, \quad k = 0, 1, \cdots, N-1. \tag{54}$$

Therefore, (19) is proved by (53) and (54).

Appendix C: Proof of Lemma 3

For $\overline{x} := Fx(t+k) + B\tilde{u}_{t+k|t} - Be(t+k) + B_d d(t+k)$, $V(\overline{x}) = \sqrt{\overline{x}^T P\, \overline{x}}$ as shown in (9). Then, the following is satisfied.

$$\begin{aligned}V(\overline{x}) &= \sqrt{\overline{x}^T P\, \overline{x}} \\ &= \|P^{\frac{1}{2}}(Fx(t+k) + B\tilde{u}_{t+k|t} - Be(t+k) \\ &\qquad + B_d d(t+k)\| \\ &\le \|P^{\frac{1}{2}}Fx(t+k)\| + \|P^{\frac{1}{2}}Bd(t+k)\| \\ &\qquad + \|P^{\frac{1}{2}}B_d d(t+k)\| + \|P^{\frac{1}{2}}B\tilde{u}_{t+k|t}\|\end{aligned} \tag{55}$$

Then, since $e(t+k) \in E$, $d(t+k) \in D$ and $\sqrt{\underline{\lambda}(P)}\|x\| \le V(x)$, the followings hold.

$$\begin{aligned}\|P^{\frac{1}{2}}Fx(t+k)\| &= \sqrt{x^T(t+k)F^T PFx(t+k)} \\ &= \sqrt{x^T(t+k)(P-Q)x(t+k)} \\ &= \sqrt{V^2(x(t+k)) - x^T(t+k)Qx(t+k)} \\ &\le V(x(t+k))\sqrt{1 - \frac{\underline{\lambda}(Q)}{\overline{\lambda}(P)}},\end{aligned} \tag{56}$$

$$\begin{aligned}\|P^{\frac{1}{2}}Be(t+k)\| &\le \overline{\sigma}(P^{\frac{1}{2}}B)\|e(t+k)\| \\ &\le \overline{\sigma}(P^{\frac{1}{2}}B)\nu_{t+k|t}\|x(t+k)\| \\ &\le \frac{\overline{\sigma}(P^{\frac{1}{2}}B)\nu_{t+k|t}}{\sqrt{\underline{\lambda}(P)}} V(x(t+k))\end{aligned} \tag{57}$$

$$\|P^{\frac{1}{2}}B_d d(t+k)\| \leq \begin{cases} \|P^{\frac{1}{2}}B_d\|\|d(t)\|, & \text{for } k=0, \\ \|P^{\frac{1}{2}}B_d\|, & \text{for } k=1,\cdots,N-1. \end{cases} \quad (58)$$

Thus, it follows from (55), (56) (57) and (58) that

$$\begin{aligned} V(\overline{x}) &\leq \left(\sqrt{1-\frac{\underline{\lambda}(Q)}{\overline{\lambda}(P)}} + \frac{\overline{\sigma}(P^{\frac{1}{2}}B)\nu_{t+k|t}}{\sqrt{\underline{\lambda}(P)}} \right) \\ &\quad \times V(x(t+k)) + \|P^{\frac{1}{2}}B_d\|g_k + \|P^{\frac{1}{2}}B\||\tilde{u}_{t+k|t}| \\ &= a_{t+k|t}V(x(t+k)) + b_0 g_k + b_1|\tilde{u}_{t+k|t}|. \end{aligned} \quad (59)$$

Therefore, (29) is shown from (28) and (59) by an inductive method.

Appendix D: Proof of Theorem 1

From (23) and (27), we have

$$\begin{aligned} x(t+k+1) &= F^{k+1}x(t) - \sum_{s=0}^{k} F^s Be(t+k-s) \\ &\quad + \sum_{s=0}^{k} F^s B_d d(t+k-s) + \sum_{s=0}^{k} F^s B\tilde{u}_{t+k-s|t}, \end{aligned} \quad (60)$$

$$\hat{x}_{t+k+1|t} = F^{k+1}x(t) + F^k B_d d(t) + \sum_{s=0}^{k} F^s B\tilde{u}_{t+k-s|t}. \quad (61)$$

