

Ternary Matrices over B_3 and (k_+, k_-) -Balanced Regular Structures

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Abstract: This paper introduces a comprehensive theory of matrices whose entries are drawn from the ternary set $B_3 = \{-1, 0, 1\}$. We introduce a base-3 coding scheme that provides a lexicographic ordering of rows and columns, which in turn leads to the definition of canonical ordered matrix classes C_3 and D_3 . Extending classical results for binary regular matrices, we define and characterize (k_+, k_-) -balanced matrices as their ternary analogues. We prove that the sequence $H_n = 3^n - 1$ represents the maximum attainable row or column code in this framework. Furthermore, we establish the core structural properties and symmetries of these matrix classes. These results provide a rigorous mathematical foundation for modeling systems with ternary interactions, including signed networks, neural connectivity patterns, and other ternary-structured systems.

Key-Words: Ternary matrices, balanced matrices, lexicographic order, signed graphs, matrix coding, combinatorial sequences, H_n numbers.

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1 Introduction

Classical studies on binary matrices over the set $B_2 = \{0, 1\}$ provide the foundation for representing k -regular bipartite graphs via the set L_k^n of $(0, 1)$ -matrices whose rows and columns each contain exactly k ones. Within this framework, the algorithmic generation of such combinatorial objects, the analysis of their factor sets, and the classification of binary matrix classes reveal fundamental structures and provide important groundwork [1], [2], [3]. This binary framework also has deep connections to combinatorial sequences, particularly Fibonacci, Lucas, and H_n numbers [4], [5], [6].

The combinatorial analysis of matrix problems [7] offers a broader context for these studies. This paper extends the established theory to matrices over the ternary set B_3 , building upon foundational work on ternary logic structures [8] and their logical properties [9]. This forms the basis for our comprehensive mathematical framework for ternary matrices with balanced regular structures. The foundational algebraic and logical structure of B_3 , which underpins our matrix coding scheme, is established in [9]. The application of ternary matrices to signed graphs further builds upon research in suborbital graphs [10]. Collectively, these works provide the necessary background for our investigation.

The motivation for this extension arises from sev-

eral application domains. Ternary matrices naturally serve as adjacency matrices for signed graphs [10], where entries 1, -1 , and 0 represent positive, negative, and absent edges, respectively. They can model excitatory ($+1$), inhibitory (-1), and absent (0) connections in neural networks; model activating, inhibiting, and neutral interactions in molecular interaction networks; and offer efficient representation in ternary coding schemes for signal processing.

Within this ternary setting, we introduce several new mathematical structures: a systematic coding scheme using base-3 numerical representation; the lexicographically ordered matrix classes C_3 and D_3 as canonical forms; (k_+, k_-) -balanced regular structures as ternary analogues of binary k -regular matrices; and the sequence $H_n = 3^n - 1$, which emerges naturally as the maximum row/column code and provides combinatorial connections.

Combinatorial sequences are treated extensively in the literature, including foundational texts on Fibonacci and Lucas numbers [11], [12], concrete mathematics [13], and enumerative combinatorics [14], [15]. The classical theory of these sequences is covered in [16], while generatingfunctionology [17] provides essential tools. Proof techniques from [18] and historical perspectives from [19] enrich this field. General combinatorics references [20], [21], [22] and analytic approaches [23] provide comprehensive

frameworks. Combinatorial proof methods [24] and sequence databases [25] support this research.

Advanced combinatorial theory [26], [27], partition theory [28], and walkthroughs of combinatorics [29] offer deeper insights. Counting methods [30], discrete mathematics handbooks [31], and introductory combinatorics texts [32] provide pedagogical foundations. Combinatorial theory [33] and number theory invitations [34] connect to broader mathematical contexts. Classical number theory references [35], [36], [37] and multiplicative number theory [38] underpin the numerical aspects.

