Sampled-Data Models for Nonlinear Systems via the Taylor-Lie Series and Multirate Input

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Abstract: It is well-known that the existence of unstable zero dynamics is recognized as a major barrier in many control problems. This paper investigates an approximate sampled-data model for nonlinear systems by the use of multirate input and hold such as a generalized sample hold function (GSHF), and further analyzes the sampling zero dynamics of the sampled-data model to show a condition which assures the stability of the sampling zero dynamics of the model. The results presented here show how sampling zero dynamics of the obtained model can be arbitrarily placed. In a word, the stability of the sampling zero dynamics is definitely improved compared with a zero-order hold (ZOH) or a fractional-order hold (FROH).

Key–Words: Nonlinear systems, Sampled-data models, Zero dynamics, Generalized sample hold function

1 Introduction

Since recent control systems usually employ digital technology for controller implementation, the research of sampled-data systems has become an important issue in control fields. When dealing with sampled-data models for nonlinear systems, the exact sampled-data model is often unavailable to the controller designers. Thus the accuracy of the approximate sampled-data model has proven to be a key issue in the context of control design, where a controller designed to stabilize an approximate model may fail to stabilize the exact discrete-time model [1].

However, the theory for the sampled-data nonlinear systems is less well developed than for linear case and the absence of the good models for sampled-data nonlinear plants is still recognized as an important issue for control design. Indeed, for linear systems we can write down an exact sampled-data model while typically for nonlinear systems we can not. Moreover, the exact discrete-time model of a linear system is linear while the exact sampled-data model for a nonlinear systems does not usually preserve important structures of the nonlinear systems [2].

Recently, Yuz and Goodwin [3] have proposed a more accurate approximate model than the simple Euler model. The resulting model includes extra zero dynamics which correspond to the case of relative degree one. Such extra zero dynamics are called sampling zero dynamics. It has been shown explicitly that they have no counterpart in the underlying continuous-time system and are the same as those for linear case [4], although an implicit characterization has been given in [5].

Further, Ishitobi et al. [6, 7] have pointed out that the Yuz and Goodwin model can not be used for discrete-time controller design, and it is necessary to derive a more accurate sampled-data model than the Yuz and Goodwin model in the case of a zero-order hold (ZOH) [8]. The reason is that we need a more accurate model in order to decide whether a controller design method based on the assumption of the stability of the zero dynamics can be applied. Therefore, it depends on stability of the sampling zero dynamics of the sampled-data model.

Ishitobi and Nishi [9, 10] have also showed that the stability of zero dynamics can be improved by using fractional order hold (FROH) stead of ZOH. It can be seen that the sampling zero dynamics always lie strictly outside the unit disc when the relative degree of a continuous-time nonlinear system is greater than or equal to three [6–10].

For linear systems, the properties of the sampling zeros corresponding to the sampling zero dynamics for nonlinear systems have been discussed in many papers [4, 11–24]. Some of the previous studies show that use of a FROH instead of a ZOH overcomes the problem of the instability of the sampling zeros when the relative degree of a continuous-time plant is two [12–14, 21, 23]. However, unstable sampling zeros may be generated by ZOH or FROH even though the continuous-time system is of minimum phase. To avoid this unstable sampling zeros problem, further ideas have been introduced such as multirate sampling control and digital control with the generalized sampled-data hold function (GSHF) [15–17, 22]. Furthermore, it has shown that the sampling zeros can be placed inside the unit circle by the parameters of GSHF. Hence, it is natural to raise the question of how the results of the linear case with GSHF can be extended to nonlinear systems.

In this paper, we present an approximate sampled-data model for nonlinear system, which is related with some order of sampling period. In particular, we also show how a particular strategy can be used to approximate the system output and its derivatives in order to obtain a local truncation error between the output of this model and the true system. Moreover, the authors analyze zero dynamics for nonlinear systems with GSHF to show a condition which ensures the stability of the sampling zero dynamics of the obtained model. An insightful interpretation of the obtained sampled-data model can be made in terms of sampling zero dynamics, and their characterizations are explicitly explored. Finally, two instructive examples are shown to illustrate the validity of the nonlinear approximations.

2 Sampled-data model with GSHF

Consider a class of the following singer-input single-output nth-order nonlinear system with the uniform relative degree $r(\leq n)$, which is expressed in its so-called normal form [25, 26]

$$\begin{cases} \dot{\boldsymbol{\zeta}} = \begin{bmatrix} \mathbf{0_{r-1}} & \mathbf{I_{r-1}} \\ 0 & \mathbf{0^T_{r-1}} \end{bmatrix} \boldsymbol{\zeta} \\ \qquad + \begin{bmatrix} \mathbf{0_{r-1}} \\ 1 \end{bmatrix} (b(\boldsymbol{\zeta},\boldsymbol{\eta}) + a(\boldsymbol{\zeta},\boldsymbol{\eta})u) \\ \dot{\boldsymbol{\eta}} = c(\boldsymbol{\zeta},\boldsymbol{\eta}) \\ y = z_1 \end{cases} \tag{1}$$

$$\boldsymbol{\zeta} = \begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix}, \quad \boldsymbol{\eta} = \begin{bmatrix} z_{r+1} \\ \vdots \\ z_n \end{bmatrix} \tag{2}$$

$$\mathbf{z} = \begin{bmatrix} \boldsymbol{\zeta} \\ \boldsymbol{\eta} \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} c_{r+1}(\boldsymbol{\zeta},\boldsymbol{\eta}) \\ \vdots \\ c_n(\boldsymbol{\zeta},\boldsymbol{\eta}) \end{bmatrix} \tag{3}$$

where $z = (\boldsymbol{\zeta}^T, \boldsymbol{\eta}^T)$ is the state evolving in an open subset $M \subset R^n$, and $a(\boldsymbol{\zeta},\boldsymbol{\eta}) \neq 0$, $b(\mathbf{0},\mathbf{0}) = 0$ and $c(\mathbf{0},\mathbf{0}) = 0$.

First, the following **Assumptions** are introduced.

Assumption 1: The unique equilibrium point lies on the origin.

Assumption 2: The scalar functions $a(\zeta,\eta)$ and $b(\zeta,\eta)$, and the output function $c(\zeta,\eta)$ are analytic on M.

Under **Assumptions 1** and **2**, the zero dynamics of (1) are determined by

$$\dot{\boldsymbol{\eta}} = c(\mathbf{0},\boldsymbol{\eta}) \tag{4}$$

We are interested in the sampled-data model for the nonlinear system (1) with GSHF. However, it is difficult to make a GSHF in practice because it is generally composed of exponential and sinusoidal functions. Thus, we consider a piecewise constant GSHF (PC GSHF) defined by piecewise constant impulse responses [15–17, 22]

$$h(t) = \begin{cases} \alpha_1, & t \in [0, \frac{T}{N}), \\ \alpha_2, & t \in [\frac{T}{N}, \frac{2T}{N}), \\ \cdots & \cdots \\ \alpha_N, & t \in [\frac{(N-1)T}{N}, T). \end{cases} \tag{5}$$

Clearly, PC GSHF keep a regular partition in time of sampling interval $[0,T)$ as in the case of the ZOH (see Fig. 1). When multiplicity output of PC GSHF showed in Fig. 2 is considered, each sampling period T is equally divided into N subperiods of length $D = \frac{T}{N}$ and the control input over the subinterval $[kT, D]$ is described by

$$u(kT+D) = u_j(kT), \quad \frac{(j-1)T}{N} \leq D < \frac{jT}{N} \tag{6}$$

From (5) and (6), it can be rewritten as

$$u_j(kT) = \alpha_j u(kT), \quad j = 1, \ldots, N \tag{7}$$

where α_j is a real constant.