It follows from (24), (28) and Lemma 3 that

$$\begin{aligned} &|x_i(t+k+1) - \hat{x}_{i,t+k+1|t}| \\ &\leq \sum_{s=0}^{k} |\zeta_{1i}(s)e(t+k-s)| \\ &\quad + \sum_{s=0}^{k-1} |\zeta_{2i}(s)d(t+k+1-s)| \\ &\leq \sum_{s=0}^{k} |\zeta_i(s)| \frac{\nu_{t+k-s|t}w_{t+k-s|t}}{\sqrt{\underline{\lambda}(P)}} + \sum_{s=0}^{k-1} \|\zeta_{2i}(s)\|. \end{aligned} \quad (62)$$

Therefore, from (30) and (62),

$$\begin{aligned} &|x_i(t+k+1)| \\ &\leq |\hat{x}_{i,t+k+1|t}| + |x_i(t+k+1) - \hat{x}_{i,t+k+1|t}| \\ &\leq |\hat{x}_{i,t+k+1|t}| + \hat{\psi}_i(t+k+1, \nu_{\cdot|t}, w_{\cdot|t}) \leq \psi_i. \end{aligned} \quad (63)$$

This implies that any $\tilde{u}_{t+k|t}$ which satisfies $\hat{x}_{t+k+1|t} \in \hat{X}(t+k+1, \nu_{\cdot|t}, w_{\cdot|t})$ also satisfies $x(t+k+1) \in X$ in the presence of all possible $e(t+k) \in E$ and $d(t+k) \in D$. Similarly, from (22) and (26), the following holds.

$$|u(t+k) - \hat{u}_{t+k|t}| \leq |K(\hat{\theta}_{t+k|t})(\Delta e(t+k) + \Delta d(t+k))| \quad (64)$$

where

$$\Delta e(t+k) := -\sum_{s=0}^{k} F^{s-1}Be(t+k-s),$$

$$\Delta d(t+k) := \sum_{s=0}^{k} F^{s-2}B_d d(t+k+1-s).$$

It follows from (24), (28) and Lemma 3 that

$$\begin{aligned} &|K(\hat{\theta}_{t+k|t})\Delta e(t+k)| \\ &\leq \sum_{s=0}^{k} |K(\hat{\theta}_{t+k|t})F^{s-1}B|\|e(t+k-s)\| \\ &\leq \sum_{s=0}^{k} |(\hat{\theta}_{t+k|t}^T + \theta_0^T)F^{s-1}B|\|e(t+k-s)\| \\ &\quad + \sum_{s=0}^{k} |K_0 F^{s-1}B|\|e(t+k-s)\| \\ &\leq \sum_{s=0}^{k} (\nu_{t+k|t} + \nu_0)\|F^{s-1}B\|\|e(t+k-s)\| \\ &\quad + \sum_{s=0}^{k} |K_0 F^{s-1}B|\|e(t+k-s)\| \\ &\leq (\nu_{t+k|t} + \nu_0) \sum_{s=0}^{k} \|\xi_1(s)\| \frac{\nu_{t+k-s|t}w_{t+k-s|t}}{\sqrt{\underline{\lambda}(P)}} \\ &\quad + \sum_{s=0}^{k} |K_0 \xi_1(s)| \frac{\nu_{t+k-s|t}w_{t+k-s|t}}{\sqrt{\underline{\lambda}(P)}}. \end{aligned} \quad (65)$$

and

$$\begin{aligned} &|K(\hat{\theta}_{t+k|t})\Delta d(t+k)| \\ &\leq \sum_{s=0}^{k} \|K(\hat{\theta}_{t+k|t})F^{s-2}B_d\|\|d(t+k+1-s)\| \\ &\leq \sum_{s=0}^{k} \left(\|(\hat{\theta}_{t+k|t}^T + \theta_0^T)F^{s-2}B_d\| + \|K_0 F^{s-2}B_d\| \right) \\ &\leq \sum_{s=0}^{k} \left((\nu_{t+k|t} + \nu_0)\|F^{s-2}B_d\| + \|K_0 F^{s-2}B_d\| \right) \\ &\leq (\nu_{t+k|t} + \nu_0) \sum_{s=0}^{k} \|\xi_2(s)\| + \sum_{s=0}^{k} \|K_0 \xi_2(s)\|. \end{aligned} \quad (66)$$