Experimental mathematics perspectives [39] and expositions on Fibonacci numbers [40] offer fresh viewpoints. Computer programming foundations [41] and set partition algorithms [42] provide computational context. Finally, Fibonacci-Catalan connections [43] demonstrate the breadth of this field's applications. The modeling of balanced systems, such as synchronous generator-infinite bus systems, has been successfully approached using multi-bond graph techniques [44], demonstrating the practical relevance of balanced structures in engineering. Furthermore, the development of customizable frameworks for specific organizational contexts, such as balanced scorecards for educational non-profits [45], highlights the broader applicability of balanced structures across domains.

This work builds a formal bridge between ternary matrix theory, combinatorial sequences (notably H_n), and the theory of signed graphs. The systematic approach developed here not only generalizes classical binary matrix results but also reveals new structural properties unique to ternary systems, such as the symmetry induced by the inversion map ϕ .

2 Ternary Matrices and Coding

This section introduces the core framework for analyzing ternary matrices with entries from the set B_3 . We define a base-3 coding system that assigns a unique integer value to each row and column, enabling a lexicographic ordering. Using this ordering, we construct the canonical classes C_3 and D_3 of ternary matrices with nondecreasing and nonincreasing codes, respectively. We establish the fundamental properties of these codes, including their sharp bounds defined by the sequence H_n and the behavior under the sign inversion operation.

We first define the primary object of study.

Definition 2.1. The set

$$B_3^{n \times m} = \{A = [a_{ij}] : a_{ij} \in B_3\} \quad (1)$$

is called the set of *ternary matrices* of size $n \times m$.

To compare rows and columns lexicographically, we assign to each row and column a numerical code

obtained by interpreting its entries as digits in base 3. This coding reduces the lexicographic comparison to simple numerical ordering.

Definition 2.2. For a matrix $A \in B_3^{n \times m}$, the *row code* is the ordered n -tuple

$$r(A) = \langle x_1, x_2, \dots, x_n \rangle, \quad (2)$$

where

$$x_i = \sum_{j=1}^m (a_{ij} + 1) 3^{m-j}, \quad i = 1, 2, \dots, n. \quad (3)$$

Since $(a_{ij} + 1) \in \{0, 1, 2\}$, every x_i is an integer satisfying $0 \leq x_i \leq 3^m - 1 = H_m$, where $H_m = 3^m - 1$.

Similarly, the *column code* is

$$c(A) = \langle y_1, y_2, \dots, y_m \rangle, \\ y_j = \sum_{i=1}^n (a_{ij} + 1) 3^{n-i}, \quad j = 1, 2, \dots, m.$$

Each y_j satisfies $0 \leq y_j \leq 3^n - 1 = H_n$.

These codings provide a bijection between rows (columns) and integers in $[0, H_m]$ ($[0, H_n]$), allowing lexicographic comparison via the natural integer order.

Remark 2.3. The sequence H_n emerges as the maximum attainable code for any row or column of a ternary matrix of size $n \times m$ (or $m \times n$). It satisfies the linear recurrence $H_{n+1} = 4H_n - 3H_{n-1}$ with initial values $H_0 = 0, H_1 = 2$.

To illustrate the sharpness of these bounds, consider the following concrete case.

Example 2.4. Consider ternary matrices with $n = 3$ rows. The maximum possible row code is $H_3 = 3^3 - 1 = 26$. This maximum is attained precisely by the row containing only 1's, i.e., $(1, 1, 1)$. Indeed, its code is computed as:

$$\begin{aligned} \sum_{k=0}^2 (1+1) 3^k &= 2 \sum_{k=0}^2 3^k \\ &= 2(9+3+1) \\ &= 18+6+2 = 26 = H_3. \end{aligned} \quad (4)$$

Conversely, the minimum code 0 corresponds to the row $(-1, -1, -1)$.

The following proposition establishes the fundamental range of these codes.