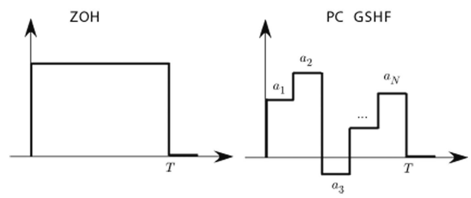

Fig. 1 Pulse response of a ZOH and a PC GSHF

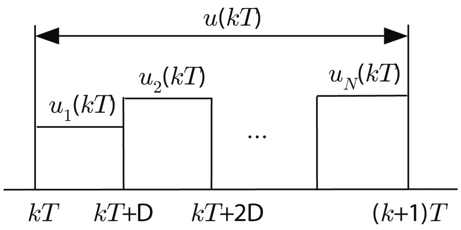

Fig. 2 Multiplicity output of a PC GSHF

The sampled-data model for (1) is derived below with PC GSHF.

First, the following **Assumption** is needed here.

Assumption 3:

$$\frac{\partial a(\boldsymbol{\zeta},\boldsymbol{\eta})}{\partial z_r}=0 \tag{8}$$

The **Assumption 3** ensures that a new sampled-data system is also an affine one.

It can be seen from Fig. 1 that PC GSHF keep a regular partition in time of sampling interval $[0,T)$ as in the case of the ZOH. Therefore, we present an approximate sampled-data model for nonlinear system using PC GSHF by means of multiple step approach. Namely, applying the Taylor's expansion formula in the sampling subperiods. When the input is a $u_{1,k}(kT\le u_k<(k+\frac{1}{N})T)$, one can obtain for sufficiently small sampling periods that

$$\begin{cases} y_{k+\frac{1}{N}}\approx y_k+\frac{T}{N}\dot y_k+\cdots+\frac{1}{r!}(\frac{T}{N})^r y_k^{(r)} \\ \qquad\qquad +\frac{1}{(r+1)!}(\frac{T}{N})^{r+1}y_k^{(r+1)} \\ \qquad\qquad \vdots \\ y_{k+\frac{1}{N}}^{(r-1)}\approx y_k^{(r-1)}+\frac{T}{N}y_k^{(r)}+\frac{1}{2!}(\frac{T}{N})^2 y_k^{(r+1)} \end{cases} \tag{9}$$

where

$$y_k^{(r)}=b_k+a_k\alpha_1 u_k,\quad y_k^{(r+1)}=\overline{b}_k+\overline{a}_k\alpha_1 u_k \tag{10}$$

$$b_k\equiv b(\boldsymbol{\zeta}_k,\boldsymbol{\eta}_k),\quad a_k\equiv a(\boldsymbol{\zeta}_k,\boldsymbol{\eta}_k) \tag{11}$$

$$\overline{b}_k=\sum_{i=1}^{r-1}\frac{\partial b_k}{\partial z_i}z_{i+1}+\frac{\partial b_k}{\partial z_r}b_k+\sum_{i=r+1}^{n}\frac{\partial b_k}{\partial z_i}c_i \tag{12}$$

$$\overline{a}_k=\frac{\partial b_k}{\partial z_r}a_k+\sum_{i=1}^{r-1}\frac{\partial a_k}{\partial z_i}z_{i+1}+\sum_{i=r+1}^{n}\frac{\partial a_k}{\partial z_i}c_i \tag{13}$$

Denote

$$\mathbf{Y}_{k+\frac{1}{N}}=[y_{k+\frac{1}{N}}\quad \dot y_{k+\frac{1}{N}}\quad \cdots\quad y_{k+\frac{1}{N}}^{(r-1)}]^T \tag{14}$$

Then

$$\begin{aligned}\mathbf{Y}_{k+\frac{1}{N}} \;=&\; A_{k,\frac{1}{N}}\mathbf{Y}_k+B_{k,\frac{1}{N}}(b_k+a_k\alpha_1 u_k)\\ &+\overline{B}_{k,\frac{1}{N}}(\overline{b}_k+\overline{a}_k\alpha_1 u_k)\end{aligned} \tag{15}$$

where

$$A_{k,\frac{1}{N}}=\begin{bmatrix} 1 & \frac{T}{N} & \cdots & \frac{1}{(r-1)!}(\frac{T}{N})^{r-1} \\ 0 & 1 & \cdots & \frac{1}{(r-2)!}(\frac{T}{N})^{r-2} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \frac{T}{N} \\ 0 & \cdots & 0 & 1 \end{bmatrix} \tag{16}$$

$$B_{k,\frac{1}{N}}=\begin{bmatrix} \frac{1}{r!}(\frac{T}{N})^r & \frac{1}{(r-1)!}(\frac{T}{N})^{r-1} & \cdots & \frac{T}{N}\end{bmatrix}^T \tag{17}$$

$$\overline{B}_{k,\frac{1}{N}}=\begin{bmatrix} \frac{1}{(r+1)!}(\frac{T}{N})^{r+1} & \frac{1}{r!}(\frac{T}{N})^{r} & \cdots & \frac{1}{2!}(\frac{T}{N})^2\end{bmatrix}^T \tag{18}$$

Similarly, we give approximate asymptotic expressions of the output $\mathbf{Y}_{k+\frac{2}{N}},\cdots,\mathbf{Y}_{k+1}$ as power series expansions with respect to a sufficiently small sampling period when the input is the $u_{2,k},\cdots,u_{N,k}$, respectively. By deleting $\mathbf{Y}_{k+\frac{1}{N}},\cdots,\mathbf{Y}_{k+\frac{N-1}{N}}$, and lead the approximate expressions of $\mathbf{Y}_{k+1}$:

$$\begin{aligned}\mathbf{Y}_{k+1}=&A_k^N\mathbf{Y}_k+(A_k^{N-1}+\cdots+A_k+I_r)(B_k b_k\\ &+\overline{B}_k\overline{b}_k)+(\alpha_1 A_k^{N-1}+\cdots+\alpha_{N-1}A_k\\ &+\alpha_N I_r)(B_k a_k+\overline{B}_k\overline{a}_k)u_k\end{aligned} \tag{19}$$

where

$$A_k=A_{k,1}=\cdots=A_{k,\frac{1}{N}} \tag{20}$$

$$B_k=B_{k,1}=\cdots=B_{k,\frac{1}{N}},\overline{B}_k=\overline{B}_{k,1}=\cdots=\overline{B}_{k,\frac{1}{N}} \tag{21}$$