Therefore, from (31), (65) and (66), we have

$$\begin{aligned}|u(t+k)| &\le |\hat{u}_{t+k|t}| + |u(t+k) - \hat{u}_{t+k|t}| \\ &\le |\hat{u}_{t+k|t}| \\ &+ \sum_{s=0}^{k}\left(|K_0\xi_1(s)|\frac{\nu_{t+k-s|t}w_{t+k-s|t}}{\sqrt{\underline{\lambda}(P)}} + \|K_0\xi_2(s)\|\right) \\ &+ (\nu_{t+k|t}+\nu_0)\sum_{s=0}^{k}\|\xi_1(s)\|\frac{\nu_{t+k-s|t}w_{t+k-s|t}}{\sqrt{\underline{\lambda}(P)}} \\ &+ (\nu_{t+k|t}+\nu_0)\sum_{s=0}^{k}\|\xi_2(s)\|\end{aligned} \tag{67}$$

Also, it follows from (26) that

$$|\hat{u}_{t+k|t}| \le |K_0\hat{x}_{t+k|t}+\tilde{u}_{t+k|t}|+|(-\hat{\theta}^T_{t+k|t}+\theta^T_0)\hat{x}_{t+k|t}|. \tag{68}$$

From (61) and Lemmas 1 and 3, the last term of (68) can be written as follows

$$\begin{aligned}&|(-\hat{\theta}^T_{t+k|t}+\theta^T_0)\hat{x}_{t+k|t}| \\ &\le (\nu_{t+k|t}+\nu_0)\sum_{s=0}^{k}\|x(t+k)\| \\ &\quad +(\nu_{t+k|t}+\nu_0)\sum_{s=0}^{k}\|F^{s-1}B\|\|e(t+k-s)\| \\ &\quad +(\nu_{t+k|t}+\nu_0)\sum_{s=0}^{k}\|F^{s-2}B_d\|\|d(t+k+1-s)\| \\ &\le (\nu_{t+k|t}+\nu_0)\frac{w_{t+k|t}}{\sqrt{\underline{\lambda}(P)}} + (\nu_{t+k|t}+\nu_0) \\ &\quad \times\sum_{s=0}^{k}\left(\|\xi_1(s)\|\frac{\nu_{t+k-s|t}w_{t+k-s|t}}{\sqrt{\underline{\lambda}(P)}}+\|\xi_2(s)\|\right).\end{aligned} \tag{69}$$

Therefore, from (67), (68) and (69), it follows that

$$\begin{aligned}|u(t+k)| &\le |K_0\hat{x}_{t+k|t}+\tilde{u}_{t+k|t}| \\ &+2(\nu_{t+k|t}+\nu_0) \\ &\times\sum_{s=0}^{k}\left(\|\xi_1(s)\|\frac{\nu_{t+k-s|t}w_{t+k-s|t}}{\sqrt{\underline{\lambda}(P)}}+\|\xi_2(s)\|\right) \\ &+\sum_{s=0}^{k}\left(|K_0\xi_1(s)|\frac{\nu_{t+k-s|t}w_{t+k-s|t}}{\sqrt{\underline{\lambda}(P)}} + \|K_0\xi_2(s)\|\right) \\ &+(\nu_{t+k|t}+\nu_0)\frac{w_{t+k|t}}{\sqrt{\underline{\lambda}(P)}} \\ &= |K_0\hat{x}_{t+k|t}+\tilde{u}_{t+k|t}| + \hat{\eta}(t+k,\nu_{\cdot|t},w_{\cdot|t}) \le \eta\end{aligned} \tag{70}$$

This implies that any $\tilde{u}_{t+k|t}$ satisfying (32) also satisfies (33) for all possible $d(t+k)\in D$.

Appendix E: Proof of Lemma 4

In order to prove Lemma 4, we use the following two facts, Lemmas 5 and 6.