Proposition 2.5. For any $A \in B_3^{n \times m}$, the row and column codes satisfy:

$$0 \leq x_i \leq H_m \quad \text{for all } i = 1, \dots, n,$$

$$0 \leq y_j \leq H_n \quad \text{for all } j = 1, \dots, m,$$

where $H_k = 3^k - 1$. Moreover, the bounds are sharp: for each k , there exist matrices attaining the minimum code 0 and the maximum code H_k .

Proof. The minimum code 0 corresponds to a row/column containing only -1 entries (mapped to 0). The maximum code $H_k = 3^k - 1 = 2(3^{k-1} + 3^{k-2} + \dots + 1)$ corresponds to a row/column containing only 1 entries (mapped to 2). $\square$

With the numerical codes for rows and columns established, we can now define lexicographically ordered ternary matrices. These sets provide canonical forms for ternary matrices under row and column permutations.

Let $A \in B_3^{n \times m}$, $r(A) = \langle x_1, x_2, \dots, x_n \rangle$ and $c(A) = \langle y_1, y_2, \dots, y_m \rangle$.

Definition 2.6. The set of lexicographically nondecreasing ternary matrices is:

$$C_3^{n \times m} = \{A \in B_3^{n \times m} \mid x_1 \leq x_2 \leq \dots \leq x_n \text{ and } y_1 \leq y_2 \leq \dots \leq y_m\} \quad (5)$$

The set of lexicographically nonincreasing ternary matrices is:

$$D_3^{n \times m} = \{A \in B_3^{n \times m} \mid x_1 \geq x_2 \geq \dots \geq x_n \text{ and } y_1 \geq y_2 \geq \dots \geq y_m\}. \quad (6)$$

In other words, $A \in C_3^{n \times m}$ if and only if the rows and columns of A are sorted in lexicographical nondecreasing order, and $A \in D_3^{n \times m}$ if and only if the rows and columns of A are sorted in lexicographical nonincreasing order.

We illustrate this definition with a concrete example in Example 2.7.

Example 2.7. For the matrix:

$$A = \begin{pmatrix} -1 & -1 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

Row codes with $m = 4$, $H_4 = 3^4 - 1 = 80$: $r(A) = \langle 5, 17, 53 \rangle$. Column codes with $n = 3$, $H_3 = 3^3 - 1 = 26$: $c(A) = \langle 1, 5, 17, 26 \rangle$.

Since both sequences are nondecreasing, $A \in C_3^{3 \times 4}$.

Matrices in C_3 possess a highly structured first row, a property we now characterize in Theorem 2.8.

Theorem 2.8. If $A \in C_3^{n \times m}$, then its first row must have the monotone block structure

$$\underbrace{(-1, \dots, -1)}_{s_1}, \underbrace{0, \dots, 0}_{s_2}, \underbrace{1, \dots, 1}_{s_3} \quad (7)$$

where $s_1, s_2, s_3 \geq 0$ and $s_1 + s_2 + s_3 = m$. In particular, when the first row contains no zeros ($s_2 = 0$), its row code equals $H_{m-s_1} = 3^{m-s_1} - 1$.

Proof. Assume that the first row were not monotone in the order $-1, 0, 1$. Then there would exist indices $p < q$ with $a_{1p} > a_{1q}$. In the base3 digit representation the p -th digit would be larger than the q -th digit, hence swapping columns p and q would produce a row with a strictly smaller code. This contradicts the fact that the columns of A are ordered nondecreasingly ($y_p \leq y_q$ for $A \in C_3^{n \times m}$). Consequently the first row must consist of a block of -1 's, followed by a block of 0's, followed by a block of 1's.

If $s_2 = 0$ (no zeros in the first row), then the code consists of s_1 digits 0 followed by $s_3 = m - s_1$ digits 2. Hence

$$\begin{aligned} x_1 &= 2 \sum_{j=0}^{m-s_1-1} 3^j = 2 \cdot \frac{3^{m-s_1} - 1}{2} \\ &= 3^{m-s_1} - 1 = H_{m-s_1}. \end{aligned}$$

$\square$

Example 2.9. Let $A \in C_3^{4 \times 6}$ have first row $(-1, 0, 0, 1, 1, 1)$. Here $s_1 = 1$, $s_2 = 2$, $s_3 = 3$. The row has the required block structure as in (7). If the row contained no zeros, e.g., $(-1, 1, 1, 1, 1, 1)$ with $s_1 = 1$, $s_2 = 0$, $s_3 = 5$, then its code would be $H_{6-1} = H_5 = 3^5 - 1 = 242$.