Next, one can also obtain the asymptotic expressions of η_{k+1} for sufficiently small sampling periods:

$$\boldsymbol{\eta}_{k+1}=\boldsymbol{\eta}_k+T\dot{\boldsymbol{\eta}}_k=\boldsymbol{\eta}_k+Tc(\boldsymbol{\zeta}_k,\boldsymbol{\eta}_k) \tag{22}$$

Hence, a sampled-data model for (1) is obtained with PC GSHF as follows:

$$\begin{cases}\boldsymbol{\zeta}_{k+1}=\Phi_r\boldsymbol{\zeta}_k+eb_k+\overline{e}\overline{b}_k+(\mathbf{d}a_k+\overline{\mathbf{d}}\overline{a}_k)u_k\\ \boldsymbol{\eta}_{k+1}=\boldsymbol{\eta}_k+Tc(\boldsymbol{\zeta}_k,\boldsymbol{\eta}_k)\\ y_k=[1\quad 0_{r-1}^k]\boldsymbol{\zeta}_k\end{cases} \tag{23}$$

where

$$\Phi_r=A_k^N \tag{24}$$

$$(A_k^{N-1}+\cdots+A_k+I_r)B_k=e \tag{25}$$

$$(A_k^{N-1}+\cdots+A_k+I_r)\overline{B}_k=\overline{e} \tag{26}$$

$$(\alpha_1 A_k^{N-1} + \cdots + \alpha_{N-1} A_k + \alpha_N I_r) B_k = \mathbf{d} \quad (27)$$

$$(\alpha_1 A_k^{N-1} + \cdots + \alpha_{N-1} A_k + \alpha_N I_r) \overline{B}_k = \overline{\mathbf{d}} \quad (28)$$

Then

$$\Phi_r = \begin{bmatrix} 1 & T & \cdots & \frac{T^{r-1}}{(r-1)!} \\ 0 & 1 & \cdots & \frac{T^{r-2}}{(r-2)!} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix} \quad (29)$$

$$e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \overline{e} = \begin{bmatrix} \overline{e}_1 \\ \overline{e}_2 \end{bmatrix}, \mathbf{d} = \begin{bmatrix} d \\ d' \end{bmatrix}, \overline{\mathbf{d}} = \begin{bmatrix} \overline{d} \\ \overline{d}' \end{bmatrix} \quad (30)$$

$$d' = \begin{bmatrix} d'_1 & d'_2 & \cdots & d'_{r-1} \end{bmatrix}^T \quad (31)$$

$$\overline{d}' = \begin{bmatrix} \overline{d}'_1 & \overline{d}'_2 & \cdots & \overline{d}'_{r-1} \end{bmatrix}^T \quad (32)$$

$$e_1 = \frac{T^r}{r!}, e_2 = \begin{bmatrix} \frac{T^{r-1}}{(r-1)!} & \cdots & \frac{T^2}{2!} & T \end{bmatrix}^T \quad (33)$$

$$\overline{e}_1 = \frac{T^{r+1}}{(r+1)!}, \overline{e}_2 = \begin{bmatrix} \frac{T^r}{r!} & \cdots & \frac{T^3}{3!} & \frac{T^2}{2!} \end{bmatrix}^T \quad (34)$$

$$d = \frac{T^r}{r!} c_N^r(\alpha), d'_k = \frac{T^{r-k}}{(r-k)!} c_N^{r-k}(\alpha), \quad (35)$$

$$\overline{d} = \frac{T^{r+1}}{(r+1)!} c_N^{r+1}(\alpha), \overline{d}'_l = \frac{T^{r-l+1}}{(r-l+1)!} c_N^{r-l+1}(\alpha), \quad (36)$$

$$k, l = 1, 2, \cdots, r-1$$

where

$$c_N^p(\alpha) = \sum_{j=1}^{N} (\Lambda_{1j}^p - \Lambda_{2j}^p) \alpha_j \quad (37)$$

$$\Lambda_{1j} = 1 - \frac{j-1}{N}, \Lambda_{2j} = 1 - \frac{j}{N} \quad (38)$$

In addition, we calculate the local truncation error between the true system output and the output of the obtained sampled-data model. It is assumed that the state of the sampled-data model is identical to the true system state at $t = kT$. At the end of the sampling interval $t = (k+1)T$, we compare the true system output $y((k+1)T)$ and the first state $z_{1,k+1}$ of the approximate sampled-data model.

First, on the basis of the result in [3], the true system output $y((k+1)T)$ can be expressed as

$$y((k+1)T) = z_1(kT) + T z_2(kT) + \cdots + \frac{T^{r+1}}{(r+1)!} [\overline{b}_k + c_N^{r+1}(\alpha) \overline{a}_k u_k]_{t=\xi_1} \quad (39)$$

with $kT < \xi_1 < (k+1)T$

This yields the following local truncation output error:

$$\begin{aligned} e &= |y((k+1)T) - z_{1,k+1}| \\ &= \frac{T^{r+1}}{(r+1)!} |[\overline{b}_k + c_N^{r+1}(\alpha)\overline{a}_k u_k]_{t=\xi_1} \\ &\quad - [\overline{b}_k + c_N^{r+1}(\alpha)\overline{a}_k u_k]_{t=kT}| \\ &\leq \frac{T^{r+1}}{(r+1)!} \times L \| z(\xi_1) - z(kT) \| \end{aligned} \quad (40)$$

where L is the Lipschitz constant. Further, the Lipschitz constant guarantees that [27]

$$\begin{aligned} \|z(\xi_1) - z(kT)\| &\leq C \times \frac{e^{L|\xi_1 - kT|} - 1}{L} \\ &< C \times \frac{e^{LT} - 1}{L} = O(T) \end{aligned} \quad (41)$$

Thus, the local truncation error between the true system output and the output of the sampled-data model is of order T^{r+2} which implies that the accuracy of the obtained sampled-data model is more accurate than that of the Yuz and Goodwin's model [3].

3 Zero dynamics of the sampled-data model with GSHF

The zero dynamics of the sampled-data model (23) consist of the sampled counterpart of the continuous-time zero dynamics and the additional zero dynamics produced by the sampling process [3]. The former are called intrinsic zero dynamics which have counterpart in the underlying continuous-time system. The latter are called the sampling zero dynamics, and turn out to be the same as those which appear asymptotically for the linear case when the sampling period goes to zero.

First, we consider the following **Assumption** so that one can further derive the asymptotic expressions of the intrinsic zero dynamics of the sampled-data model (23).

Assumption 4:

$$\frac{\partial c(\boldsymbol{\zeta}, \boldsymbol{\eta})}{\partial z_i} = 0, i = 2, \cdots, r \quad (42)$$

The **Assumption 4** implies that the vector $c(\zeta, \eta)$ does not include a term of $z_i (i = 2, \cdots, r)$.

Since $c(\boldsymbol{\zeta}, \boldsymbol{\eta})$ is independent of $z_i (i = 2, \cdots, r)$ under the **Assumption 4**, the sampled-data system (23) has the sampled counterpart of the continuous-time zero dynamics given by

$$\boldsymbol{\eta}_{k+1} = \boldsymbol{\eta}_k + T c(\mathbf{0}, \boldsymbol{\eta}_k) \quad (43)$$

In the following, the sampling zero dynamics of the sampled-data model are analyzed, and the main result of this paper is given by the following Theorem 1.