Lemma 5: For a given trajectory $\tilde{u}^*_{t+k|t}$, $k = 0,1,\cdots,N-1$, and

$$\tilde{u}_{t+k|t+1} = \tilde{u}^*_{t+k|t}, \quad k=1,2,\cdots,N-1, \tag{71}$$

the scalar system (28) satisfies

$$w_{t+k|t+1} \le w_{t+k|t}, \quad k=1,2,\cdots,N-1. \tag{72}$$

Proof: From (28) and Lemma 2, it follows that

$$a_{t+k|t+1} \le a_{t+k|t}. \tag{73}$$

Therefore, we have

$$\begin{aligned}&w_{t+k+1|t+1} - w_{t+k+1|t} \\ &\quad = a_{t+k|t+1}w_{t+k|t+1} - a_{t+k|t}w_{t+k|t} \\ &\quad \le a_{t+k|t}(w_{t+k|t+1} - w_{t+k|t})\end{aligned} \tag{74}$$

where $k = 1,2,\cdots,N-1$. Since $w_{t+1|t+1} = V(x(t+1)) \le w_{t+1|t}$ from (28) and Lemma 3, it follows from (74) that

$$w_{t+k+1|t+1} - w_{t+k+1|t} \le 0$$

where $k=1,2,\cdots,N-1$. Thus (72) is proved. □

Lemma 6: Assume a predicted trajectory $w_{\cdot|t}$ in (28) satisfies

$$w_{t+k|t} \le \omega, \quad k=0,1,\cdots,N \tag{75}$$

under $\tilde{u}_{t+k|t}$ at the current time t. Then, for the control $\tilde{u}_{t+k|t+1}$, $k=1,2,\cdots,N-1$, in (39), we have

$$\underline{\psi} \le \psi_i - \hat{\psi}_i(t+k+1,\nu_{\cdot|t+1},w_{\cdot|t+1}) \tag{76}$$

and

$$\underline{\eta} \le \eta_i - \hat{\eta}(t+k,\nu_{\cdot|t+1},w_{\cdot|t+1}) \tag{77}$$

at the next time instant $t+1$ where

$$\begin{aligned}\underline{\psi} &:= \max_i \psi_i - \frac{\omega\nu(t)c_{\zeta_1}}{\sqrt{\underline{\lambda}(P)}} - c_{\zeta_2} \\ \underline{\eta} &:= \eta - \frac{\omega}{\sqrt{\underline{\lambda}(P)}} \\ &\quad \times\big((\nu(t)+\nu_0)(2\nu(t)c_{\xi_1}+1)+\nu(t)c_{\xi_2}\big) \\ &\quad -2(\nu(t)+\nu_0)c_{\xi_3} - c_{\xi_4}\end{aligned} \tag{78}$$

Proof: From (18), (75) and the definition of $\hat{\psi}_i$ in (31),

$$\min_i \psi_i+\hat{\psi}_i(t+k+1,\nu_{\cdot|t},w_{\cdot|t}) \le \psi_i+\frac{\omega\nu(t)c_{\zeta_1}}{\sqrt{\underline{\lambda}(P)}}+c_{\zeta_2} \tag{79}$$

which implies

$$\underline{\psi} \le \psi_i - \hat{\psi}_i(t+k+1,\nu_{\cdot|t},w_{\cdot|t}). \tag{80}$$

Also, from (31) and Lemmas 2 and 5, it follows that

$$\hat{\psi}_i(t+k+1,\nu_{\cdot|t+1},w_{\cdot|t+1}) \le \hat{\psi}_i(t+k+1,\nu_{\cdot|t},w_{\cdot|t}). \tag{81}$$

Thus (76) is proved by (80) and (81).

Similarly, from the definition of $\hat{\eta}$ in (31),

$$\eta+\hat{\eta}(t+k,\nu_{\cdot|t},w_{\cdot|t}) \le \eta+\frac{\nu(t)\omega}{\sqrt{\underline{\lambda}(P)}}c_{\xi_2}+c_{\xi_4} + (\nu(t)+\nu_0)\left(\frac{2\nu(t)\omega}{\sqrt{\underline{\lambda}(P)}}c_{\xi_1}+2c_{\xi_3}+\frac{\omega}{\sqrt{\underline{\lambda}(P)}}\right) \tag{82}$$

which implies

$$\underline{\eta} \le \eta - \hat{\eta}(t+k,\nu_{\cdot|t},w_{\cdot|t}). \tag{83}$$

Also, from (31), Lemmas 2 and 5, it follows that

$$\hat{\eta}(t+k,\nu_{\cdot|t+1},w_{\cdot|t+1}) \le \hat{\eta}(t+k,\nu_{\cdot|t},w_{\cdot|t}). \tag{84}$$