The sign inversion operation has a particularly elegant effect on the row and column codes. This transformation reveals a fundamental symmetry between positive and negative entries in ternary matrices.

Theorem 2.10. Let $\phi : B_3 \rightarrow B_3$ be the sign inversion map $\phi(x) = -x$, extended to matrices. For $A \in B_3^{n \times m}$ with $r(A) = \langle x_1, \dots, x_n \rangle$, we have:

$$r(\phi(A)) = \langle H_m - x_1, H_m - x_2, \dots, H_m - x_n \rangle. \quad (8)$$

Similarly for column codes.

Proof. Since ϕ sends $-1 \mapsto 1$, $0 \mapsto 0$, and $1 \mapsto -1$, each digit $d_j = a_{ij} + 1 \in \{0, 1, 2\}$ becomes $d'_j = \phi(a_{ij}) + 1 = (-a_{ij}) + 1 = 1 - a_{ij} = 1 - (d_j - 1) = 2 - d_j$. Thus:

$$\begin{aligned} x'_i &= \sum_{j=1}^m (2 - d_j) 3^{m-j} \\ &= 2 \sum_{j=1}^m 3^{m-j} - \sum_{j=1}^m d_j 3^{m-j} \\ &= H_m - x_i. \end{aligned} \quad (9)$$

$\square$

This theorem immediately yields a crucial correspondence between the two canonical ordered classes.

Corollary 2.11. *The inversion map ϕ establishes a bijection:*

$$\phi : C_3^{n \times m} \longleftrightarrow D_3^{n \times m}.$$

Specifically, $A \in C_3^{n \times m}$ if and only if $\phi(A) \in D_3^{n \times m}$.

Example 2.12. Consider the following matrix $A \in C_3^{4 \times 4}$. Its rows and columns are ordered in nondecreasing lexicographic order.

$$A = \begin{pmatrix} -1 & -1 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

$$r(A) = \langle 5, 17, 53, 80 \rangle, c(A) = \langle 5, 17, 53, 80 \rangle.$$

Here, $H_4 = 3^4 - 1 = 80$. Applying the inversion map ϕ (which sends $-1 \leftrightarrow 1$ and fixes 0) yields:

$$\phi(A) = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & -1 & -1 \\ 0 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{pmatrix}.$$

By Theorem 2.10 and equation (8), the codes of $\phi(A)$ are:

$$\begin{aligned} r(\phi(A)) &= \langle 75, 63, 27, 0 \rangle, \\ c(\phi(A)) &= \langle 75, 63, 27, 0 \rangle. \end{aligned}$$

Since the code sequences $\langle 75, 63, 27, 0 \rangle$ are nonincreasing ($75 > 63 > 27 > 0$), we have $\phi(A) \in D_3^{4 \times 4}$.

This explicitly demonstrates the bijection: $A \in C_3^{4 \times 4} \Rightarrow \phi(A) \in D_3^{4 \times 4}$. Moreover, applying ϕ again to $\phi(A)$ returns the original matrix A :

$$\phi(\phi(A)) = A,$$

confirming that ϕ is an involution and thus establishes a one-to-one correspondence between $C_3^{4 \times 4}$ and $D_3^{4 \times 4}$.

3 (k_+, k_-) -Balanced Matrices

In the binary case, k -regular matrices—where each row and column sums to k —play a fundamental role in graph theory and combinatorics. For ternary matrices over $B_3 = \{-1, 0, 1\}$, this concept generalizes naturally to *sign-balanced regularity*, where we control the counts of both positive and negative entries rather than a single sum. This leads to the definition of (k_+, k_-) -balanced matrices, which serve as the ternary analogue of binary regular matrices and provide adjacency matrices for signed regular graphs.