Theorem 1 *Assume that both a_{k0} and $\overline{a}_{k0}$ are constant. Here, a_{k0} and $\overline{a}_{k0}$ denote the values of a_k and $\overline{a}_k$, respectively. Then, for sufficiently small sampling periods, the sampling zero dynamics of the sampled-data model (23) with PC GSHF are given by*

$$(-1)^r a_{k0}\frac{T^r}{r!}D_r(z;\alpha) + T\overline{a}_{k0}\overline{f}_r(z) = 0 \tag{44}$$

where

$$\begin{aligned}\overline{f}_r(z) &= \sum_{i=1}^{r}(-1)^i\frac{T^i}{i!}(z-1)^{i-1}\overline{f}_{r-i}(z)\\ \overline{f}_0(z) &= \frac{c_N^{r+1}(\alpha)}{r+1}\end{aligned} \tag{45}$$

and $D_r(z;\alpha)$ is a monic polynomial with the order $r-1$ defined by [16,28]

$$D_r(z;\alpha) = d_r z^{r-1} + d_{r-1}z^{r-2} + \cdots + d_1 \tag{46}$$

where d_i are real coefficients defined by

$$d_i = \sum_{j=1}^{N} a_{ij}\alpha_j, i = 1,\cdots,r \tag{47}$$

where a_{ij} are constants composed of Λ_{1j} and Λ_{2j}.

Proof: On the basis of the result in [3], the sampling zero dynamics of the model (23) are calculated below. First, when set $y_{k+1} = 0$ and $y_k = 0$, then (23) leads to

$$0 = p_{11}^T\overline{\zeta}_k + p_{12}u_k + p_{13}b_{k0} \tag{48}$$

$$\overline{\zeta}_{k+1} = p_{21}\overline{\zeta}_k + p_{22}u_k + p_{23}b_{k0} \tag{49}$$

where

$$p_{11} = \left[\begin{array}{cccc} T & \frac{T^2}{2!} & \cdots & \frac{T^{r-1}}{(r-1)!}\end{array}\right]^T \tag{50}$$

$$p_{12} = \frac{T^r}{r!}c_N^r(\alpha)a_{k0} + \frac{T^{r+1}}{(r+1)!}c_N^{r+1}(\alpha)\overline{a}_{k0} \tag{51}$$

$$p_{13} = \frac{T^r}{r!}b_{k0} + \frac{T^{r+1}}{(r+1)!}\overline{b}_{k0} \tag{52}$$

$$p_{21} = \begin{bmatrix} 1 & T & \cdots & \frac{T^{r-2}}{(r-2)!}\\ 0 & 1 & \cdots & \vdots\\ \vdots & \ddots & \ddots & T\\ 0 & \cdots & 0 & 1\end{bmatrix} \tag{53}$$

$$p_{22} = \begin{bmatrix} \frac{T^{r-1}}{(r-1)!}c_N^{r-1}(\alpha)a_{k0} + \frac{T^r}{r!}c_N^r(\alpha)\overline{a}_{k0}\\ \frac{T^{r-2}}{(r-2)!}c_N^{r-2}(\alpha)a_{k0} + \frac{T^{r-1}}{(r-1)!}c_N^{r-1}(\alpha)\overline{a}_{k0}\\ \vdots\\ Ta_{k0} + \frac{T^2}{2!}c_N^2(\alpha)\overline{a}_{k0}\end{bmatrix} \tag{54}$$

$$p_{23} = \begin{bmatrix} \frac{T^{r-1}}{(r-1)!}b_{k0} + \frac{T^r}{r!}\overline{b}_{k0}\\ \frac{T^{r-2}}{(r-2)!}b_{k0} + \frac{T^{r-1}}{(r-1)!}\overline{b}_{k0}\\ \vdots\\ Tb_{k0} + \frac{T^2}{2!}\overline{b}_{k0}\end{bmatrix} \tag{55}$$

Hence, noting that the order of p_{13} and p_{23} is higher with respect to T than the corresponding one of p_{11} and p_{21}, and leads to the sampling zero dynamics

$$\Phi_r(z)\upsilon = 0 \tag{56}$$

where

$$\Phi_r(z) = \begin{bmatrix} -p_{11}^T & -p_{12}\\ zI - p_{21} & -p_{22}\end{bmatrix}, \upsilon = \begin{bmatrix} Z[\overline{\zeta}_k]\\ Z[u_k]\end{bmatrix} \tag{57}$$

where $Z[.]$ denotes the z-transform.

As a result, the sampling zero dynamics are obtained from $|\Phi_r(z)| = 0$, where

$$|\Phi_r(z)| = \begin{vmatrix} -T & \cdots & -\frac{T^{r-1}}{(r-1)!} & \delta_r\\ z-1 & \cdots & -\frac{T^{r-2}}{(r-2)!} & \delta_{r-1}\\ 0 & \ddots & \vdots & \vdots\\ \vdots & \ddots & -T & \delta_2\\ 0 & \cdots & z-1 & \delta_1\end{vmatrix} \tag{58}$$

where

$$\delta_i = -\frac{T^i}{i!}c_N^i(\alpha)a_{k0} - \frac{T^{i+1}}{(i+1)!}c_N^{i+1}(\alpha)\overline{a}_{k0}\quad i = 1,\cdots,r$$

The determinant $|\Phi_r(z)|$ can be expanded along the first column as follows:

$$\begin{aligned}|\Phi_r(z)| =& -T|\Phi_{r-1}(z)| - (z-1)\\ &\times\begin{vmatrix} -\frac{T^2}{2!} & \cdots & -\frac{T^{r-1}}{(r-1)!} & \delta_r\\ z-1 & \cdots & -\frac{T^{r-3}}{(r-3)!} & \delta_{r-2}\\ 0 & \ddots & \vdots & \vdots\\ \vdots & \ddots & -T & \delta_2\\ 0 & \cdots & z-1 & \delta_1\end{vmatrix}\\ =& \sum_{i=1}^{r}(-1)^i\frac{T^i}{i!}(z-1)^{i-1}|\Phi_{r-i}(z)|\end{aligned} \tag{59}$$

where

$$|\Phi_0(z)| = c_N^r(\alpha)a_{k0} + \frac{Tc_N^{r+1}(\alpha)}{r+1}\overline{a}_{k0} \quad (60)$$

Hence, the following result is obtained

$$|\Phi_r(z)| = a_{k0}f_r(z) + T\overline{a}_{k0}\overline{f}_r(z) \quad (61)$$

where

$$\begin{aligned} f_r(z) &= \sum_{i=1}^{r}(-1)^i\frac{T^i}{i!}(z-1)^{i-1}f_{r-i}(z) \\ f_0(z) &= c_N^r(\alpha) \end{aligned} \quad (62)$$