Thus (77) is proved by (83) and (84). □

Based on the results above, we first prove the feasibility of $\tilde{u}_{t+k|t+1}$ for $k=1,2,\cdots,N-1$ in (39). From (39) and (61), $\hat{x}_{t+k+1|t+1}$ is described as

$$\begin{aligned}
&\hat{x}_{t+k+1|t+1}\\
&= F^k x(t+1) + \sum_{s=0}^{k-1} F^s B\tilde{u}_{t+k-s|t+1} + F^{k-1}B_d d(t+1)\\
&= F^k x(t+1) + \sum_{s=0}^{k-1} F^s B\tilde{u}^*_{t+k-s|t} + F^{k-1}B_d d(t+1)\\
&= F^k \hat{x}_{t+1|t} + \sum_{s=0}^{k-1} F^s B\tilde{u}^*_{t+k-s|t} + F^k(x(t+1)-\hat{x}_{t+1|t}) + F^{k-1}B_d d(t+1)\\
&= \hat{x}_{t+k+1|t} + F^k(x(t+1)-\hat{x}_{t+1|t}) + F^{k-1}B_d d(t+1).
\end{aligned} \tag{85}$$

Thus, from (60) and (61), it follows that

$$\hat{x}_{t+k+1|t+1} = \hat{x}_{t+k+1|t} - F^k Be(t) + F^{k-1}B_d d(t+1). \tag{86}$$

Therefore, from $e(t)\in E$, $d(t)\in D$ and Lemma 3,

$$\begin{aligned}
&|\hat{x}_{i,t+k+1|t+1}| + \hat{\psi}_i(t+k+1,\nu_{\cdot|t+1},w_{\cdot|t+1})\\
&\le |\hat{x}_{i,t+k+1|t}| + |\zeta_{1i}(k)|\frac{\nu(t)w(t)}{\sqrt{\underline{\lambda}(P)}} + |\zeta_{2i}(k)d(t+1)| + \hat{\psi}_i(t+k+1,\nu_{\cdot|t+1},w_{\cdot|t+1})\\
&\le |\hat{x}_{i,t+k+1|t}| + |\zeta_{1i}(k)|\frac{\nu(t)w(t)}{\sqrt{\underline{\lambda}(P)}} + \|\zeta_{2i}(k)\| + \hat{\psi}_i(t+k+1,\nu_{\cdot|t+1},w_{\cdot|t+1}).
\end{aligned} \tag{87}$$

From the definition of $\hat{\psi}_i$ in (31) and Lemmas 2 and 5,

$$\begin{aligned}
&\hat{\psi}_i(t+k+1,\nu_{\cdot|t+1},w_{\cdot|t+1})\\
&= \sum_{s=0}^{k-1}\left(|\zeta_{1i}(s)|\frac{\nu_{t+k-s|t+1}w_{t+k-s|t+1}}{\sqrt{\underline{\lambda}(P)}} + \|\zeta_{2i}(s)\|\right)\\
&\le \sum_{s=0}^{k-1}\left(|\zeta_{1i}(s)|\frac{\nu_{t+k-s|t}w_{t+k-s|t}}{\sqrt{\underline{\lambda}(P)}} + \|\zeta_{2i}(s)\|\right)\\
&= \hat{\psi}_i(t+k+1,\nu_{\cdot|t},w_{\cdot|t}) - |\zeta_i(k)|\frac{\nu(t)w(t)}{\sqrt{\underline{\lambda}(P)}} - \|\zeta_{2i}(k)\|.
\end{aligned} \tag{88}$$

Therefore, from (87) and (88),

$$\begin{aligned}
&|\hat{x}_{i,t+k+1|t+1}| + \hat{\psi}_i(t+k+1,\nu_{\cdot|t+1},w_{\cdot|t+1})\\
&\quad \le |\hat{x}_{i,t+k+1|t}| + \hat{\psi}_i(t+k+1,\nu_{\cdot|t},w_{\cdot|t}).
\end{aligned} \tag{89}$$

This implies that, if the solution $\tilde{u}^*_{t+k|t}$ satisfies

$$\hat{x}_{t+k+1|t} \in \hat{X}(t+k+1,\nu_{\cdot|t},w_{\cdot|t}) \tag{90}$$

at the current time t, then $\tilde{u}_{t+k|t+1}$ in (39) satisfies

$$\hat{x}_{t+k+1|t+1} \in \hat{X}(t+k+1,\nu_{\cdot|t+1},w_{\cdot|t+1}) \tag{91}$$

where $k=1,2,\cdots,N-1$ at the next time step $t+1$.