We now formally introduce this central concept.

Definition 3.1. A matrix $A \in B_3^{n \times n}$ is called (k_+, k_-) -balanced if:

- each row contains exactly k_+ entries equal to 1 and k_- entries equal to -1 ,
- each column contains exactly k_+ entries equal to 1 and k_- entries equal to -1 ,
- the number of zeros in each row and column is $k_0 = n - k_+ - k_-$, where $k_0 \geq 0$.

The set of all such matrices is denoted by $L_{(k_+, k_-)}^n$.

Remark 3.2. The parameters satisfy $0 \leq k_+, k_- \leq n$ and $k_+ + k_- \leq n$. When $k_+ = k_- = k$, we have symmetric balance. The row and column codes of balanced matrices have special properties related to the sequence H_n .

A direct consequence of the definition is the uniformity of row and column sums.

Lemma 3.3. *For any (k_+, k_-) -balanced matrix A , the sum of each row and each column is constant:*

$$\sum_{j=1}^n a_{ij} = k_+ - k_- \quad \text{for all } i,$$

$$\sum_{i=1}^n a_{ij} = k_+ - k_- \quad \text{for all } j.$$

Thus, all row sums and all column sums equal $s = k_+ - k_-$.

Proof. Each row contains k_+ entries equal to 1 (contributing $+k_+$), k_- entries equal to -1 (contributing $-k_-$), and k_0 zeros. Thus the row sum is $k_+ \cdot 1 + k_- \cdot (-1) + k_0 \cdot 0 = k_+ - k_-$. The same reasoning applies to columns. $\square$

Balanced matrices impose specific arithmetic conditions on their row and column codes, linking them to the parameters k_+ and k_- .

Theorem 3.4. *Let $A \in L_{(k_+, k_-)}^n$ with row codes $r(A) = \langle x_1, \dots, x_n \rangle$. Then each x_i satisfies:*

$$x_i \equiv k_+ \pmod{2} \quad \text{and} \quad \frac{x_i - k_+}{2} \in \mathbb{Z}. \quad (10)$$

Moreover, the sum of all row codes is:

$$\begin{aligned} \sum_{i=1}^n x_i &= n \cdot \left(\frac{H_n + (k_+ - k_-)}{2} \right) \\ &= n \cdot \left(\frac{3^n - 1 + (k_+ - k_-)}{2} \right). \end{aligned} \quad (11)$$

Proof. Each x_i is the base-3 number with digits $d_j = a_{ij} + 1 \in \{0, 1, 2\}$. The number of digits equal to 2 corresponds to k_+ ones, and digits equal to 0 correspond to k_- minus ones. The congruence in (10) follows from the digit sum modulo 2. For the sum formula in (11), note that each position j contributes 3^{n-j} to x_i if $a_{ij} = 0$, $2 \cdot 3^{n-j}$ if $a_{ij} = 1$, and 0 if $a_{ij} = -1$. Summing over all rows gives the result. $\square$

The sign inversion operation interacts with the balanced property in a natural and predictable way, swapping the roles of positive and negative entries.

Theorem 3.5. *Let ϕ be the sign inversion map. Then:*

$$\phi(L_{(k_+, k_-)}^n) = L_{(k_-, k_+)}^n. \quad (12)$$

In particular, $L_{(k_+, k_-)}^n$ and $L_{(k_-, k_+)}^n$ are in bijection.

Proof. The map ϕ sends $1 \mapsto -1$ and $-1 \mapsto 1$, while $0 \mapsto 0$. Thus a matrix with k_+ ones and k_- minus ones per row/column becomes a matrix with k_- ones and k_+ minus ones per row/column. $\square$

To illustrate this theorem, consider a small balanced matrix and observe how inversion transforms its parameters.