It is obvious that the order of any term of $f_r(z)$ is r with respect of T. Here, the following relation holds from Appendix

$$f_r(z) = (-1)^r\frac{T^r}{r!}D_r(z;\alpha) \quad (63)$$

Therefore, the result (44) follows from (61) and (63). □

Remark 2 *The polynomials $\overline{f}_r(z)$ are listed below for a few values of r.*

$$\begin{aligned}
\overline{f}_1(z) &= -\frac{T}{2}c_N^2(\alpha), r=1 \\
\overline{f}_2(z) &= \frac{T^2}{3!}(c_N^3(\alpha)z + 3c_N^2(\alpha) - c_N^3(\alpha)), r=2 \\
\overline{f}_3(z) &= -\frac{T^3}{4!}\{c_N^4(\alpha)z^2 + [-2c_N^4(\alpha) + 4c_N^3(\alpha) \\
&\quad + 6c_N^2(\alpha)]z + [c_N^4(\alpha) - 4c_N^3(\alpha) + 6c_N^2(\alpha)]\}, \\
&\qquad r=3 \\
\overline{f}_4(z) &= \frac{T^4}{5!}\{c_N^5(\alpha)z^3 + [-3c_N^5(\alpha) + 5c_N^4(\alpha) \\
&\quad + 10c_N^3(\alpha) + 10c_N^2(\alpha)]z^2 + [3c_N^5(\alpha) \\
&\quad - 10c_N^4(\alpha) + 40c_N^2(\alpha)]z + [-c_N^5(\alpha) \\
&\quad + 5c_N^4(\alpha) - 30c_N^3(\alpha) + 70c_N^2(\alpha)]\}, r=4 \\
\overline{f}_5(z) &= -\frac{T^5}{6!}\{c_N^6(\alpha)z^4 + [-4c_N^6(\alpha) + 6c_N^5(\alpha) \\
&\quad + 15c_N^4(\alpha) + 20c_N^3(\alpha) + 15c_N^2(\alpha)]z^3 \\
&\quad + [6c_N^6(\alpha) - 18c_N^5(\alpha) - 15c_N^4(\alpha) \\
&\quad + 60c_N^3(\alpha) + 165c_N^2(\alpha)]z^2 + [-4c_N^6(\alpha) \\
&\quad + 18c_N^5(\alpha) - 15c_N^4(\alpha) - 60c_N^3(\alpha) \\
&\quad + 165c_N^2(\alpha)]z + [c_N^6(\alpha) - 6c_N^5(\alpha) + 15c_N^4(\alpha) \\
&\quad - 140c_N^3(\alpha) + 375c_N^2(\alpha)]\}, r=5
\end{aligned} \quad (64)$$

Remark 3 *When the* **Assumption 3** *is not fulfilled, the term $(b+au)\partial a/\partial z_r$ is included in $\overline{b}_k$ of (12) and in $\overline{a}_k$ of (13), and then there may appear the term u^2 in (23).*

Remark 4 *From (43) and (44), it is found that the sampled counterpart of the continuous-time zero dynamics and the sampling zero dynamics can not be determined separately when the* **Assumption 4** *is not satisfied.*

Remark 5 *The proposed model (23) is more accurate than the Euler model so that a controller design by the Euler model [29–31] is easier but better performance could be obtained if the sampled-data model (23) is useful for controller design.*

In the particular case when the relative degree r of nonlinear system is two or three and $N = r+1$, the stability conditon of zero dynamics for a sufficiently small T is shown by the following Theorem 6.

Theorem 6 *Consider an affine nonlinear system (1). Case(a) $r=2$ and $N=3$: The sampling zero dynamics of the sampled-data model (23) with PC GSHF for sufficiently small sampling periods are stable if both a_{k0} and $\overline{a}_{k0}$ are constant and*

$$\begin{cases}
(2a_{k0} + \frac{5}{3}T\overline{a}_{k0})\alpha_1 + (2a_{k0} + T\overline{a}_{k0})\alpha_2 \\
+(2a_{k0} + \frac{1}{3}T\overline{a}_{k0})\alpha_3 > 0 \\
(\frac{7}{9}T\overline{a}_{k0} - 4a_{k0})\alpha_1 + \frac{17}{9}T\overline{a}_{k0}\alpha_2 \\
+(4a_{k0} + \frac{1}{3}T\overline{a}_{k0})\alpha_3 > 0 \\
|(\frac{1}{3}a_{k0} + \frac{26}{27}T\overline{a}_{k0})\alpha_1 + (a_{k0} + \frac{22}{27}T\overline{a}_{k0})\alpha_2 \\
+(\frac{5}{3}a_{k0} + \frac{2}{9}T\overline{a}_{k0})\alpha_3| < (\frac{5}{3}a_{k0} + \frac{19}{27}T\overline{a}_{k0})\alpha_1 \\
+(a_{k0} + \frac{5}{27}T\overline{a}_{k0})\alpha_2 + (\frac{1}{3}a_{k0} + \frac{1}{9}T\overline{a}_{k0})\alpha_3
\end{cases} \quad (65)$$

Case(b) $r=3$ and $N=4$: The sampling zero dynamics of the sampled-data model (23) with PC GSHF for sufficiently small sampling periods are stable if both a_{k0} and $\overline{a}_{k0}$ are constant and

$$\begin{cases}
(37a_{k0} - \frac{175}{16}T\overline{a}_{k0})\alpha_1 + (19a_{k0} - \frac{65}{16}T\overline{a}_{k0})\alpha_2 \\
+(7a_{k0} - \frac{15}{16}T\overline{a}_{k0})\alpha_3 + (a_{k0} - \frac{1}{16}T\overline{a}_{k0})\alpha_4 > 0 \\
|(a_{k0} - \frac{95}{16}T\overline{a}_{k0})\alpha_1 + (7a_{k0} + \frac{111}{16}T\overline{a}_{k0})\alpha_2 \\
+(19a_{k0} + \frac{161}{16}T\overline{a}_{k0})\alpha_3 + (37a_{k0} + \frac{79}{16}T\overline{a}_{k0})\alpha_4| \\
< (37a_{k0} - \frac{175}{16}T\overline{a}_{k0})\alpha_1 + (19a_{k0} - \frac{65}{16}T\overline{a}_{k0})\alpha_2 \\
+(7a_{k0} - \frac{15}{16}T\overline{a}_{k0})\alpha_3 + (a_{k0} - \frac{1}{16}T\overline{a}_{k0})\alpha_4 \\
(3a_{k0} - \frac{37}{16}T\overline{a}_{k0})\alpha_1 + (3a_{k0} - \frac{19}{16}T\overline{a}_{k0})\alpha_2 \\
+(3a_{k0} - \frac{7}{16}T\overline{a}_{k0})\alpha_3 + (3a_{k0} - \frac{1}{16}T\overline{a}_{k0})\alpha_4 > 0 \\
(\frac{161}{16}T\overline{a}_{k0} - 5a_{k0})\alpha_1 + (\frac{175}{16}T\overline{a}_{k0} - 11a_{k0})\alpha_2 + \\
(\frac{129}{16}T\overline{a}_{k0} - 11a_{k0})\alpha_3 + (\frac{47}{16}T\overline{a}_{k0} - 5a_{k0})\alpha_4 > 0
\end{cases} \quad (66)$$