Likewise, from (39) and (86), it follows that

$$\begin{aligned}
&\tilde{u}_{t+k|t+1} + K_0\hat{x}_{t+k|t+1}\\
&\quad = \tilde{u}^*_{t+k|t} + K_0\hat{x}_{t+k|t} - K_0F^{k-1}Be(t)\\
&\qquad + K_0F^{k-2}B_d d(t+1)
\end{aligned} \tag{92}$$

Thus, from (30) and (92),

$$\begin{aligned}&|\tilde{u}_{t+k|t+1}+K_0\hat{x}_{t+k|t+1}|+\hat{\eta}(t+k,\nu_{\cdot|t+1},w_{\cdot|t+1})\\&\le|\tilde{u}^*_{t+k|t}+K_0\hat{x}_{t+k|t}|+|K_0\xi_1(k)|\frac{\nu(t)w(t)}{\sqrt{\underline{\lambda}(P)}}\\&\quad+|K_0\xi_2(k)d(t+1)|+\hat{\eta}(t+k,\nu_{\cdot|t+1},w_{\cdot|t+1})\\&\le|\tilde{u}^*_{t+k|t}+K_0\hat{x}_{t+k|t}|+|K_0\xi_1(k)|\frac{\nu(t)w(t)}{\sqrt{\underline{\lambda}(P)}}\\&\quad+\|K_0\xi_2(k)\|+\hat{\eta}(t+k,\nu_{\cdot|t+1},w_{\cdot|t+1}).\end{aligned}\tag{93}$$

From the definition of $\hat{\eta}$ in (31), Lemmas 2 and 5,

$$\begin{aligned}&\hat{\eta}(t+k,\nu_{\cdot|t+1},w_{\cdot|t+1})\\&\le 2(\nu_{t+k|t}+\nu_0)\\&\quad\times\sum_{s=0}^{k-1}\left(\|\xi_1(s)\|\frac{\nu_{t+k-s|t}w_{t+k-s|t}}{\sqrt{\underline{\lambda}(P)}}+\|\xi_2(s)\|\right)\\&\quad+\sum_{s=0}^{k-1}\left(|K_0\xi_1(s)|\frac{\nu_{t+k-s|t}w_{t+k-s|t}}{\sqrt{\underline{\lambda}(P)}}+\|K_0\xi_2(s)\|\right)\\&\quad+(\nu_{t+k|t}+\nu_0)\frac{w_{t+k|t}}{\sqrt{\underline{\lambda}(P)}}\\&=\hat{\eta}(t+k,\nu_{\cdot|t},w_{\cdot|t})\\&\quad-2(\nu_{t+k|t}+\nu_0)\left(\|\xi_1(k)\|\frac{\nu(t)w(t)}{\sqrt{\underline{\lambda}(P)}}+\|\xi_2(k)\|\right)\\&\quad-\left(|K_0\xi_1(k)|\frac{\nu(t)w(t)}{\sqrt{\underline{\lambda}(P)}}+\|K_0\xi_2(k)\|\right).\end{aligned}\tag{94}$$

Therefore, from (93) and (94),

$$\begin{aligned}&|\tilde{u}_{t+k|t+1}+K_0\hat{x}_{t+k|t+1}|+\hat{\eta}(t+k,\nu_{\cdot|t+1},w_{\cdot|t+1})\\&\le|\tilde{u}^*_{t+k|t}+K_0\hat{x}_{t+k|t}|+\hat{\eta}(t+k,\nu_{\cdot|t},w_{\cdot|t})\\&\quad-2(\nu_{t+k|t}+\nu_0)\left(\|\xi_1(k)\|\frac{\nu(t)w(t)}{\sqrt{\underline{\lambda}(P)}}+\|\xi_2(k)\|\right)\\&\le|\tilde{u}^*_{t+k|t}+K_0\hat{x}_{t+k|t}|+\hat{\eta}(t+k,\nu_{\cdot|t},w_{\cdot|t}).\end{aligned}\tag{95}$$