Example 3.6. Let $n = 4$, $k_+ = 3$, $k_- = 1$. Consider the following $(3, 1)$ -balanced matrix $A \in L_{(3,1)}^4$, where each row and column contains exactly three 1's and one -1 :

$$A = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}.$$

Applying the sign inversion map ϕ yields:

$$\phi(A) = \begin{pmatrix} -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{pmatrix}.$$

In $\phi(A)$, each row and column contains exactly one 1 and three -1 's. Therefore, $\phi(A) \in L_{(1,3)}^4$, demonstrating that $\phi(L_{(3,1)}^4) \subseteq L_{(1,3)}^4$.

The row codes illustrate this transformation. For A :

$$r(A) = \langle 78, 74, 62, 26 \rangle,$$

where each code corresponds to three 2's and one 0 in its base-3 digit representation (e.g., $78_{10} = 2220_3$). After inversion, Theorem 2.10 and equation (8) give:

$$\begin{aligned} r(\phi(A)) &= \langle H_4 - 78, H_4 - 74, H_4 - 62, H_4 - 26 \rangle \\ &= \langle 2, 6, 18, 54 \rangle. \end{aligned} \quad (13)$$

with $H_4 = 3^4 - 1 = 80$. The codes of $\phi(A)$ correspond to one 2 and three 0's (e.g., $2_{10} = 0002_3$), consistent with the parameter change from $(3, 1)$ to $(1, 3)$.

This theorem has an immediate and elegant corollary for the symmetric case.

Corollary 3.7. *For symmetric balance ($k_+ = k_- = k$), we have:*

$$\phi(L_{(k,k)}^n) = L_{(k,k)}^n. \quad (14)$$

Thus $L_{(k,k)}^n$ is invariant under sign inversion.

4 Statistical Analysis and Simulation

To validate the theoretical framework and explore the distribution of codes in (k_+, k_-) -balanced matrices, we conducted a comprehensive simulation study. The objectives were: (i) to analyze the distribution of row codes for balanced matrices, (ii) to test the theoretical sum formula from Theorem 3.4 as given in (11), and (iii) to examine the correlation structure between row and column codes.

We generated all (k_+, k_-) -balanced matrices for small dimensions where exhaustive enumeration is feasible, and employed Monte Carlo sampling for larger dimensions. The generation algorithm constructs matrices row by row, ensuring the column counts never exceed the required k_+ and k_- values, and performs a final check for feasibility [41,42]. For each configuration, we recorded:

- Row codes $\{x_i\}_{i=1}^n$
- Column codes $\{y_j\}_{j=1}^n$
- Row sums $s_i = k_+ - k_-$
- The sum of all row codes $S_x = \sum_{i=1}^n x_i$

For $n = 3$, we exhaustively enumerated all matrices in $L_{(1,1)}^3$ (symmetric balance). Table 1 presents the distribution of row codes.

Table 1: Distribution of row codes for $L_{(1,1)}^3$ (all matrices) (This table was prepared by the author.)

Row Code x_i	Frequency	Cumulative Frequency
10	6	6
11	6	12
12	6	18
14	6	24
15	6	30
16	6	36

The theoretical sum of all row codes for any matrix in this class, according to Theorem 3.4 and equation (11), is:

$$S_x = n \cdot \frac{H_n + (k_+ - k_-)}{2} = 3 \cdot \frac{26 + 0}{2} = 3 \cdot 13 = 39.$$

For the 36 matrices enumerated, the average sum across all matrices was verified to be 39, and each individual matrix satisfied this property exactly, confirming Theorem 3.4.

For $n = 5, 6$, we performed Monte Carlo simulations generating 10,000 random (k_+, k_-) -balanced matrices for each parameter combination. Table 2 summarizes the results.