Proof: Case (a): For $r = 2$ and $N = 3$, we have [28]

$$\begin{aligned} c_N^1(\alpha) &= \frac{1}{3}(\alpha_1+\alpha_2+\alpha_3) \\ c_N^2(\alpha) &= \frac{5}{9}\alpha_1+\frac{1}{3}\alpha_2+\frac{1}{9}\alpha_3 \\ c_N^3(\alpha) &= \frac{19}{27}\alpha_1+\frac{7}{27}\alpha_2+\frac{1}{9}\alpha_3 \end{aligned}$$

Further

$$\begin{aligned} &A(z;\alpha) = \frac{1}{3}\times\{ \\ &\left[\frac{1}{3}\left(5a_{k0}+\frac{19}{9}T\overline{a}_{k0}\right)\alpha_1+\left(a_{k0}+\frac{5}{27}T\overline{a}_{k0}\right)\alpha_2 \right.\\ &\left.+\frac{1}{3}\left(a_{k0}+\frac{1}{3}T\overline{a}_{k0}\right)\alpha_3\right]z+\frac{1}{3}\left(a_{k0}+\frac{26}{9}T\overline{a}_{k0}\right)\alpha_1 \\ &+\left(a_{k0}+\frac{22}{27}T\overline{a}_{k0}\right)\alpha_2+\frac{1}{3}\left(5a_{k0}+\frac{2}{3}T\overline{a}_{k0}\right)\alpha_3\Big\} \end{aligned} \tag{67}$$

Thus by the Jury stability test [32], the roots of the polynomial $A(z;\alpha) = 0$ are located inside the unit circle if and only if (65) holds.

Case (b): Similarly, For $r = 3$ and $N = 4$, we have [28]

$$\begin{aligned} c_N^1(\alpha) &= \frac{1}{4}(\alpha_1+\alpha_2+\alpha_3+\alpha_4) \\ c_N^2(\alpha) &= \frac{7}{16}\alpha_1+\frac{5}{16}\alpha_2+\frac{3}{16}\alpha_3+\frac{1}{16}\alpha_4 \\ c_N^3(\alpha) &= \frac{37}{64}\alpha_1+\frac{19}{64}\alpha_2+\frac{7}{64}\alpha_3+\frac{1}{64}\alpha_4 \\ c_N^4(\alpha) &= \frac{175}{256}\alpha_1+\frac{65}{256}\alpha_2+\frac{15}{256}\alpha_3+\frac{1}{256}\alpha_4 \end{aligned}$$

Then

$$\begin{aligned} &B(z;\alpha) = \frac{1}{32}\times\{ \\ &\left[\left(\frac{37}{2}a_{k0}-\frac{175}{16}T\overline{a}_{k0}\right)\alpha_1+\left(\frac{19}{2}a_{k0}-\frac{65}{16}T\overline{a}_{k0}\right)\alpha_2\right. \\ &\left.+\left(\frac{7}{2}a_{k0}-\frac{15}{16}T\overline{a}_{k0}\right)\alpha_3+\left(\frac{1}{2}a_{k0}-\frac{1}{16}T\overline{a}_{k0}\right)\alpha_4\right]z^2 \\ &+\left[\left(29a_{k0}-\frac{457}{16}T\overline{a}_{k0}\right)\alpha_1+\left(35a_{k0}-\frac{327}{16}T\overline{a}_{k0}\right)\alpha_2\right. \\ &\left.+\left(35a_{k0}-\frac{185}{16}T\overline{a}_{k0}\right)\alpha_3+\left(29a_{k0}-\frac{55}{16}T\overline{a}_{k0}\right)\alpha_4\right]z \\ &+\left(\frac{1}{2}a_{k0}-\frac{95}{16}T\overline{a}_{k0}\right)\alpha_1+\left(\frac{7}{2}a_{k0}+\frac{111}{16}T\overline{a}_{k0}\right)\alpha_2 \\ &+\left(\frac{19}{2}a_{k0}+\frac{161}{16}T\overline{a}_{k0}\right)\alpha_3+\left(\frac{37}{2}a_{k0}+\frac{79}{16}T\overline{a}_{k0}\right)\alpha_4\Big\} \end{aligned} \tag{68}$$

When we perform the bilinear transformation $z = \frac{\omega+1}{\omega-1}$ on the above equation, by the Jury stability test [32], thus the roots of the polynomial $B(z;\alpha) = 0$ lie inside the unit circle if and only if (66) holds. □

Remark 7 *As in the foregoing statement, the sampling zero dynamics can be assigned inside the unit circle for a sufficiently small T by choosing design parameters $\alpha_j(j = 1,\cdots,N, N \geq r)$ so that the sampling zero dynamics polynomial of (44) is identical to a desired stable one, though it seems difficult to derive explicit inequality relations for $r \geq 4$.*

4 Examples

This section presents two interesting examples to analysis the stability of the sampling zero dynamics of a more accurate sampled-data model than that of Yuz and Goodwin model with PC GSHF.

Consider a controlled Van der pol system with the following equation [25, 26].

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1+\varepsilon(1-x_1^2)x_2+u, \varepsilon > 0 \\ y = x_1 \end{cases} \tag{69}$$

where the parameter ε is constant.

It is obvious that the relative degree of the system (69) is two, and that the system does not have any zero dynamics. From (12), (13) and (69), It is easy to obtain

$$a(\boldsymbol{\zeta}_k,\boldsymbol{\eta}_k) = 1, b(\boldsymbol{\zeta}_k,\boldsymbol{\eta}_k) = -x_1+\varepsilon(1-x_1^2)x_2 \tag{70}$$

$$\overline{a}(\boldsymbol{\zeta}_k,\boldsymbol{\eta}_k) = \frac{\partial b}{\partial x_2}a+\frac{\partial a}{\partial x_1}x_2 = \varepsilon(1-x_1^2) \tag{71}$$

$$\begin{aligned} \overline{b}(\boldsymbol{\zeta}_k,\boldsymbol{\eta}_k) &= \frac{\partial b}{\partial x_1}x_2+\frac{\partial b}{\partial x_2}b \\ &= (-1-2\varepsilon x_1x_2)x_2+\varepsilon(1-x_1^2)b \end{aligned} \tag{72}$$

A further more accurate sampled-data model [6–8] than that of Yuz and Goodwin is expressed as

$$\begin{cases} x_{1,k+1} = x_{1,k}+Tx_{2,k} \\ \qquad +\frac{T^2}{2}[-x_{1,k}+\varepsilon(1-x_{1,k}^2)x_{2,k}+u_k] \\ \qquad +\frac{T^3}{3!}[-x_{2,k}-2\varepsilon x_{1,k}x_{2,k}^2+\varepsilon(1-x_{1,k}^2) \\ \qquad \times\{-x_{1,k}+\varepsilon(1-x_{1,k}^2)x_{2,k}+u_k\}] \\ x_{2,k+1} = x_{2,k}+T[-x_{1,k}+\varepsilon(1-x_{1,k}^2)x_{2,k}+u_k] \\ \qquad +\frac{T^2}{2!}[-x_{2,k}-2\varepsilon x_{1,k}x_{2,k}^2+\varepsilon(1-x_{1,k}^2) \\ \qquad \times\{-x_{1,k}+\varepsilon(1-x_{1,k}^2)x_{2,k}+u_k\}] \\ y = x_{1,k} \end{cases} \tag{73}$$

The sampling zero dynamics of (73) is expressed as

$$z + 1 + \frac{\varepsilon}{3}T = 0 \tag{74}$$

Therefore, the sampling zero dynamics are obviously unstable.