Thus, if $\tilde{u}^*_{t+k|t}$, $k=0,1,\cdots,N-1$, satisfies

$$\tilde{u}^*_{t+k|t}+K_0\hat{x}_{t+k|t}\in\hat{U}(t+k,\nu_{\cdot|t},w_{\cdot|t}),\tag{96}$$

then $\tilde{u}_{t+k|t+1}$, $k=1,2,\cdots,N-1$, satisfies that

$$\tilde{u}_{t+k|t+1}+K_0\hat{x}_{t+k|t+1}\in\hat{U}(t+k,\nu_{\cdot|t+1},w_{\cdot|t+1}).\tag{97}$$

Moreover, it is clear from Lemma 5 that if the constraint $w_{t+k|t}\le\omega$, $k=0,1,\cdots,N-1$, is satisfied, then

$$w_{t+k|t+1}\le\omega,\ \ k=1,2,\cdots,N-1,\tag{98}$$

which conclude the proof for the feasibility at $k=1,2,\cdots,N-1$. Notice that, similarly to (98), it follows from $w_{t+N|t}\le 1$ and Lemma 5 that

$$w_{t+N|t+1}\le 1.\tag{99}$$

Next, we prove

$$w_{t+N+1|t+1}\le 1\tag{100}$$

as follows: From (28) and Assumption 2, $w_{t+N+1|t+1}$ is decreasing for $\tilde{u}_{t+N|t+1}=0$. Therefore, (100) is obviously satisfied from (99). Note that for the same ω satisfying Assumption 3, the following conditions are also satisfied.

$$\begin{aligned}&\omega\nu(t)c_{\zeta_1}\le\sqrt{\underline{\lambda}(P)}\left(\min_i\psi_i-c_{\zeta_2}\right)-1,\\&\omega\big((\nu(t)+\nu_0)(2\nu(t)c_{\xi_1}+1)+\nu(t)c_{\xi_2}\big)\\&\quad\le\sqrt{\underline{\lambda}(P)}\,(\eta-2(\nu(t)+\nu_0)c_{\xi_3}-c_{\xi_4})-\|K_0\|,\end{aligned}\tag{101}$$

which are the sufficient conditions for Assumption 1 [11, 12]. It can be easily verified using the result of Section 3; i.e., $\nu(t)\le\nu_0$ is guaranteed for $t\ge 0$. Since the above condition (101) can be rewritten as $1\le\underline{\psi}\sqrt{\underline{\lambda}(P)}$ and $\|K_0\|\le\underline{\eta}\sqrt{\underline{\lambda}(P)}$ by using $\underline{\psi}$ and $\underline{\eta}$ in Lemma 6, we have

$$\begin{aligned}&|x_i|\le\|x\|\le\frac{1}{\sqrt{\underline{\lambda}(P)}}\le\underline{\psi},\\&|K_0x|\le\frac{\|K_0\|}{\sqrt{\underline{\lambda}(P)}}\le\underline{\eta},\ \ \forall x\in X_f.\end{aligned}\tag{102}$$

Since it is clear from (100) and Lemma 3 that $\hat{x}_{t+N+1|t+1}\in X_f$, we have

$$\begin{aligned}&\hat{x}_{t+N+1|t+1}\in\hat{X}(t+k+1,\nu_{\cdot|t+1},w_{\cdot|t+1}),\\&K_0\hat{x}_{t+N+1|t+1}\in\hat{U}(t+k,\nu_{\cdot|t},w_{\cdot|t})\end{aligned}\tag{103}$$

from (102) and $\tilde{u}_{t+N|t+1}=0$. Therefore, (100) and (103) prove the feasibility for $k=N$, which concludes the proof.

Appendix F: Proof of Theorem 2

We show (i) by induction. The AMPC optimization problem is feasible at $t=0$ by the assumption. Assume now it is feasible at each $t=i$ $(i=1,\cdots,k)$. Then, from Lemma 4, the AMPC optimization problem is feasible at $t=i+1$. Therefore (i) is proved. In order to prove (ii), we next show that the optimal cost $J(x(t),\tilde{u}^*)$ is nonincreasing. At the time step $t+1$, the feasible solution in Lemma 4 satisfies

$$J(x(t+1),\tilde{u}_{t+k|t+1})\le J(x(t),\tilde{u}^*_{t+k|t})\tag{104}$$

since

$$\begin{aligned} &J(x(t+1), \tilde{u}_{t+k|t+1}) - J(x(t), \tilde{u}^*_{t+k|t}) \\ &= \sum_{k=1}^{N} \tilde{u}^T_{t+k|t+1} R \tilde{u}_{t+k|t+1} - \sum_{k=0}^{N-1} \tilde{u}^{*T}_{t+k|t} R \tilde{u}^*_{t+k|t} \\ &= -\tilde{u}^{*T}_{t|t} R \tilde{u}^*_{t|t} \leq 0. \end{aligned} \quad (105)$$