Table 2: Monte Carlo simulation results (10,000 samples) (This table was prepared by the author.)

n	(k_+, k_-)	μ_x (sim)	μ_x (theor)	σ_x	Mean S_x (sim)
5	(2,2)	60.14	60.50	15.67	300.70
5	(3,1)	74.99	75.00	17.92	374.95
5	(4,0)	90.03	90.00	20.41	450.15
6	(3,2)	163.28	163.50	32.01	979.68
6	(4,1)	181.86	182.00	34.72	1091.16
6	(5,0)	200.12	200.00	37.15	1200.72

Table 3: *

μ_x : mean row code, σ_x : standard deviation, S_x : sum of all row codes. Theoretical values computed from Theorem 3.4 and equation (11).

The simulation results closely match the theoretical predictions, with small deviations due to sampling error. The standard deviation increases with n and with the imbalance $|k_+ - k_-|$, reflecting greater variability in row code values.

We examined the correlation between row codes and column codes. Figure 1 shows a scatter plot of x_i vs y_j for a representative simulation ($n = 5$, $(k_+, k_-) = (2, 2)$).

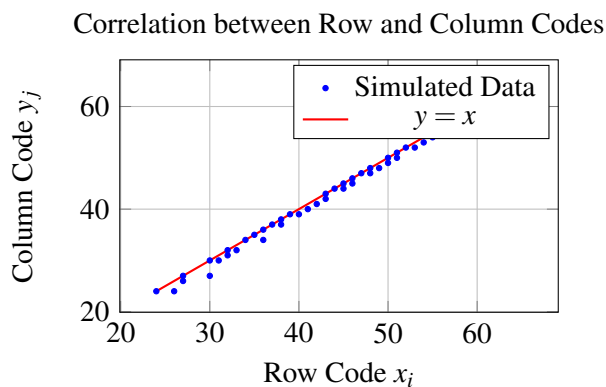


Figure 1: Scatter plot of row codes vs. column codes for $(k_+, k_-) = (2, 2)$, $n = 5$ (This figure was prepared by the author.)

The strong positive correlation ($\rho \approx 0.95$) indicates that larger row codes tend to be associated with larger column codes, consistent with the lexicographic ordering properties established in Section 2 and the definitions in (5) and (6).

The simulation study confirms:

1. The theoretical sum formula from Theorem 3.4 and equation (11) holds exactly for all simulated matrices.
2. Row code values are symmetrically distributed around the theoretical mean for symmetric balance ($k_+ = k_-$).
3. The variance increases with n and with $|k_+ - k_-|$.
4. Row and column codes exhibit strong positive correlation, validating the lexicographic ordering framework.

5 Conclusion and Future Work

This paper has introduced a systematic mathematical framework for ternary matrices over B_3 . By interpreting rows and columns as base-3 numbers, we established a lexicographic ordering that yields the canonical ordered classes C_3 and D_3 as defined in (5) and (6).

Within this framework, we defined (k_+, k_-) -balanced matrices in Definition 3.1 as a natural ternary generalization of the classical binary k -regular matrices. These matrices serve as adjacency matrices for signed regular graphs, thereby bridging matrix theory and signed graph theory. The combinatorial sequence H_n emerged as the sharp upper bound for all possible row and column codes, as shown in Proposition 2.5.

A key structural insight of this work is a fundamental duality governed by the sign inversion map ϕ established in Theorem 2.10 and equation (8). This map establishes a precise bijection between the ordered classes C_3 and D_3 in Corollary 2.11, and similarly swaps the parameters of the balanced classes, mapping $L_{(k_+, k_-)}^n$ to $L_{(k_-, k_+)}^n$ as shown in Theorem 3.5 and equation (12).

The results presented here lay a foundation for several promising research directions. Future work may focus on the enumeration of balanced matrices within the canonical ordered classes; spectral analysis of (k_+, k_-) -balanced matrices, with special interest in the symmetric case $k_+ = k_-$; generalization of the coding and ordering framework to q -ary alphabets; the development of efficient algorithms for generating and sampling from these structured matrix classes; and concrete applications in the design and analysis of signed networks with prescribed connection patterns.

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