A sampled-data model with PC GSHF is represented as

$$\begin{cases} x_{1,k+1} = x_{1,k} + Tx_{2,k} + \frac{T^2}{2}[-x_{1,k} \\ \quad +\varepsilon(1-x_{1,k}^2)x_{2,k} + c_N^2(\alpha)u_k] + \frac{T^3}{3!}[-x_{2,k} \\ \quad -2\varepsilon x_{1,k}x_{2,k}^2 + \varepsilon(1-x_{1,k}^2)\{-x_{1,k} \\ \quad +\varepsilon(1-x_{1,k}^2)x_{2,k} + c_N^3(\alpha)\varepsilon(1-x_{1,k}^2)u_k\}] \\ x_{2,k+1} = x_{2,k} + T[-x_{1,k} + \varepsilon(1-x_{1,k}^2)x_{2,k} \\ \quad +c_N^2(\alpha)u_k] + \frac{T^2}{2!}[-x_{2,k} - 2\varepsilon x_{1,k}x_{2,k}^2 \\ \quad +\varepsilon(1-x_{1,k}^2)\{-x_{1,k} + \varepsilon(1-x_{1,k}^2)x_{2,k} \\ \quad +c_N^3(\alpha)\varepsilon(1-x_{1,k}^2)u_k\}] \\ y = x_{1,k} \end{cases} \tag{75}$$

The sampling zero dynamics of (75) is expressed as

$$z - 1 + \frac{2c_N^1(\alpha)}{c_N^2(\alpha)} = 0 \tag{76}$$

From (76), the sampling zero dynamics of the sampled-data model (75) are stable for sufficiently small sampling periods if $(\alpha_1 + \alpha_2 + \alpha_3)(5\alpha_1 + 3\alpha_2 + \alpha_3) > 0$. Thus, there exists a set of solutions $\alpha_1 = 1, \alpha_2 = -0.202$ and $\alpha_3 = -0.624$ such that the sampling zero dynamics are stable for sampling period $T = 0.01$ at $N = 3$. On the other hand, There is also the rest of the solution to meet the relationship (76) for Van der pol system with PC GSHF. In addtion, we should preserve the stability of zero dynamics by choosing the parameters $\alpha_j (j = 1, \cdots, N)$ while satisfying other performance requirements.

Based on above researching analysis, we consider here model following control such that the output converge to the origin. Moreover, some phenomenon can seems that the stability of the closed-loop system is related directly to that of the zero dynamics of a more accurate sampled-data model. A discrete-time model following controller is designed using the model as

$$\begin{aligned} u_k &= \frac{1}{c_N^2(\alpha)}\{x_{1,k} - \varepsilon(1-x_{1,k}^2)x_{2,k} \\ &\quad + \frac{2}{T^2}[(\beta-1)x_{1,k} - Tx_{2,k}]\}, \quad 0 < \beta < 1 \end{aligned}$$

The parameters $T = 0.01$, $\varepsilon = 1$, $k = 1$ and $\beta = 0.8$ are used in simulation, and the simulation results are shown in Fig. 3 and Fig. 4.

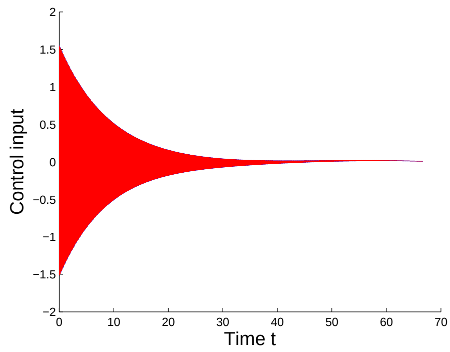

Fig. 3 Input of Van der pol in sampled-data model with a PC GSHF

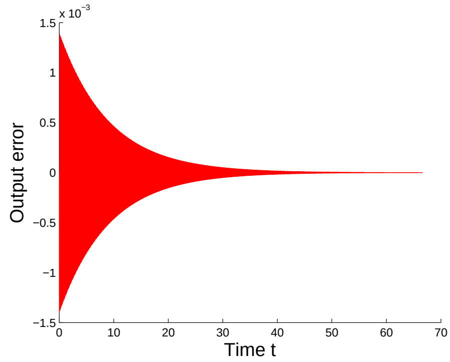

Fig. 4 Output error of Van der pol in sampled-data model with a PC GSHF

Next, we consider an extended pendulum system with the relative degree three

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -cx_2 - dsinx_1 + au \\ y = x_1 \end{cases} \tag{77}$$

From (12), (13) and (77), It is easy to obtain

$$a(\boldsymbol{\zeta}_k, \boldsymbol{\eta}_k) = a, b(\boldsymbol{\zeta}_k, \boldsymbol{\eta}_k) = -cx_2 - dsinx_1 \tag{78}$$

$$\overline{a}(\boldsymbol{\zeta}_k, \boldsymbol{\eta}_k) = \frac{\partial b}{\partial x_3}a + \frac{\partial a}{\partial x_1}x_2 + \frac{\partial a}{\partial x_2}x_3 = -ac \tag{79}$$

$$\begin{aligned} \overline{b}(\boldsymbol{\zeta}_k, \boldsymbol{\eta}_k) &= \frac{\partial b}{\partial x_1}x_2 + \frac{\partial b}{\partial x_3}b + \frac{\partial b}{\partial x_2}x_3 \\ &= -dx_2cosx_1 - cx_3 \end{aligned} \tag{80}$$

A sampled-data model with PC GSHF is represented as

$$\begin{cases} x_{1,k+1} = x_{1,k} + Tx_{2,k} + \frac{T^2}{2!}x_{3,k} \\ \qquad + \frac{T^3}{3!}[x_{1,k}x_{3,k} + c_N^3(\alpha)u_k] \\ \qquad + \frac{T^4}{4!}[x_{3,k}(x_{1,k}^2 + x_{2,k}) + c_N^4(\alpha)x_{1,k}u_k] \\ x_{2,k+1} = x_{2,k} + Tx_{3,k} + \frac{T^2}{2!}[x_{1,k}x_{3,k} + c_N^3(\alpha)u_k] \\ \qquad + \frac{T^3}{3!}[x_{3,k}(x_{1,k}^2 + x_{2,k}) + c_N^4(\alpha)x_{1,k}u_k] \\ x_{3,k+1} = x_{3,k} + T[x_{1,k}x_{3,k} + c_N^3(\alpha)u_k] \\ \qquad + \frac{T^2}{2!}[x_{3,k}(x_{1,k}^2 + x_{2,k}) + c_N^4(\alpha)x_{1,k}u_k] \\ y = x_{1,k} \end{cases} \tag{81}$$