It also holds that

$$J(x(t+1), \tilde{u}^*_{t+k|t+1}) \leq J(x(t), \tilde{u}_{t+k|t+1}) \quad (106)$$

from the optimality of $J(x(t+1), \tilde{u}^*_{t+k|t+1})$. From (104)and (106), the optimal cost is nonincreasing; i.e.,

$$J(x(t+1), \tilde{u}^*_{t+k|t+1}) \leq J(x(t), \tilde{u}^*_{t+k|t}). \quad (107)$$

Since the optimal cost is nonincreasing and bounded by 0 from below, it satisfies $J(x(t+1), \tilde{u}^*_{t+k|t+1}) \to c$ as $t \to \infty$ for a constant $c \geq 0$. This implies that, for each $\epsilon > 0$, there exists $t_\epsilon \geq 0$ such that

$$0 \leq J(x(t), \tilde{u}^*_{t+k|t}) - J(x(t+1), \tilde{u}^*_{t+k|t+1}) < \epsilon \quad (108)$$

for $\forall t \geq t_\epsilon$. But, from (104), (105) and (106), we have

$$\begin{aligned} \tilde{u}^{*T}_{t|t} R \tilde{u}^*_{t|t} &= J(x(t), \tilde{u}^*_{t+k|t}) - J(x(t+1), \tilde{u}_{t+k|t+1}) \\ &\leq J(x(t), \tilde{u}^*_{t+k|t}) - J(x(t+1), \tilde{u}^*_{t+k|t+1}). \end{aligned} \quad (109)$$

Thus, (108) and (109) imply

$$\tilde{u}^*_{t|t} \to 0 \text{ as } t \to \infty. \quad (110)$$

On the other hand, $\nu_{t|t} \to 0$ as $t \to \infty$ from (18). Hence, given ϵ_u and ϵ_ν satisfying

$$\frac{b_0 + b_1 \epsilon_c}{1 - \beta_0 - \beta_1 \epsilon_\nu} < \mu \quad (111)$$

for $\mu > b_0/(1-\beta_0)$, we can choose t_ϵ such that

$$|\tilde{u}^*_{t|t}| \leq \epsilon_u, \ \ \nu_{t|t} \leq \epsilon_\nu, \ \ \forall t \geq t_\epsilon. \quad (112)$$

From (59), (112) and Lemma 2, it follows that

$$\begin{aligned} V(\bar{x}) &\leq a_{t|t} V(x(t)) + b_0 + b_1 |\tilde{u}^*_{t|t}| \\ &\leq (\beta_0 + \beta_1 \epsilon_\nu) V(x(t)) + b_0 + b_1 \epsilon_u \\ &= \hat{a} V(x(t)) + \hat{b} \end{aligned} \quad (113)$$

for $\forall t \geq t_\epsilon$ where $\hat{a} := \beta_0 + \beta_1 \epsilon_\nu$ and $\hat{b} := b_0 + b_1 \epsilon_c$. Thus, it is satisfied that

$$\begin{aligned} V(x(t_\epsilon + k)) &\leq \hat{a}^k V(x(t_\epsilon)) + \sum_{s=0}^{k-1} \hat{a}^s \hat{b} \\ &= \hat{a}^k V(x(t_\epsilon)) + \frac{\hat{b}(1-\hat{a}^k)}{1-\hat{a}} \end{aligned} \quad (114)$$

and the right-hand side of (114) converges to $\hat{b}/(1-\hat{a})$ as $k \to \infty$. Therefore, from (111) and (114), there exists a finite time t_c ($\geq t_\epsilon$) satisfying

$$V(x(t)) \leq \mu, \ \ \forall t \geq t_c \quad (115)$$

Thus, (40) follows from (115). Finally, we prove (iii). It is easily verified that the feasible solution $\tilde{u}_{t+k|t+1}$ of the AMPC optimization problem in Lemma 4 ensures that the constraints on state and control given in (8) are fulfilled, since the condition (32) in Theorem 1 is always satisfied.