The sampling zero dynamics of (81) is expressed as

$$\begin{vmatrix} z + 2 + T\beta_{11} & T\beta_{12} \\ \frac{1}{T}\beta_{21} & z + 2 + T\beta_{22} \end{vmatrix} = 0 \tag{82}$$

where

$$\beta_{11} = \frac{c_N^4(\alpha)ac}{\delta}, \beta_{12} = \frac{-2c_N^3(\alpha) + Tc_N^4(\alpha)ac}{\delta}$$
$$\beta_{21} = \frac{-24c_N^3(\alpha) + 12Tc_N^4(\alpha)ac}{\delta}, \beta_{22} = \frac{3c_N^4(\alpha)ac}{\delta}$$
$$\delta = 4c_N^3(\alpha) - Tc_N^4(\alpha)ac$$

From (82), the sampling zero dynamics of the sampled-data model (81) are stable for sufficiently small sampling periods if $\beta_{12}\beta_{21} > 9$ or $\beta_{12}\beta_{21} < 1$, i.e.,

$$\begin{cases} -96[c_N^3(\alpha)]^2 + 24Tc_N^3(\alpha)c_N^4(\alpha)ac \\ +3T^2[c_N^4(\alpha)ac]^2 > 0 \\ 32[c_N^3(\alpha)]^2 - 40Tc_N^3(\alpha)c_N^4(\alpha)ac \\ +11T^2[c_N^4(\alpha)ac]^2 < 0 \end{cases} \tag{83}$$

where the parameter a and c are constant.

Thus, simple straightforward calculation verify that the sampling zero dynamics (82) can be arbitrarily assigned inside the unit circle by choosing design parameters α_j of the $c_N^3(\alpha)$ and $c_N^4(\alpha)$ which satisfy the inequality (83).

When the relative degree of the continuous-time system is three, the sampling zero dynamics with FROH are unstable [9, 10]. However, we can choose the parameters $\alpha_j(j = 1, \cdots, N)$ so that the sampling zero dynamics of the sampled-data model with PC GSHF are located into a stable region from (66). Thus, the use of PC GSHF instead of ZOH or FROH overcomes the problem of the instability of the sampling zero dynamics when the relative degree of a continuous-time plant is greater than or equal to three.

5 Conclusion

This paper derives a good approximate sampled-data model for nonlinear continuous-time systems in the case of a PC GSHF, which the obtained discrete model uses a more sophisticated derivative approximation than the simple Euler approach. In addition, we also analyze the stability conditions of zero dynamics when the relative degree is two or three and $N = r+1$. We further show how the sampling zero dynamics of the model obtained can be expressed in such a way that the discrete zero dynamics are given by the sampled counterpart of the continuous zero dynamics, together with extra zero dynamics produced by the sampling process. As a result of this work, it has been shown that the zero dynamics of the sampled-data model with PC GSHF can be located inside the stability region by choosing suitable values of the design parameters $\alpha_j(j = 1, \cdots, N)$. For a future study, it is still necessary to preserve the stability of sampling zero dynamics by appropriately selecting design parameters of PC GSHF while satisfying other performance requirements, such as gain margin, intersample ripples, etc.

Appendix

Proof of (63).

Consider a linear transfer function

$$G(s) = \frac{1}{s^r} \tag{84}$$

with an input $u(t)$ and an output $y(t)$. In the following, we discuss the sampled-data model obtained from the linear system (84) by use of a PC GSHF.

First, we define the state variables such as $x_1(t) = y(t), x_2(t) = \dot{y}(t), \cdots, x_r(t) = y^{r-1}(t)$.

Applying a higher-order Taylor expansion such as

$$x_j(k+1) = y^{j-1}(k+1) = \sum_{i=0}^{\infty} \frac{T^i}{i!} y_k^{(i+j-1)} \tag{85}$$
$$j = 1, 2, \cdots, r$$

Thus, the matrix state equation is expressed as

$$\begin{cases} x_{k+1} = \begin{bmatrix} 1 & T & \cdots & \frac{T^{r-1}}{(r-1)!} \\ & \ddots & & \vdots \\ & O & \ddots & T \\ & & & 1 \end{bmatrix} x_k + \begin{bmatrix} \frac{T^r}{r!}c_N^r(\alpha) \\ \vdots \\ \frac{T^2}{2!}c_N^2(\alpha) \\ Tc_N^1(\alpha) \end{bmatrix} u_k \\ y_k = [\ 1 \quad 0 \quad \cdots \quad 0\ ] x_k \end{cases} \tag{86}$$

where $x_k = [x_1(k), x_2(k), \cdots, x_r(k)]^T$.

The zeros of the sampled-data model above are determined by $y(k+1) = y(k) = 0$ in (86) as follows:

$$\begin{cases} 0 = [\ T \quad \frac{T^2}{2!} \quad \cdots \quad \frac{T^{r-1}}{(r-1)!}\ ]\overline{x}_k + \frac{T^r}{r!}c_N^r(\alpha)u_k \\ \overline{x}_{k+1} = \begin{bmatrix} 1 & T & \cdots & \frac{T^{r-2}}{(r-2)!} \\ & O & \ddots & \vdots \\ & & & 1 \end{bmatrix}\overline{x}_k + \begin{bmatrix} \frac{T^{r-1}}{(r-1)!}c_N^{r-1}(\alpha) \\ \vdots \\ Tc_N^1(\alpha) \end{bmatrix}u_k \end{cases} \tag{87}$$

where $\overline{x}_k = [x_2(k), \cdots, x_r(k)]^T$.

Apply z-transform to (87), then we see that the zeros are solution of $|\phi_r(z)| = 0$, where

$$\phi_r(z) = \begin{vmatrix} -T & -\frac{T^2}{2!} & \cdots & -\frac{T^{r-1}}{(r-1)!} & -\frac{T^r}{r!}c_N^r(\alpha) \\ z-1 & -T & \cdots & -\frac{T^{r-2}}{(r-2)!} & -\frac{T^{r-1}}{(r-1)!}c_N^{r-1}(\alpha) \\ & \ddots & & & \vdots \\ O & & & z-1 & -Tc_N^1(\alpha) \end{vmatrix} \tag{88}$$

Expanding $\phi_r(z)$ along the first column and repeating this operation lead to

$$\begin{aligned} \phi_r(z) &= \sum_{i=1}^{r}(-1)^i\frac{T^i}{i!}(z-1)^{i-1}\phi_{r-i}(z) \\ \phi_0(z) &= c_N^r(\alpha) \end{aligned} \tag{89}$$

which the order of $\phi_i(z)$ is obviously $i-1$ from the definition.

When the transfer function (84) is discretized by a PC GSHF, the pulse transfer function $G_d(z)$ is given by

$$G_d(z) = \frac{T^r D_r(z;\alpha)}{r!(z-1)^r} \tag{90}$$

where $D_r(z;\alpha)$ is a monic polynomial with the order $r-1$ [16]. Therefore, we have obtained (63).

As a result, the proof is complete.

Acknowledgments

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